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Rapid Communications

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Quantum nucleation of false-vacuum bubbles

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We show that a small bubble of false vacuum can tunnel to the critical size for inflation, and calculate the amplitude in leading WKB approximation. An initially nonsingular space becomes an exterior space plus a baby universe (which contains the bubble), joined by a black-hole singularity. We work in a Hamiltonian formalism; the corresponding Euclidean bounce is shown to have a degenerate vierbein.

I. INTRODUCTION

In this Rapid Communication we study the nucleation of false-vacuum bubbles by quantum tunneling. Our interest in the process is twofold: as the possible origin of matter in our Universe and as a thought experiment for investigating various technical and conceptual issues in quantum gravity.

Various mechanisms have been proposed[1,2] to explain the vanishing of the cosmological constant, but only at the cost of predicting an empty universe, with no matter or radiation.[3] It has been suggested that the observed Universe may have arisen from such an empty or nearly empty state by a gravitational instability, negative gravitational potential energy offsetting the positive energy of matter.[4–10] In particular, a large enough bubble of *false* vacuum ($\Lambda_F > 0$) embedded in *true* vacuum ($\Lambda_T = 0$) will grow indefinitely, the tendency of the false vacuum to inflate overcoming the inward force on the bubble from pressure and tension.[4–6] The observed Universe arises from subsequent decay in the interior of the bubble. Notice that this is the reverse of the familiar process of *false*-vacuum decay by the growth of bubbles of *true* vacuum.

It has been shown that such a growing bubble cannot develop classically from a nonsingular initial configuration.[7] It might, however, be produced in a quantum process:[5–10] an initial bubble, too small to inflate, tunneling to the critical size for growth and then evolving classically. It is this process that we study, using Hamiltonian WKB methods. We find that occurs, and we calculate the exponential suppression factor. Other groups have studied this process using different methods: the Euclidean path integral[8,9] and a quantum-mechanical description of the bubble.[8,10] In the Conclusion we discuss the relation of these papers to our work.

The final state after false-vacuum bubble nucleation is two spaces connected by an Einstein-Rosen bridge, which collapses classically to form a black-hole singularity. Whether the spacetimes become topologically disconnected due to black-hole evaporation is an open question.

In a future publication we will discuss this work in greater detail, and give further applications to quantum cosmology.

II. WKB GRAVITY

We are interested in the amplitude for a small false-vacuum bubble to tunnel to the critical size for inflation. We will assume that the initial and final bubbles, and the path of least action between, are spherically symmetric, so in the leading WKB approximation we need only consider metrics of the form

$$ds^2 = -N^t(t,r)^2 dt^2 + L(t,r)^2[dr + N^r(t,r)dt]^2 + R(t,r)^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{1}$$

In (1), N^t is the lapse, N^r the shift, L the ratio of proper to coordinate length in the radial direction, and R the transverse radius [e.g., the area of a shell at fixed t,r is $4\pi R^2(t,r)$]. Quantum gravity restricted to spherically symmetric fields is a version of minisuperspace known as the Berger-Chitre-Moncrief-Nutku (BCMN) model.[11] Construction of the Hamiltonian formalism is standard:[11–13] the action for gravity plus a general matter system, with metric (1), can be brought to the form

$$S = p_i \dot q^i + \int dr\, dt (\pi_L \dot L + \pi_R \dot R - N^t \mathcal{H}_t - N^r \mathcal{H}_r)\,, \tag{2}$$

where q^i are general matter degrees of freedom. The primary constraints are $\pi_{N^t} = 0$ and $\pi_{N^r} = 0$, and the secondary constraints are $\mathcal{H}_t = 0$ and $\mathcal{H}_r = 0$. In Dirac's canonical quantization,[12] these are imposed on the state:

$$\pi_{N^t,N^r}(r)\psi = \mathcal{H}_{t,r}(r;\pi_L,\pi_R,p,L,R,q)\psi = 0\,. \quad (3)$$

These four equations essentially fix the dependence of ψ on N^t, N^r, L, and R, leaving the matter coordinates q as dynamical degrees of freedom. By the primary constraints, $\psi(N^t,N^r,L,R,q)$ is independent of N^t and N^r, so that ψ depends only on the spatial geometry and the matter configuration. The $\mathcal{H}_r$ constraint is also simple, setting the amplitudes equal for configurations which differ only by an r reparametrization. The $\mathcal{H}_t$ constraint is more complicated because it is second order in momenta.

In the WKB approximation, set

$$\psi(L,R,q) = \exp[i\Sigma_0(L,R,q)/\hbar + O(\hbar^0)]\,.$$

To leading order in $\hbar$, the $\mathcal{H}_t$ and $\mathcal{H}_r$ constraints become Hamilton-Jacobi equations:

$$\mathcal{H}_{t,r}\left[r;\frac{\delta\Sigma_0}{\delta L},\frac{\delta\Sigma_0}{\delta R},\frac{\delta\Sigma_0}{\delta q},L,R,q\right] = 0\,. \quad (4)$$

One can solve these equations to determine (up to the usual sign choices) the gradients of Σ_0 with respect to the gravitational fields, $\delta\Sigma_0/\delta L$ and $\delta\Sigma_0/\delta R$, and integrate to obtain the L and R dependence at fixed q. (The equations are integrable because the algebra of constraints closes.) The next-order WKB approximation would require treatment of nonspherical metrics, operator ordering, and central charges in the constraint algebra. We will not attempt this; it is vastly simpler in the path-integral formalism, if the latter can be developed.

For

$$S = (16\pi G)^{-1}\int d^4x\sqrt{g}(R-2\Lambda) + S_M,$$

where M denotes matter, the secondary constraints become

$$\mathcal{H}_t = \frac{GL\pi_L^2}{2R^2} - \frac{G\pi_L\pi_R}{R} + \frac{1}{2G}\left[\left(\frac{2RR'}{L}\right)' - \frac{R'R'}{L} - L + \Lambda LR^2\right] + \mathcal{H}_{tM}\,,$$
$$\mathcal{H}_r = R'\pi_R - L\pi_L' + \mathcal{H}_{rM}\,, \quad (5)$$

where a prime denotes d/dr. Vanishing of the linear combination $R'\mathcal{H}_t/L + G\pi_L\mathcal{H}_r/RL$ implies $\mathcal{M}' = R'\mathcal{H}_{tM} + G\pi_L\mathcal{H}_{rM}/R$, where

$$\mathcal{M} = \frac{G\pi_L^2}{2R} + \frac{R}{2G}\left[1 - \left(\frac{R'}{L}\right)^2 + \frac{\Lambda R^2}{3}\right]\,. \quad (6)$$

By considering a static slice ($\pi = 0$) in the absence of matter, one identifies $\mathcal{M}$ as the mass parameter in Schwarzschild or Schwarzschild–de Sitter geometry. For the problem at hand, the spatial topology is R^3, with the origin a nonsingular point and with cosmological constant zero and Schwarzschild mass M at infinity. This gives the boundary conditions $\mathcal{M}(r) \to 0$ as $r \to 0$, $\mathcal{M}(r) \to M$ as $r \to \infty$.

III. VACUUM BUBBLES

For a spherically symmetric bubble in the thin-wall approximation, the only matter degree of freedom is the coordinate radius $\hat{r}$. The cosmological constant takes a positive value Λ for $r < \hat{r}$ and vanishes for $r > \hat{r}$. In addition, the energy density of the bubble wall gives $\mathcal{H}_{tM}(r) = \delta(r-\hat{r})(L^{-2}\hat{p}^2 + m^2)^{1/2}$ and $\mathcal{H}_{rM}(r) = -\delta(r-\hat{r})\hat{p}$, where $m = 4\pi\sigma R^2(\hat{r})$, σ being the tension in the bubble wall. Equation (6) and the boundary conditions enable us to solve for $\pi_L(r)$ in the interior and the exterior. The solution can be written as

$$\mathbf{V}\cdot\mathbf{V} = \begin{cases} 1 - \Lambda R^2/3, & r < \hat{r}\,, \\ 1 - 2GM/R, & r > \hat{r}\,, \end{cases} \quad (7)$$

where $\mathbf{V} = (R'/L, G\pi_L/R)$ and $\mathbf{V}\cdot\mathbf{V} = V_1^2 - V_2^2$. The constraints also imply matching conditions at the domain wall. Requiring the metric to be continuous at the wall and π_L and π_R to be nonsingular there leads to

$$\mathbf{V}(\hat{r}+\epsilon) - \mathbf{V}(\hat{r}-\epsilon) = -\mathbf{V}_M\,, \quad (8)$$

where $\mathbf{V}_M = (G(\hat{p}^2+m^2)^{1/2}/\hat{R}, G\hat{p}/\hat{R})$, $\hat{R} = R(\hat{r})$, and $\mathbf{V}_M\cdot\mathbf{V}_M = G^2m^2/\hat{R}^2$.

Equations (7) and (8) allow a simple description of the allowed and forbidden regions as a function of $\hat{R} = R(\hat{r})$. If $\mathbf{V}(\hat{r}+\epsilon)$, $\mathbf{V}(\hat{r}-\epsilon)$, and $\mathbf{V}_M$ satisfy the Lorentzian triangle inequality (one length *greater than* the sum of the other two), the matching conditions have a solution with real momenta; if they satisfy the usual Euclidean triangle inequality, the momenta are imaginary.

IV. TUNNELING

The classical motions of the false-vacuum bubble have been studied extensively in Refs. 4–6. One conclusion, which can be obtained readily from Eqs. (7) and (8) above, is the following: For M sufficiently small ($M < M_S$ in the notation of Ref. 6), there are two allowed regions, $\hat{R} \le R_1$ and $\hat{R} \ge R_2$, where the turning radii R_1 and R_2 are related to the exterior Schwarzschild and interior de Sitter radii by $R_S < R_1 < R_2 < R_D$. At $\hat{R} = R_1$ and R_2, there are spacelike slices which are static de Sitter inside and static Schwarzschild outside. These geometries are shown in Figs. 1(a) and 1(b). For the R_1 turning point, $R(r)$ is monotonically increasing. For the R_2 turning point, $R'(\hat{r}+\epsilon)$ is negative, and $R(r)$ in the exterior region decreases to $R_S = 2GM$ before increasing. The subsequent classical motion of the R_1 bubble is to collapse into a Schwarzschild singularity. The subsequent classical motion of the R_2 bubble is to inflate indefinitely, while the neck in the geometry collapses into a Schwarzschild singularity.

It is shown in Ref. 7 that the R_1 geometry is *buildable*—it can be constructed classically in a universe without past singularity—and that the R_2 geometry is not buildable. We are interested in the amplitude to tunnel from the former geometry to the latter. The relative amplitude for these two geometries is obtained by integrating $\nabla\Sigma_0$ along any smooth path connecting the two. An example of such a path is provided (for $\Lambda < 48\pi^2\sigma^2G^2$) by choosing $L=1$ and by the minimal slicing[14] $\pi_l^l = L\pi_L + 2R\pi_R = 0$. With the $\mathcal{H}_r$ constraint and the matching conditions this implies $\pi_L(r)=0$ for $r<\hat{r}$ and $\pi_L(r) = -\hat{p}[\hat{R}/R(r)]^{1/2}$ for $r>\hat{r}$. Solving the matching conditions, $\hat{p}^2$ and $R'(\hat{r}+\epsilon)$ are smooth functions of $\hat{R}$ for $R_1<\hat{R}<R_2$. For example,

$$R'(\hat{r}+\epsilon) = (1-\Lambda\hat{R}^2/3)^{-1/2} \times [1 - GM/\hat{R} - G(8\pi^2G\sigma^2 + \Lambda/6G)\hat{R}^2]\,. \quad (9)$$

Equation (7), the minimal gauge condition, and Eq. (9) determine R' as a function of R. This can be integrated to fix the whole geometry as a function of $\hat{R}$; it interpolates continuously between Figs. 1(a) and 1(b). The amplitude is now found from

$$-i\hbar\delta\ln\psi = \hat{p}\delta\hat{r} + \int_0^\infty dr(\pi_L\delta L + \pi_R\delta R)\,,$$

which, with some effort, integrates to

$$\hbar G\ln\frac{\psi_\pm(\hat{R})}{\psi_\pm(R_1)} = \mp\int_{\hat{r}}^\infty dr\left[R(1-R'^2+2GM/R)^{1/2} - RR'\arccos\frac{R'}{(1-2GM/R)^{1/2}}\right] \pm\int_{R_1}^{\hat{R}} d\hat{R}\,\hat{R}\arccos\frac{R'(\hat{r}+\epsilon)}{(1-2GM/\hat{R})^{1/2}}\,, \quad (10)$$

where $R'(\hat{r}+\epsilon)$ is given as a function of $\hat{R}$ by Eq. (9), and where the arccos always lies between 0 and π. The upper sign represents tunneling from R_1 to R_2. The first WKB approximation to the tunneling rate is e^{-B}:

$$B = -2\ln\frac{\psi_+(R_2)}{\psi_+(R_1)} = \frac{\pi}{\hbar G}(R_2^2 - 4G^2M^2) - \frac{2}{\hbar G}\int_{R_1}^{R_2} d\hat{R}\,\hat{R}\arccos\frac{R'(\hat{r}+\epsilon)}{(1-2GM/\hat{R})^{1/2}}\,. \quad (11)$$

This is in the range

$$\pi(R_1^2 - 4M^2G^2)/\hbar G < B < \pi(r_2^2 - 4M^2G^2)/\hbar G\,;$$

ϵ^{-B} is $\ll 1$ unless the bubble parameters are near the Planck scale.

As a check, the derivation of (10) also applies in a case studied by Coleman and de Lucchia,[15] namely $M=0$ and $\Lambda < -48\pi^2G^2\sigma^2$, in which case our Hamiltonian analysis agrees with their Euclidean result.

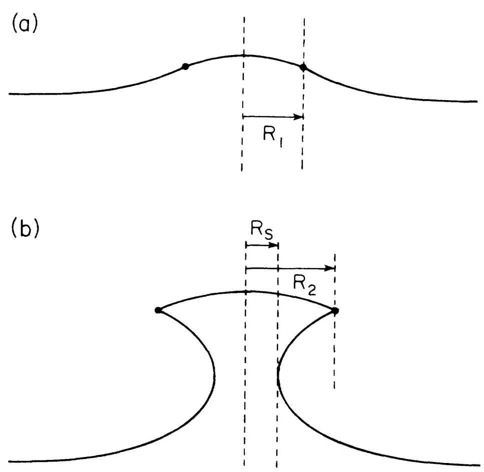

FIG. 1. Turning-point geometries: de Sitter interior, Schwarzschild exterior, with a kink produced by the bubble wall. These figures are obtained embedding the three-geometry in Euclidean four-space, and drawing a slice at fixed θ,ϕ. (a) The buildable geometry: the bubble subsequently collapses. (b) The unbuildable geometry: the bubble inflates indefinitely.

V. CONCLUSION

We have shown that false-vacuum bubble nucleation does occur, given an appropriate seed. The cosmological implications of this will be considered in future work. Note that, provided the bubble parameters are small compared to the Planck mass, all curvatures are small and the calculation should be trustworthy. However, the subsequent classical evolution leads to a singularity, and the final state consists of two universes: an interior universe and a closed "baby universe" with the inflating bubble, joined by a black-hole singularity.[4-10] If black-hole evaporation proceeds to completion then topology change has occurred; it is not clear that there is any other consistent final state.

Farhi, Guth, and Guven[8] have studied this process using the Euclidean path integral. (Euclidean solutions have also been considered in Ref. 9 but these do not represent false-vacuum nucleation; they are closed, without initial and final hypersurfaces, and so their interpretation is unclear.) A puzzle arises because they show that there is no (nondegenerate) Euclidean bounce solution for this process. On the other hand, given any Minkowski tunneling process such as we have found, one can always construct a Euclidean bounce solution: label the successive configurations in the history by a parameter τ, which fixes the spatial metric components $L(r,\tau)$, $R(r,\tau)$. The lapse and shift, $N^t(r,\tau)$ and $N^r(r,\tau)$, are derived from the relation between velocities and momenta. When this is done in the present case, the Euclidean vierbein is found to be *degenerate* in places. In particular, the determinant of the vierbein must change sign on the final slice, Fig. 1(b), between the bubble wall and the narrow point on the neck. Reference 8 describes this Euclidean spacetime in terms of a bounce solution which is nondegenerate but double valued, and gives a prescription for the bounce action which agrees with our tunneling result. From the Euclidean point of view, it is not clear whether such spacetimes should be allowed, but the Hamiltonian analysis indicates unambiguously that the process occurs: we have simply integrated the constraints, and it is not possible to forbid the process without doing violence to the constraints. This is evidence that one must necessarily include degenerate vierbeins in the path integral; this possibility has been considered from a different point of view in Ref. 16.

Because spherically symmetric gravitational fields have no dynamics, one would expect to be able to reduce the system to a quantum mechanics of the bubble radius. This has been attempted in Refs. 8 and 10, but the resulting quantum-mechanics problem has certain pathologies. Our Dirac quantization can be reduced to a quantum-mechanics problem by choosing a gauge—the pathologies in Refs. 8 and 10 are due to the gauge choice. This will be discussed in more detail in our forthcoming work.

Goncharov, Linde, and Mukhanov[17] have discussed the nucleation of baby universes in a somewhat different context, a chaotic-inflationary potential, whereas we are implicitly considering an old-inflationary potential. It seems likely that this process will occur in any kind of inflationary system.

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