Quaternionic Dirac equation

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The quaternionic generalization of the Dirac equation is investigated. From elementary considerations of unitarity and Lorentz invariance it is demonstrated that potentials with quaternionic parts are not consistent with representation independence. This result leads to the conclusion that either quaternionic quantum mechanics singles out a special class of Dirac representations, or that allowed potentials appearing in the problem have no j or k components. If the latter alternative is the case, consideration of nonrelativistic wave mechanics casts doubt on the existence of simple experimental tests of quaternionic quantum mechanics.

Given certain reasonable assumptions it was shown by Birkhoff and von Neumann[1] that quantum mechanics may be formulated over real numbers, complex numbers, or quaternions. However, as shown by Finkelstein *et al.*,[2] real quantum mechanics is not rich enough to accommodate the required properties, which leaves the choice between the complex numbers and the quaternions. One would hope that some experimental test may differentiate between the two theories, to give a unique quantum theory (although recently the idea of formulating p-adic quantum mechanics has been advanced[3]).

In terms of calculations of physical quantities, some attention has been paid to nonrelativistic quaternionic quantum mechanics (QQM). It has been found that some interesting effects occur when quaternion-valued potentials are present in the Schrödinger equation for quaternionic fields. Specifically, we write the potential in the form

$$V(x)=V_0(x)+iV_1(x)+jV_2(x)+kV_3(x)\ , \tag{1}$$

where the units i, j, and k satisfy the quaternion algebra $i^2=j^2=k^2=-1$, $ij=k=-ji$ and cyclic. Adler[4] has shown that if $V_2(x)$ and $V_3(x)$ are linearly independent functions, then time-reversal-violating effects may be present in the scattering S matrix. A model of CP nonconservation in K-meson decays due to the presence of $V_2(x)$ and $V_3(x)$ terms in the K mass matrix has been constructed, also by Adler.[5] It has also recently been shown that a necessary condition for inversion of reflection to give a reflection coefficient which differs in magnitude is the existence of a spatially varying phase difference between $V_2(x)$ and $V_3(x)$ (Ref. 6).

It is thus obviously important as far as possible experimental tests of QQM versus the standard complex theory are concerned to examine possible restrictions on the values of potentials which appear in the theory. It is the purpose of this note to consider the introduction of potentials into the relativistic theory.

We begin with the Dirac equation[7,8]

$$\frac{\partial\psi}{\partial t}=H\psi\ , \tag{2}$$

with H now anti-Hermitian (i.e., $H^\dagger=-H$) (Ref. 9). The free Hamiltonian takes the form

$$H=-i(\boldsymbol{\alpha}\cdot\mathbf{P}+\beta m)\ , \tag{3}$$

where α and β are to be determined. We will henceforth work in $1+1$ dimensions for ease of exposition. It is not too difficult to show that the conclusions hold in $3+1$ dimensions. To maintain complete generality we make no assumptions regarding the form of the γ matrices which will appear. In particular, the γ matrices may have quaternionic entries, which to our knowledge has not been investigated previously, and hence we will detail the steps required to obtain the correct properties of the Dirac equation in this case. Also, the spinor ψ is in general quaternion valued, which leads to a "doubling" of solutions[7,9] and has been used by Adler in a novel formulation of QED.[7]

Anti-Hermiticity of H then leads to the relations

$$\alpha=\alpha^\dagger \quad\text{and}\quad \beta^\dagger i=i\beta\ . \tag{4}$$

We also take the standard definitions of the operators P^μ,[10] the unit i being chosen with no loss of generality:

$$P^0\psi=i\frac{\partial\psi}{\partial t},\quad P^1\psi=-i\frac{\partial\psi}{\partial x}\ . \tag{5}$$

(For the metric, etc., we follow the conventions of Itzykson and Zuber.[10])

The Dirac equation must lead to the correct relativistic relation between energy and momentum. Requiring the Klein-Gordon equation to arise from twice applying the Dirac operators to the field ψ leads us to the quaternionic version of the Dirac algebra which must be satisfied by α and β:

$$\alpha^2=1,\quad i\beta i\beta=-1,\quad\text{and}\quad \alpha\beta=\beta i\alpha i\ . \tag{6}$$

These relations reduce to the usual Dirac algebra when the unit i commutes with the matrices α and β, which will be true whenever α and β are real or complex. An example of a representation which satisfies (6) but not the usual Dirac algebra is $\alpha=\sigma_3$, $\beta=j\sigma_1$, which has $\beta^2=-1$, where the σ are two of the Pauli matrices. An important corollary of the conditions on α and β is that β must be unitary. (To see this consider $i\beta i\beta=-1$, which leads to $\beta i\beta=i$. Then taking the Hermitian conjugate of this gives $i=\beta^\dagger i\beta^\dagger=\beta^\dagger\beta i$, or $\beta^\dagger\beta=1$.)

Introducing the γ matrices $\gamma^0=\beta$ and $\gamma^1=-\alpha\beta$, we can write the free particle equation in the form

$$\gamma^\mu\partial_\mu\psi=-im\psi \ . \tag{7}$$

We now turn to the question of Lorentz invariance of the Dirac equation. We follow the construction of Sec. 2-1-3 of Ref. 10. Writing (7) in another frame as

$$\gamma^\mu\partial'_\mu\psi'=-im\psi' \tag{8}$$

and assuming the relation $\psi'(x')=S(\Lambda)\psi(x)$ to hold, we may derive the form of $S(\Lambda)$. We write

$$S(\Lambda)=1-\tfrac{1}{4}\omega^{\mu\nu}\Sigma_{\mu\nu} \ ,$$

where the infinitesimal matrix $\omega^{\mu\nu}$ is antisymmetric and real, and the operator Σ is to be determined. Imposing Lorentz invariance on (7) and (8) yields the relation

$$S(\Lambda)i\gamma^\mu S^{-1}(\Lambda)=i\gamma^\nu(\Lambda^{-1})_\nu{}^\mu \ ,$$

with $(\Lambda^{-1})_\nu{}^\mu=g_\nu{}^\mu-\omega_\nu{}^\mu$, g being the metric tensor. In terms of Σ, this becomes

$$[i\gamma_\alpha,\Sigma_{\mu\nu}]=2i(\gamma_\mu g_{\alpha\nu}-\gamma_\nu g_{\alpha\mu}) \ . \tag{9}$$

In 1+1 dimensions there is only one component of $\Sigma_{\mu\nu}=\Sigma_{01}\equiv\Sigma$. Then relations (9) become

$$[\Sigma,i\gamma_0]=2i\gamma_1 \quad \text{and} \quad [\Sigma,i\gamma_1]=2i\gamma_0 \ ,$$

which are satisfied by $\Sigma=\frac{1}{2}[i\gamma_0,i\gamma_1]$. A little algebra gives the result [using properties (4) and (6) of α and β] $\Sigma=-i\alpha i$. Of course, in the case where i commutes with the γ_μ, Σ becomes equal to $i\sigma_{01}=-\frac{1}{2}[\gamma_0,\gamma_1]$, as per the complex case.

We now turn to constructing a conserved current. Defining the conjugate spinor $\bar\psi\equiv\psi^\dagger\gamma_0^\dagger$, and multiplying (7) from the left we obtain the equation

$$\bar\psi\gamma^\mu\partial_\mu\psi=-m\bar\psi i\psi \ . \tag{10}$$

Taking the Hermitian conjugate of this equation and using the identity $\gamma_0\gamma_\mu^\dagger\gamma_0=\gamma_\mu$ (obtained by considering separately $\mu=0,1$), we get the equation

$$(\partial_\mu\bar\psi)\gamma^\mu\psi=m\bar\psi i\psi \ . \tag{11}$$

Adding (10) and (11) we obtain a candidate current for the Dirac equation:

$$\partial_\mu(\bar\psi\gamma^\mu\psi)\equiv\partial_\mu J^\mu=0 \ . \tag{12}$$

It can be checked that, using the operator $S(\Lambda)$ derived earlier, J^μ transforms as a Lorentz vector, as we require. Consideration of the quantity $\bar\psi\psi$, which in the complex Dirac theory is a Lorentz scalar, leads us to a new result. Consider infinitesimal transformations, where $S(\Lambda)$ takes the form (ω infinitesimal)

$$S(\Lambda)=1+\frac{\omega}{2}i\alpha i=[S(\Lambda)]^\dagger \ .$$

It is easy to check that

$$\bar\psi\psi\to\bar\psi\left[1+\frac{\omega}{2}(\alpha+i\alpha i)\right]\psi \ ,$$

again using the properties of α and β. So $\bar\psi\psi$ is a Lorentz scalar if and only if α commutes with the unit i. Thus, a consequence of demanding that $\bar\psi\psi$ be a Lorentz scalar is that the matrix α must be complex—that is, it has no entries which have j or k components.

We now turn to the case of a Dirac particle interacting with an external potential, which may be quaternion valued. We will deal with scattering problems with no source terms; hence, we require a conserved current as in (12).

For simplicity, first consider a potential which is added to the mass term in (7). In the usual complex theory such a potential transforms as a Lorentz scalar, just as the mass does. Introducing the potential $V(x)$ into the Hamiltonian we find

$$H=-i[-\alpha P_x+\beta m+\beta V(x)] \ . \tag{13}$$

For Eq. (7) to be Lorentz invariant with this potential we would require $V(x)$ to satisfy

$$S^{-1}(\Lambda)V'(x')S(\Lambda)=V(x) \ .$$

For infinitesimal transformations this implies

$$i\alpha iV(x)=V(x)i\alpha i \ . \tag{14}$$

This equation is not true in general when $V(x)$ has j and k components, since even if we require α to be complex only, Eq. (14) reads $\alpha V(x)=V(x)\alpha$, which is representation dependent. For instance, in a representation where α is real, V can have any quaternionic form, whereas if α is pure i imaginary we find that the j and k components must vanish.

We can also ask about unitarity with $V(x)$ present. Anti-Hermiticity of H then implies

$$i\beta V(x)=[V(x)]^*\beta^\dagger i=[V(x)]^*i\beta \ , \tag{15}$$

where the asterisk represents quaternionic conjugation. Again, this equation is representation dependent unless $V(x)$ is real. To see this, consider a representation where β is real, which leads to (15) becoming $iV(x)=[V(x)]^*i$, which implies that the i component of V must vanish, but the j and k components are allowed. This is the same as the restriction on potentials in the nonrelativistic theory.[4-6] However, if we consider a representation in which β is pure i imaginary, then (15) becomes $V(x)=[V(x)]^*$, which implies that V is real. Clearly, if V is real (15) is always satisfied, regardless of the representation. As a consistency check, working through the derivation leading to Eq. (12) again, this time with the potential V included, one finds

$$\partial_\mu(\bar\psi\gamma^\mu\psi)=\bar\psi(\beta V^*i\beta-iV)\psi \ ,$$

which reduces to (12) whenever (15) is satisfied.

The same considerations hold for a Lorentz-vector potential. We assume that the minimal substitution $P_\mu\to P_\mu-eA_\mu$ is the correct prescription in the quaternionic theory.[8] In this case

$$H=H_{\text{free}}-e\gamma_0^\dagger\gamma_\mu iA^\mu \ . \tag{16}$$

Anti-Hermiticity implies that $iA_0=A_0^* i$ and $i\alpha A^1 = A^{1*}\alpha i$. The first condition is satisfied if the i component of A^0 vanishes, $A^0=A^0{}_0+jA^0{}_2+kA^0{}_3$, but the other is representation dependent. Again the only representation-independent potential A which satisfies this requirement is real. To maintain this condition under Lorentz transformations we require A_0 to be real as well; otherwise quaternionic parts of A^1 will mix into A^0. Thus we are not allowed to have any quaternionic parts in the four-potential, unless we restrict ourselves to certain classes of representations. Also, assuming that A_μ transforms as a Lorentz vector, the Lorentz invariance of the Dirac equation requires $A_\mu i\alpha i = i\alpha i A_\mu$, which is again representation dependent, but is always satisfied if the j and k components of A vanish.

If we were to demand representation independence of the Dirac equation, the situation would be somewhat worrisome regarding the nonrelativistic quaternionic theory, which must emerge in the appropriate limit of the Dirac theory. The Schrödinger equation takes the form[4–6]

$$\left[-i\frac{d^2}{dx^2}+V(x)\right]\Psi(x)=\Psi(x)iE \ , \tag{17}$$

where $\Psi(x)$ is the quaternion-valued wave function, $V(x)$ is a quaternion-valued potential, and E is the real energy. We can use the so-called symplectic representation of quaternions to write all of our equations in terms of complex numbers:

$$V(x)=i[V_\alpha(x)+jV_\beta(x)]$$

and (18)

$$\Psi(x)=\Psi_\alpha(x)+j\Psi_\beta(x) \ ,$$

with V_β, Ψ_α, and Ψ_β complex, and V_α real, by unitarity. The Schrödinger equation (17) may now be written as a pair of coupled complex equations for the symplectic components:

$$\left[-\frac{d^2}{dx^2}+V_\alpha\right]\Psi_\alpha-V_\beta^*\Psi_\beta=E\Psi_\alpha \ , \tag{19a}$$

$$\left[\frac{d^2}{dx^2}-V_\alpha\right]\Psi_\beta-V_\beta\Psi_\alpha=E\Psi_\beta \ . \tag{19b}$$

If the j and k part of the potential V_β is vanishing, then the quaternionic part of the wave function Ψ_β completely decouples from the complex part Ψ_α and hence cannot modify the results of complex quantum mechanics. Clearly this is somewhat alarming regarding the possibility of observing effects due to QQM. One thus concludes that either the quaternionic Dirac equation is representation dependent, which is not a problem of principle, but lacks the elegance of the complex theory, or that the quaternionic part decouples from the i-complex part of the wave function.

In conclusion, we have argued from unitarity that the potentials appearing in the Dirac equation for quaternions must not have any j and k components if the physics is to be representation independent. If this is the case, the quaternionic part of the wave function decouples in the nonrelativistic limit, which raises questions regarding the possibility of observable effects from QQM. Otherwise, the quaternionic Dirac equation admits the most general form of potentials only in specific classes of representations.

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[8]For a gauge theory involving quaternion-valued fields, see A. Govorkov, Theor. Math. Phys. **68**, 893 (1987). Note that this work is performed in the Majorana representation, in which the α_i and $i\beta$ are real, and the problems raised in this paper do not arise. In other representations this will not be so. Adler in Ref. 7 also works in the Majorana representation, but his gauge fields are complex, not quaternionic, and so the problems addressed here do not arise.

[9]A variant of the quaternionic Dirac equation has recently appeared: P. Rotelli, Mod. Phys. Lett. A **4**, 933 (1989); **4**, 1763 (1989). To maintain unitarity in this formalism it is required that the P_μ take a nonstandard form, which leads to a complex projection being required in the scalar product. The formalism herein avoids this requirement, but the conclusion regarding representation dependence still holds in Rotelli's formalism.

[10]C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).