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# Possible resolution of the lattice Gribov ambiguity

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The Gribov ambiguity in lattice gauge theory is discussed. The Landau gauge and the finite-temperature temporal gauge ($\partial_4 A_4 = 0$) are formulated as maximization conditions on the lattice. This formulation is shown to eliminate Gribov copies from the temporal gauge. The possibility that it also eliminates copies from the Landau gauge is discussed. An algorithm which will eliminate Gribov copies from the lattice implementation of the Landau gauge, in case any remain, is introduced and studied via Monte Carlo simulation. The algorithm involves a noncovariant intermediate step and so eliminates the copies at the cost of the possible introduction of a violation of lattice Poincaré symmetry. The covariance of this algorithm is studied numerically and no evidence is found for symmetry violation, which indicates that either the maximization form of the lattice Landau gauge is free of copies, or that the modified algorithm selects one in an acceptably covariant way.

## I. INTRODUCTION

The Gribov ambiguity is a very general affliction of gauge choices for QCD. This ambiguity arises whenever multiple gauge configurations satisfy the same gauge condition yet are related to each other by a nontrivial gauge transformation. The presence of such Gribov copies means that, in principle, a gauge is not well defined. In the continuum, the most familiar gauges are afflicted by this ambiguity. It was first discovered in the Coulomb[1] gauge and was then seen to be present in the Landau[2,3] gauge as well. In fact, it has been proven that as long as the gauge potentials go to zero sufficiently quickly at infinity, there is no algorithm which depends continuously on the gauge potentials and which uniquely specifies a gauge.[2] While, strictly speaking, the theorems about the generality of the Gribov problem have not been proven for lattice systems, there is little reason to think that the situation would be any different there.

Although the result about the generality of the Gribov ambiguity is quite a powerful one, it leaves many loopholes. There are several examples of gauge conditions that avoid Gribov copies, though the potentials do not necessarily vanish at infinity.[4–7] (One of them[5] is even covariant.) Another loophole is that the formulation of a gauge condition may not satisfy the continuity condition of the theorem. One of the results in this paper is a formulation of the lattice axial gauge as a maximization condition. It is free of Gribov copies, though the gauge potentials are not continuous functions of the fields everywhere in field configuration space.

Throughout this paper we will ignore the trivial copies that are associated with global gauge transformations. All of the gauges discussed here leave a residual freedom to make the same gauge transformation at each space-time point, both on the lattice and in the continuum. This nonuniqueness is without consequence, because it simply introduces an overall finite factor of the group volume, independent of field configuration, in both the numerator and denominator of the ratio of path integrals which define any connected matrix element.

The Gribov ambiguity can be a serious problem, but only in calculations that necessarily depend on the choice of gauge. The great majority of lattice gauge Monte Carlo simulations have been directed towards the calculation of "physical" quantities, that is, quantities which are manifestly gauge invariant, and these calculations are usually carried out in a manner which is manifestly gauge invariant, that is, which explicitly preserves gauge invariance throughout the calculation. For all these calculations, the Gribov problem is irrelevant.

The Gribov problem is also irrelevant for perturbative calculations. It is a necessarily nonperturbative effect. For infinitesimal coupling, where the gauge condition becomes Abelian, there are no ambiguities, at least for standard gauges.

The Gribov problem is potentially serious for nonperturbative studies of QCD Green's functions. Recently, calculations of two, the quark[8] and gluon[9,10] propagators, have been carried out. These Green's functions provide the most direct connection between the continuum and lattice theories. They were evaluated in the Landau gauge, which, as has been noted, has Gribov copies in the continuum. The lattice implementation of this gauge condition is sufficiently different than in the continuum so that the question of whether there are copies on the lattice is an open one.

The results of those simulations were rather striking. The gluon propagator behaved like a massive particle propagator, with a mass in physical units somewhat over half a GeV. It is probably most useful to think of this mass as describing a finite gluon field correlation length.

 

The quark propagator also behaved like a massive particle propagator, with a mass consistent with the "constituent quark mass" of the nonrelativistic quark model, even though the quark mass parameter in the lattice Dirac operator was small.

## II. THE LATTICE GRIBOV AMBIGUITY

It is useful to review the way these simulations are carried out, in order to understand why the Gribov ambiguity may be treacherous, and also to see why it may be eliminated by a lattice implementation. Gauge-invariant lattice Monte Carlo calculations are carried out by simulating the Feynman path integral

$$\langle O \rangle = \frac{1}{Z} \int [\mathcal{D}U] e^{-S(U)} O(U) , \tag{1}$$

where

$$Z = \int [\mathcal{D}U] e^{-S(U)} . \tag{2}$$

The functional measure is explicitly gauge invariant and the integration domain effectively includes a sum over all gauges. This is not a good starting point for computing gauge-variant quantities, such as quark or gluon propagators. To compute quantities that refer to a specific gauge, one begins with the Faddeev-Popov form of the path integral:[11]

$$\langle O \rangle = \frac{1}{Z} \int [\mathcal{D}U] \Delta(U) \delta(f(U)) e^{-S(U)} O(U) , \tag{3}$$

where $Z$ is the same expression without $O(U)$, the gauge condition is $f(U)=0$, and the Faddeev-Popov determinant $\Delta$ is the functional Jacobian of $f$ with respect to gauge transformations, formally given by

$$1 = \Delta(U) \int [\mathcal{D}g] \delta(f(U^g)) . \tag{4}$$

If the gauge specification is complete, so that for each $U$ only one $g$ satisfies $f(U^g)=0$, the Faddeev-Popov path integral can be simplified to

$$\langle O \rangle = \frac{1}{Z} \int [\mathcal{D}U] e^{-S(U)} O(U^{g(U)}) , \tag{5}$$

where $g(U)$ is the gauge transformation that maps $U$ into the $f=0$ gauge, $f(U^{g(U)})=0$. This formula is the basis for simulations of gauge-dependent quantities. It reduces to the Wilson form if $O$ is explicitly gauge invariant. Its validity depends, however, on the uniqueness of the gauge condition, that is, on the absence or elimination of Gribov copies. If, on the contrary, there are copies, then the formula becomes ill defined at best, because it does not include a recipe telling which copy to pick. Even worse, the recipe of summing over all copies does not correspond to the formula at all, because the weighting of the contribution from the different copies (the functional Jacobian evaluated at that copy) is, in general, different for each copy.

The lattice implementations of the Landau and Coulomb gauges involve stronger conditions than in the continuum, and so they have the possibility of being free of Gribov copies. On the lattice those gauges are implemented by an extremum condition, rather than a differential condition. The lattice gauge condition is[9,10,12,13]

$$\operatorname*{Max}_{g} \sum_{x,\mu} \mathrm{Re}\,\mathrm{Tr}\, U_\mu^g(x) , \tag{6}$$

where the sum on directions $\mu$ goes from 1 to 3 for the Coulomb and from 1 to 4 for the Landau gauge. These conditions imply the finite difference forms of the familiar continuum conditions, but being global maximization conditions they are stronger. Zwanziger has considered the equivalent modification of the differential Landau-gauge condition in the continuum.[14]

The simplest way to see that the maximization condition really is stronger than the differential one is to note that it excludes the copies which were originally found by Gribov, namely, copies of the vacuum configuration. The choice $U_\mu(x)=1$ is the unique absolute maximum of the condition Eq. (6).

## III. THE TEMPORAL GAUGE AT FINITE TEMPERATURE

An example of a gauge condition which becomes unique by being implemented as an extremum is the finite-temperature temporal gauge $\partial_4 A_4 = 0$. One can explicitly show that the finite-difference form of this condition has Gribov copies, but its extremal form is unique. The extremal form is identical to Eq. (6), but with $\mu$ restricted to the single value $\mu=4$ rather than being summed:

$$\operatorname*{Max}_{g} \sum_{x_4} U_4^g(x_1,x_2,x_3,x_4) \equiv \operatorname*{Max}_{g} \sum_{\substack{x_4 \\ x_{1,2,3}\ \text{fixed}}} g(x+\hat{4}) U_4(x) g^\dagger(x) . \tag{7}$$

The notation $x+\hat{4}$ means that the lattice site $x$ is displaced by one lattice spacing along the 4 axis. The solution of this condition is straightforward to find. If $U$ satisfies the maximization condition, then stability with respect to an infinitesimal gauge transformation at a single site $x$ gives the finite-difference condition

$$\Delta_4 A_4(x) = 0 , \tag{8}$$

where the finite-difference operation is $\Delta_\mu \phi(x) \equiv \phi(x+\hat{\mu}) - \phi(x)$, and the lattice gauge potential is given by

$$A_\mu(x) = \frac{1}{2i} [U_\mu(x) - U_\mu^\dagger(x)]_{\text{traceless}} . \tag{9}$$

Note that because of the periodic boundary conditions of the lattice (specifically along the Euclidean time direction), it is not consistent to require the usual zero-temperature form of the temporal gauge, $A_4=0$. The reason is that this condition would imply that the product of all the timelike links at a given spatial site is 1, whereas in fact that product can be any SU(3) matrix. This is exactly the situation that occurs at finite temperature, in the continuum as well as on the lattice.

The constancy of $A_4$ along the four-direction is satisfied if $U_4$ is similarly constant, and so after the application of the gauge transformation, the timelike links will be

$$U_4^g(x_1,x_2,x_3,x_4)=\left[\prod_{x_4} U_4(x_1,x_2,x_3,x_4)\right]^{1/N_t} . \quad (10)$$

The gauge transformation which accomplishes this is

$$g(x_1,x_2,x_3,n)=[U_4^g(x_1,x_2,x_3,0)]^n \prod_{i=0}^{n} U_4^\dagger(x_1,x_2,x_3,i) , \quad (11)$$

where the $x_4$ coordinate runs from 0 to $N_t-1$ and $g(x_1,x_2,x_3,0)$ is the unit matrix. This gauge transformation is not unique, because of the $N_t$th root that appears in Eq. (10). An SU(3) matrix has $N^2$ $N$th roots which are unitary and unimodular. These are all nontrivial gauge copies, which can be chosen independently at each spatial site on the lattice. They all satisfy the finite difference gauge condition, but only one satisfies the maximization condition.

The only one of the copies implicit in Eq. (10) which maximizes the sum of traces in Eq. (9) depends on the eigenvalues $e^{i\theta}$ (we take their phases to lie in the interval $-\pi<\theta_i\leq\pi$) of the matrix whose $N_t$th root is taken in Eq. (10) as follows: If the sum of the phases, $\sum\theta_i$, is zero, the eigenvalues of $U_4^g$, $e^{i\phi}$, are each given by $\phi_i=\theta_i/N_t$. If the sum of the phases is $\pm 2\pi$, which is the other possible case, then two of the eigenvalues are still given by $\phi_i=\theta_i/N_t$, but one, corresponding to the phase $\theta_i$ which is largest in magnitude, is given by $\phi_i=(\theta_i\mp 2\pi)/N_t$. This specifies the gauge uniquely. In accord with the general result that no continuous unique gauge can be specified on a compact space, the extremum condition has discontinuities as a function of the link variables. They occur when the sum of the phases, $\sum\theta_i$, is $\pm 2\pi$ and the two largest phases (in magnitude) cross. These discontinuities are finite jumps in the gauge transformation. They are associated with a set of measure zero in the path integral, and do not appear to have any harmful consequences.

## IV. THE LANDAU GAUGE

The question of whether the maximization form of the Landau or Coulomb gauges is unique is more difficult than for the temporal gauge, and we do not have an analytic answer to it. The intrinsic difficulty of the problem can perhaps be understood if we note that it is analogous to the problem of finding whether a spin glass has a degenerate (classical) ground state. The relation to a spin glass is expressed through the gauge condition Eq. (6). The expression whose maximum is demanded can be thought of as (the negative of) a Hamiltonian in which the dynamical variables are the SU(3) matrices $g(x)$, one for each space-time point on the lattice, the analogues of spins. The analogues of the irregular couplings of the spins are the SU(3) matrices $U_\mu(x)$, one for each link of the lattice, which take different values on each link, and which furthermore take different values in each configuration in a complete simulation. This analogy indicates that the question of whether there are Gribov copies in the lattice implementations of the Landau and Coulomb gauges is at least as difficult as the question of whether or not the classical minimum of a quenched spin glass is unique.

Although difficult to resolve analytically, the Gribov problem on the lattice can be examined in a systematic way. The means of doing so uses a gauge which is unambiguously free of Gribov copies on the lattice, the maximal tree axial gauge, as an intermediate step.[15] Recall that the maximal tree gauge is that in which all temporal links are 1 except at one reference time, conventionally taken to be the interval between slices $n_t=0$ and $n_t=1$. At time 1 all $z$ links are 1 except at a reference $n_z$; at the reference time and $z$ coordinate, all $y$ links are 1 except at a reference value of $n_y$; and on the reference values of the time, $z$, and $y$ coordinates, all $x$ links but one are 1. This is a unique gauge—it is straightforward to show that if two configurations differ by a gauge transformation, and each is transformed to this gauge, the resulting configurations will be the same to within a global gauge transformation.

A modification of the straightforward Landau gauge algorithm which is free of the Gribov copies is then the following: First transform the gauge configuration to the maximal tree axial gauge; then apply some definite algorithm to transform it to the Landau (or Coulomb) gauge.[16] The algorithm we have used in this analysis is simply to sweep through the lattice and at each site effect the gauge transformation that maximizes the real part of the sum of the traces of the link variables emanating from that site.[9] This relaxes toward the global maximum of Eq. (6). The convergence criterion used in the present work is to continue the process until the lattice average of $\mathrm{Tr}(\Delta_\mu A_\mu)^2$ is less than $10^{-6}$. This certainly eliminates copies, because the maximal tree axial gauge is unique to within global gauge transformations. The problem is that this intermediate gauge makes reference to a specific preferred direction and preferred time slice, so the possibility exists that one has only traded one kind of problem for another. Perhaps one has eliminated Gribov copies at the cost of defining a noncovariant "Landau" gauge.

We have tested, via a moderate statistics simulation, the covariance of the unique Landau gauge defined via prefixing to the maximal tree axial gauge. Specifically we have evaluated two quantities, one of which would be sensitive to the preferred direction, and would reflect a violation of lattice rotation symmetry, while the other would be sensitive to the preferred time slice, and would reflect a violation of lattice translation symmetry. The first is the zero four-momentum gluon propagator:

$$\Delta_{\mu\nu}=\frac{1}{N_{\text{sites}}}\sum_{x,y}\langle A_\mu(x)A_\nu(y)\rangle . \quad (12)$$

If this Landau gauge is covariant, the result should be proportional to $\delta_{\mu\nu}$. On the other hand, if the four-direction is singled out, we would expect the last row and column to differ (or at least the 44 element). The result of the simulation is shown in Table I. As can be seen, there is no sign of noncovariance. As expected in any statisti-

TABLE I. The gluon propagator $\Delta_{\mu\nu}$ at zero four-momentum (i.e., averaged over all space-time sites) on a $4^4$ lattice with $\beta=5.6$, normalized so that the averaged diagonal value is 1. Errors are statistical, based on 2500 measurements.

| $\mu$ \ $\nu$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1.010 (0.035) | 0.022 (0.016) | 0.010 (0.017) | 0.006 (0.016) |
| 2 | 0.022 (0.016) | 0.921 (0.036) | 0.013 (0.017) | −0.006 (0.015) |
| 3 | 0.010 (0.017) | 0.013 (0.017) | 1.016 (0.045) | −0.002 (0.016) |
| 4 | 0.006 (0.016) | −0.006 (0.015) | −0.002 (0.016) | 1.053 (0.043) |

cal simulation, some quantities are more than one standard deviation from the norm, but these are not those associated with the $\hat{4}$-direction, which is singled out in the definition of the axial gauge.

The second quantity we have simulated is the zero three-momentum gluon propagator, summed over three-polarizations:

$$\Delta(t,\Delta t)=\frac{1}{N_{\text{site}}^3}\sum_{\mathbf{x},\mathbf{y},i}\langle A_i(\mathbf{x},t)A_i(\mathbf{y},t+\Delta t)\rangle \ . \tag{13}$$

This is the dynamical part of the gluon propagator. Covariance (translational invariance) would imply this is independent of $t$ and only depends on $\Delta t$. If the time interval between 0 and 1 is singled out, we would expect propagation involving those slices to be different. The results of the simulation are presented in Table II and in Fig. 1. Again, there is no sign of a violation of covariance. To within the precision of the calculation, there is no dependence in the propagator on the base time slice, from which propagation is measured, despite the fact that the interval between the $n_t=0$ and $n_t=1$ slices is singled out in the definition of the maximal tree axial gauge.

TABLE II. The gluon propagator on the same lattice as in Table I, evaluated at zero three-momentum and averaged over spatial polarizations [i.e., $\sum_{x,i}\Delta_{ii}(x,t)$], normalized so that the average zero-separation value is 1. Errors are statistical, based on 2400 measurements.

| Base slice $t$ \ Separation $\Delta t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 1 | 1.002 (0.012) | 0.738 (0.014) | 0.586 (0.018) | 0.738 (0.014) | 1.002 (0.012) |
| 2 | 1.001 (0.011) | 0.734 (0.014) | 0.583 (0.019) | 0.734 (0.014) | 1.001 (0.011) |
| 3 | 1.000 (0.012) | 0.734 (0.014) | 0.586 (0.018) | 0.734 (0.014) | 1.000 (0.012) |
| 4 | 0.997 (0.011) | 0.734 (0.014) | 0.583 (0.019) | 0.734 (0.014) | 0.997 (0.011) |

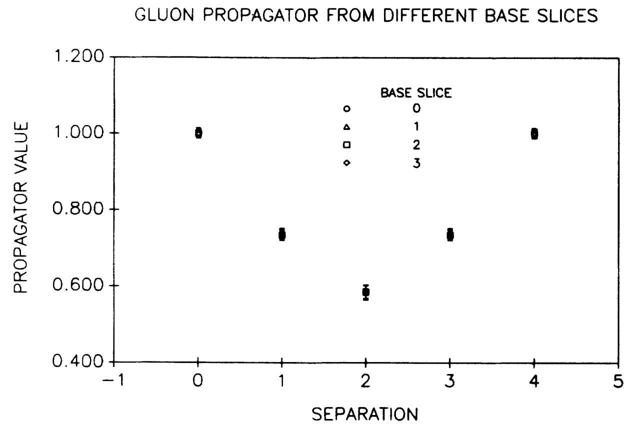


FIG. 1. A graphical display of the gluon propagator of Table II. The zero-momentum gluon propagator $\Delta(t,\Delta t)$ is plotted vs the time separation $\Delta t$ for each value of the base time slice $t=0,1,2,3$.

To the precision of the simulation, the maximal tree prefixed Landau gauge is covariant.

If the Landau gauge, defined by maximizing the sum of the real parts of the traces of the links, were free of Gribov copies, then its maximal tree prefixed version would be covariant, because it would be the same as the unprefixed version. Of course, the converse need not be true. Nonetheless, testing the covariance of the maximal tree prefixed Landau gauge amounts to looking for indirect evidence that the lattice Landau gauge (defined by maximization but without prefixing) has Gribov copies. The covariance (to within statistics) of the gluon propagator in this gauge represents a failure to find evidence of copies.

## V. SUMMARY

In this paper we have discussed a number of aspects of the Gribov problem in the Landau gauge. We have shown that the lattice axial gauge, which is like a one-dimensional Landau gauge, is rendered Gribov-copy-free by formulating it as a maximization condition. We have also noted that when the Landau gauge is formulated as a maximization condition, the vacuum field configuration, at least, has no Gribov copies. Finally, we looked for indirect evidence of Gribov copies in the lattice Landau gauge by looking for a failure of covariance when the Landau gauge was reached by prefixing to the maximal tree gauge, and failed to find such evidence. All this suggests that the lattice Landau gauge may be Gribov-copy-free, but certainly does not prove it. However, even if the lattice Landau gauge has copies, prefixing to the maximal tree axial gauge seems to be a covariant way of eliminating them and resolving the Gribov ambiguity.

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