Smooth-rough transition in Polyakov-Kleinert string

Zhou Xiaoan
Department of Physics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6
(Received 12 May 1989; revised manuscript received 27 December 1989)

A nontrivial saddle-point value of the dynamically generated string tension is found which, near the critical temperature, approaches zero and depends on the zero-temperature string tension linearly, implying that the Hausdorff dimension of the string surface is two. A smooth-rough transition is then found in the Polyakov-Kleinert string with the critical temperature in numerical agreement with the previous result.

A kind of phase transition in the Polyakov-Kleinert string[1,2] has been predicted by Tse,[3], Kleinert,[4] and the present author and Viswanathan.[5] The critical temperature of the transition has been found. However, the nature of the phase transition remains unclear. The reason is that it is unclear if the Polyakov-Kleinert string could be smooth at large distances below some finite critical temperature. This problem has fundamental importance since if a string is not smooth at large distances it seems unlikely that it belongs to the correct universality class for QCD strings to describe the large-N QCD.[1] The same question can also be asked in the context of membranes.[6]

A persistence length has been implied by the property of asymptotic freedom of the Polyakov-Kleinert string[1,2,7,8] as well as membranes[9]:

$$\xi=\lambda^{-1/2}=\frac{1}{\Lambda}\exp\left[\frac{4\pi}{d\alpha(\Lambda)}\right]=\frac{1}{\mu}\exp\left[\frac{4\pi}{d\alpha(\mu)}\right], \tag{1}$$

where Λ is the ultraviolet momentum cutoff, $\alpha(\Lambda)$ is the extrinsic coupling (inverse rigidity) at this scale, μ a renormalized energy scale, and $\alpha(\mu)$ the corresponding extrinsic coupling. It has been claimed that[7–9] the string surface appears smooth at scales smaller than ξ and crumpled at scales larger than ξ. The question is if it is possible for ξ to be infinite at or below some finite critical temperature. Polyakov argued that,[1,10] if there were a fixed point in the theory, at a critical point λ would go to zero with a critical exponent ν which is related to the Hausdorff dimension of the string surface. There would be long-ranged correlation among the normals to the surface, implying that the string is essentially smooth. The average size of the surface would increase as some power of its area.

Although the question of an infrared fixed point is now difficult to answer we are going to deal, in this Brief Report, with this problem in terms of the saddle-point approximation in the large-d limit. A nontrivial saddle-point value of the dynamically generated string tension is found which, near the critical temperature, approaches zero and depends on the zero-temperature string tension linearly, implying that the Hausdorff dimension of the string surface equals its topological dimension. We then arrive at the conclusion that the Polyakov-Kleinert string is smooth at large distances at or below the critical temperature. A critical temperature is found which agrees with the previous result numerically.[3,4] Above the critical temperature, the generated string tension (which has the physical significance of being the inverse-squared persistence length) becomes finite, implying that the string is rough. It is thus shown that there is a smooth-rough transition in the Polyakov-Kleinert string. The essential difference between the Polyakov-Kleinert string and the Nambu-Goto string is also discussed.

We start with the string action in the first-order form[1]:

$$S_0(X^\mu,g_{ab},\lambda_{ab})=\sigma_0\int g^{1/2}d^2\xi+\frac{1}{2\alpha_s}\int g^{1/2}d^2\xi[(\Delta X^\mu)^2-\lambda^{ab}(g_{ab}-\partial_a X^\mu\partial_b X^\mu)], \tag{2}$$

where ΔX^μ is defined by

$$\Delta X^\mu=\frac{1}{g^{1/2}}\partial_a g^{1/2}g^{ab}\partial_b X^\mu \tag{3}$$

and λ^{ab} is the Lagrange multiplier which ensures the constraint that the metric is the induced metric of the surface. We notice that the path integral

$$Z=\int[dg_{ab}d\lambda_{ab}dX^\mu]\exp[-S_0(X^\mu,g_{ab},\lambda_{ab})] \tag{4}$$

contains the integral over X^μ which is Gaussian and can then be calculated in standard fashion. We find

$$Z=\int[dg_{ab}d\lambda_{ab}]\exp[-S_{\text{eff}}(g_{ab},\lambda_{ab})] \tag{5}$$

and

$$S_{\text{eff}}(g_{ab},\lambda_{ab})=\frac{d}{2}\ln\det{}'(\Delta^2-D_a\lambda^{ab}D_b)+\int g^{1/2}d^2\xi\left[\sigma_0-\frac{\lambda^{ab}g_{ab}}{2\alpha_0}\right], \tag{6}$$

where d is the dimension of the bulk space and the prime on the determinant omits the zero modes. If we take d to be large, then since it enters the exponent of (5), we have reason to expect that the saddle-point approximation will be applicable. This is indeed the case. For the same reasons we neglect the ghost contribution for our purposes. To find a reasonable saddle point, we make the *Ansätze*.

(a) λ^{ab} is proportional to g^{ab} (Refs. 7 and 8):

$$\lambda^{ab}(\xi)=\lambda g^{ab}(\xi). \tag{7}$$

[In this work we only discuss an isotropic Ansatz as an approximation. In Refs. 3 and 4 it is shown that an anisotropic $\lambda^{ab}=\lambda^a g^{ab}$ (without the sum over a here) is necessary for an exact solution but the anisotropy turns out to be small. This is why we shall neglect it.]

(b) The Liouville mode does not propagate.

The *Ansatz* (a) says that λ^{ab} is isotropic which is consistent with the reparametrization invariance of the theory. The *Ansatz* (b) can be made possible by working the theory in the critical dimensions if they exist.[7]

We now have to solve a set of saddle-point equations corresponding to the conformal changes of the metric g_{ab} and the changes of λ_{ab}:

$$g_{ab}(\xi)\frac{\delta S_{\text{eff}}}{\delta\lambda_{ab}(\xi)}=0 \tag{8}$$

and

$$g_{ab}(\xi)\frac{\delta S_{\text{eff}}}{\delta g_{ab}(\xi)}=0 \ . \tag{9}$$

Under the *Ansätze* (a) and (b) and in the "conformal gauge"

$$g_{ab}=\rho\delta_{ab} \ , \tag{10}$$

these equations reduce to

$$\frac{\delta S_{\text{eff}}}{\delta\lambda}=0 \tag{8$'$}$$

and

$$\frac{\delta S_{\text{eff}}}{\delta\rho}=0 \ . \tag{9$'$}$$

[We use the term "conformal gauge" although the theory is not conformally invariant.[11] In Refs. 3 and 4, one needs $g_{ab}=\rho_a\delta_{ab}$ (without the sum over a here) because of the anisotropy of λ^{ab}. See also the parenthetical remark following Eq. (7).] Equation (8) is easy to solve; we find

$$\lambda_*=\Lambda^2\exp(-8\pi/d\alpha_0) , \tag{11}$$

where Λ is the ultraviolet momentum cutoff. The result of (11) agrees with (1).[1,2,7–9]

To solve Eq. (9$'$) we note that, in the effective action (6),

$$\frac{d}{2}\ln\det{}'(\Delta^2-D_a\lambda^{ab}D_b)=\frac{d}{2}\,\text{Tr}'\ln(-\Delta)+\frac{d}{2}\,\text{Tr}'\ln\left[-\Delta+\frac{D_a\lambda^{ab}D_b}{\Delta}\right] . \tag{12}$$

The first term on the right-hand side (RHS) of (12) contributes to the effective action a finite value, for a rectangle,[12,13]

$$(d/2)\ln\det{}'(-\Delta)=-\rho\, d\pi/6 \ , \tag{13}$$

where we have set $R\gg T$ and $T=\rho R=\sqrt{\rho}\,\beta=\sqrt{\rho}/\theta$ and the sheet C is defined by

$$C=\{(\xi^1,\xi^2)|0\le\xi^1\le R\}/\sim \ , \tag{14}$$

where $\sim$ represents the equivalence relation defined by

$$(\xi^1,\xi^2)\sim(\xi^1,2T+\xi^2) \ . \tag{15}$$

Using the ζ-function regularization we obtain, for the second term on the RHS of (12),[13]

$$\frac{d}{2}\ln\det{}'\left[-\Delta+\frac{D_a\lambda^{ab}D_b}{\Delta}\right]=\rho\, d\pi I(a)+\frac{\rho\, da^2}{8\pi}\left[1+\ln\frac{\Lambda^2}{\lambda}\right] , \tag{16}$$

where

$$I(a)=4\int_0^\infty\frac{dy}{1-e^{2\pi(y+a/2\pi)}}y^{1/2}(y+a/\pi)^{1/2}=-\frac{a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}K_1(na) \tag{17}$$

and

$$a^2=\lambda\beta^2 \ , \tag{18}$$

while $K_1(na)$ is the modified Bessel function and $I(a)$ has the limiting values

$$I(a)=\begin{cases}-\frac{1}{6} & \text{as } a\to 0 \ , & \text{(19a)}\\ 0 & \text{as } a\to\infty \ . & \text{(19b)}\end{cases}$$

Gathering all these pieces gives

$$S_{\text{eff}}=\sigma_{\text{eff}}\rho\beta^2+(\text{terms independent of }\rho) \tag{20}$$

with

$$\sigma_{\text{eff}}=\sigma_0-\frac{\lambda}{\alpha_0}+\frac{d\lambda}{8\pi}\left[1+\ln\frac{\Lambda^2}{\lambda}\right]-\frac{\pi d}{6\beta^2}[1-6I(a)] \ . \tag{21}$$

Then the saddle-point equation (9$'$) reduces to

$$\sigma_{\text{eff}}=0 \ . \tag{22}$$

The result of (22) has also been found by Pisarski[7] and David and Guitter.[8] However, these authors discussed little on its implication on the phase transition. The reason is that, by using dimensional regularization, none of their results for σ_{eff} is as complete as our expression (21): In Eqs. (2.9) of Ref. 7 and (2.23) of Ref. 8, the β-dependence term, the last term on the RHS of our Eq. (21), was overlooked by using dimensional regularization. This led the author of Ref. 7 to claim that, for $\lambda > 0$, the bare string tension must be negative and no phase transition can be identified. [Compare Eq. (2.11) of Ref. 7 with our Eq. (24) below.] In Eq. (2.23) of Ref. 8, two extra terms are also contained. One is a quadratic divergent term which should be absent in dimensional regularization as well as in ζ-function regularization. The other is a $1/\rho$ term which should not contribute to the saddle-point equation $\delta S_{\text{eff}}/\delta\rho=0$, since $S_{\text{eff}} \propto \sigma_{\text{eff}}\rho$, the $1/\rho$ term in σ_{eff}, is a ρ-independent term in S_{eff}. We still do not understand how such a ρ-independent term could lead the author of Ref. 8 to determine the saddle-point value for ρ [see Eq. (2.24b) of Ref. 8 and below].

As argued by Pisarski,[7] the value of ρ is not determined by the saddle-point equation (9′) which is merely an expression of general coordinate invariance of the theory. (More precisely, it is a result of a global scale invariance. For details, see Ref. 13.) An effective action can only be formed from invariant quantities such as a cosmological term $\sim\sqrt{g}$, an Einstein term $\sim\sqrt{g}\,R$, etc. For a surface with a fixed topology, the Einstein term does not contribute to the saddle-point equations and, as a result, the effective string tension vanishes and $S_{\text{eff}}=\text{const}$ at the stationary point, which depends only on the topology of the surface.

Although Eqs. (9) or (20) does not determine the value of ρ it, as combined with (11), determines the value of λ. The solution of (22) is

$$\alpha_0^* = \frac{8\pi/d}{\ln\Lambda^2/\lambda_*} \tag{23}$$

and

$$\lambda_* = -\frac{\sigma_0 8\pi}{d} + \frac{4\pi^2}{3\beta^2}[1-6I(a)] \ . \tag{24}$$

Equations (23) and (24) are consistent with the famous result of asymptotic freedom found in Refs. 1 and 2. Equation (24) is reminiscent of the situation in the Ising model[14] and must be solved numerically to determine the critical temperature. However, it is easy to see that there is always one solution, by using the limiting value of $I(a)$ in (19a), if

$$\theta \ge \theta_C = \left[\frac{3\sigma_0}{\pi d}\right]^{1/2} . \tag{25}$$

As $\theta \to \theta_C$ from above, λ decreases and we may obtain its asymptotic dependence by using the limiting value of $I(a)$ in (19a), that is,

$$\lambda = -\frac{\sigma_0 8\pi}{d} + \frac{8\pi^2}{3\beta^2} \ . \tag{26}$$

We see that λ approaches zero in a singular fashion as θ approaches θ_C from above, vanishing asymptotically as

$$\lambda \sim \left[1-\frac{\theta_C^2}{\theta^2}\right]^{\nu=1} . \tag{27}$$

It is interesting to compare the critical temperature (25) with that found in Refs. 3 and 4. We find that even though we have used an isotropic *Ansatz* for the Lagrange multiplier λ_{ab} and the metric, the critical temperatures almost coincide numerically. The only difference being that $d-2$ in the formulas of Refs. 3 and 4 is replaced by d in (25). This discrepancy is because a physical gauge has been taken in Refs. 3 and 4 and is not important for large d. (We remind the reader that the isotropic and anisotropic systems under consideration are generally different in nature.)

The exponent for the power-law behavior of the order parameter in (27) is given the symbol $\nu=2/H$ according to Polyakov[10]:

$$\lambda \sim \left[1-\frac{\sigma_{0\text{cr}}}{\sigma}\right]^{\nu=2/H} \tag{28}$$

with

$$\sigma = d\pi\theta^2/3 \text{ and } \sigma_{0\text{cr}} = \sigma_0 = d\pi\theta_C^2/3 \ .$$

Here H is the Hausdorff (or fractal) dimension. Comparing (28) with (27) gives $H=2$; that is, for the Polyakov-Kleinert string, the Hausdorff dimension equals its topological dimension at or below the critical temperature. (Note that $\partial\lambda/\partial\beta|_{\beta\to\beta_c} = -2\pi^2/3\beta_c^3 < 0$, but λ cannot be negative; we conclude that λ remains zero at or below the critical temperature.)

We now show that the critical temperature (25) associates with a second-order smooth-rough transition. To prove this we need to calculate the mean fluctuations $\langle u_q \rangle$ of the string world sheet from a reference plane, say the (ξ^1,ξ^2) plane, near the transition. This issue was studied more than a decade ago by Helfrich.[15] The result is, in terms of our notation,

$$\langle |u_q|^2 \rangle = \frac{\alpha_0}{A(q^4+\lambda q^2)}$$

with

$$A = L^2 \text{ and } q = \frac{2\pi}{L} \ , \tag{29}$$

where L is the length scale of the string world sheet. In the regime of temperature $\theta \le \theta_C$, $\lambda=0$, and, therefore, $\alpha_0=0$ [e.g., (23)]. We thus have $\langle |u_q|^2 \rangle = 0$ for $\theta \le \theta_C$ which means that the string is smooth for $\theta \le \theta_C$. [We recall $H(\theta_C)=2$.]

In the regime of temperature $\theta > \theta_C$ and $0 < \lambda_0 < 1/L^2$, we have $\alpha_0 > 0$ and so that

$$\langle |u_q|^2 \rangle \sim \frac{L^2}{(2\pi)^4[1+\lambda L^2/(2\pi)^2]} \sim L^2 \ . \tag{30}$$

Equation (30) means that the string is rough for $\theta > \theta_C$

and $0<\lambda<1/L^2$.

We see from (27) and (28) that λ plays a role of an order parameter in the Polyakov-Kleinert string: At or below the critical temperature, λ approaches and remains zero, implying that the system is in the global $O(d-2)\times O(2)$-symmetric phase and the string surface is essentially smooth with Hausdorff dimension two. Above the critical temperature, on the other hand, λ increases, implying that the global $O(d-2)\times O(2)$ symmetry is broken and the system is in the rough phase with Hausdorff dimension larger than two.

One might wonder that since thermal fluctuations soften the rigidity at large distances, what mechanism controls the fluctuations? To answer this question we substitute the saddle-point solutions (23) and (24) into the expression of the effective string tension (21) to get

$$\sigma_{\text{eff}}=\sigma_0+\frac{d\lambda}{8\pi}-\frac{\pi d}{6\beta^2}[1-6I(a)] \ . \tag{31}$$

The β-dependent terms in (31) arise from the zero-point transverse oscillations (undulations) of the stretched string which tends to lower the string tension. On the other hand, the term with λ, the dynamically generated string tension, tends to increase the string tension. Here we see the important role in controlling the thermal fluctuations in the infrared region played by the dynamically generated string tension. This is quite different from the Nambu-Goto string where no such dynamically generated string tension exists which can control the fluctuations. Because of this, the string is severely creased in the infrared region at any finite temperature. Although there exists a "critical temperature" in the Nambu-Goto string,[16] it does not associate with a smooth-rough transition.

We close this Brief Report by concluding that there is a smooth-rough transition in the Polyakov-Kleinert string. This transition is second order in nature (the correlation length tends to infinity as the critical point reached). There remain two questions to be answered. First, since our result, particularly the order of the transition, arises from single-string theory, will the result change as an ensemble of strings (which QCD seems to be) is considered? In general, this is possible.[17] However, detailed investigation[13] shows that our single-string result agrees with that of an ensemble of strings if the topology of the string world sheets is fixed. Second, can we identity the smooth-crumpled transition with the QCD deconfinement transition? This question is extensively studied in Ref. 13. Here we only report the conclusion that there is a close resemblance between these two transitions.

The author is grateful to Professor K. S. Viswanathan and Dr. Madan Rao for valuable discussions. This work was supported by Simon Fraser University.

[1]A. M. Polyakov, Nucl. Phys. **B268**, 406 (1986).
[2]H. Kleinert, Phys. Lett. B **174**, 335 (1986).
[3]Sze-Man Tse, Phys. Rev. D **37**, 2337 (1988).
[4]H. Kleinert, Phys. Lett. B **189**, 187 (1987); Phys. Rev. D **40**, 473 (1989); G. German and H. Kleinert, Phys. Lett. B **220**, 133 (1989); **225**, 107 (1989).
[5]Zhou Xiaoan and K. S. Viswanathan, Mod. Phys. Lett. A **4**, 99 (1989).
[6]F. David and E. Guitter, Europhys. Lett. **5**, 709 (1988); Y. Kantor and D. R. Nelson, Phys. Rev. Lett. **58**, 2774 (1987); (1987); Phys. Rev. A **36**, 4020 (1987); D. R. Nelson and L. Peliti, J. Phys. (Paris) **48**, 1085 (1987).
[7]R. D. Pisarski, Phys. Rev. D **38**, 578 (1988); see also Phys. Rev. Lett. **58**, 1300 (1987); **58**, 2608(E) (1987).
[8]F. David and E. Guitter, Nucl. Phys. **B295**, 332 (1988); see also Europhys. Lett. **3**, 1169 (1987); **5**, 709 (1988); F. David, *ibid.* **2**, 577 (1986); H. Kleinert, Phys. Lett. B **197**, 351 (1987); Phys. Rev. Lett. **58**, 1915 (1987); F. Alonso and D. Espriu, Nucl. Phys. **B283**, 393 (1987).
[9]W. Helfrich, J. Phys. (Paris) **46**, 1263 (1985); L. Peliti and S. Leibler, Phys. Rev. Lett. **54**, 1690 (1985); D. Forster, Phys. Lett. **114A**, 115 (1986); H. Kleinert, *ibid.* **114A**, 263 (1986); S. Ami and H. Kleinert, Phys. Lett. A **120**, 207 (1987).
[10]A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic, New York, 1987).
[11]Y. S. Myung, S. H. Yi, and B. H. Cho, Phys. Rev. D **36**, 2586 (1987).
[12]M. Luscher, K. Symanzik, and P. Weiss, Nucl. Phys. **B173**, 365 (1980); O. Alvarez, Phys. Rev. D **24**, 440 (1981).
[13]Zhou Xiaoan, SFU Report, 1989; Ph.D. thesis (in preparation).
[14]S. G. Brush, Rev. Mod. Phys. **39**, 883 (1967); M. Plischke and B. Bergersen, *Equilibrium Statistical Physics* (Prentice-Hall, Englewood Cliffs, New Jersey, 1989).
[15]W. Helfrich, Z. Naturforsch. C **30**, 841 (1975).
[16]R. D. Pisaraki and O. Alvarez, Phys. Rev. D **26**, 3735 (1982); P. Olesen, Phys. Lett. **160B**, 408 (1985).
[17]H. Kleinert, Phys. Lett. B **197**, 125 (1987).