Relativistic bound-state form factors in a solvable (1+1)-dimensional model including pair creation

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The relativistic two-body bound-state form factor, recently calculated by Mankiewicz and Sawicki from a light-front formula involving analytic solutions to the Weinberg equation in a $(1+1)$-dimensional model field theory, is compared with the form factor obtained from the Mandelstam formula using analytic solutions to the corresponding Bethe-Salpeter equation. The results are different, since the former formula does not include the contribution of pair-creation diagrams which are included in the latter. When the pair-creation diagrams are included in the light-front formula in $1+1$ dimensions, both approaches yield a numerically identical form factor, not having the strange properties described by Mankiewicz and Sawicki.

I. INTRODUCTION

Recently Mankiewicz and Sawicki[1] discussed strange and unexpected properties of the relativistic two-body bound-state form factor, obtained in a solvable bound-state model, that resulted from a special limit of $(1+1)$-dimensional scalar field theory. The most striking feature was that although the bound-state charge radius as a function of the binding energy was a naturally decreasing function for a class of weakly bound systems, it was reaching its minimum value at a binding energy of the order of the constituent mass and then, against intuition, was strangely increasing to infinity for the binding energy approaching the limit of the two constituent masses, corresponding to the massless bound state. The form factor exhibited also some strange behavior for momentum transfers comparable to the mass scales involved.

In this paper we describe the origin of these unphysical features. It turns out to be quite standard. The virtue of the detailed analysis of this example is that we can explicitly calculate highly relativistic bound-state effects in quite different approaches and understand the physics behind them.

In Sec. II we summarize very briefly the results of Mankiewicz and Sawicki[1] (MS). We point out that the MS light-front Fock-space method of calculating the form factor does not give the same answer as the Mandelstam formula[2] involving Bethe-Salpeter bound-state amplitudes.

Section III describes the connection between the light-front bound-state equation (usually referred to as the Weinberg[3] equation) and the Bethe-Salpeter equation. We notice that the MS solution of the Weinberg equation corresponds to the Bethe-Salpeter bound-state wave function which is a product of two free single-particle propagators, the overall momentum-conservation Dirac delta, and a constant vertex function. Consequently, all our form-factor calculations are nothing but various ways of evaluating the Feynman perturbation-theory triangle diagram in scalar field theory.[4] On the other hand, we indeed calculate the actual bound-state diagrams. The bound-state vertices satisfy corresponding relativistic bound-state equations. The present case differs from Ref. 4 in number of dimensions. We comment on this important difference in a few places.

In Sec. IV we calculate the bound-state form factor using the Mandelstam formula in the impulse approximation. This form factor qualitatively differs from the MS one in the region where the previous calculation lost physical meaning. New results are in agreement with intuition.

Section V describes the origin of the difference between both methods from Secs. II and IV. The difference comes from the light-front diagram representing creation of a particle-antiparticle pair by the external current and successive annihilation of the antiparticle with one of the constituents. This diagram was not included by MS. In $1+1$ dimensions the pair-creation diagrams cannot be eliminated by the special choice of the reference frame, as is possible and usually done in higher dimensions. The Hamiltonian description of the corresponding dynamics is not given in the present paper.

We conclude in Sec. VI by discussing how our results may help in building models of relativistic bound states.

A short appendix completes the paper by including the most important details of our formulas.

II. MS BOUND-STATE FORM FACTOR

The two-body bound-state equation for the light-front Fock-state wave function in the MS model field theory can be solved analytically in a straightforward way. The MS model is devised as a special limit, quite analogous to the replacement of the intermediate-boson exchange by the Fermi constant in weak interactions. This limit results in a separable kernel in the Weinberg equation. Therefore, the solution is available in a simple analytic form.

Using this solution one can calculate the bound-state form factor as described by MS. The form factors thus obtained are not satisfactory in regions in which relativistic effects are important, as already mentioned in the Introduction. The question is where do the strange effects come from? It can be answered by comparing the MS results with results obtained from the standard Mandelstam method of calculating the same bound-state form factor using solutions to the Bethe-Salpeter equation, which are also analytically obtainable in the model under consideration.

The Mandelstam formula yields results different from those obtained from the MS formula. Moreover, the new results are free from problems found by MS. Thus, the Mandelstam formula offers a straightforward explanation of where the MS effects come from. This is the subject of the present paper.

III. WEINBERG AND BETHE-SALPETER SOLUTIONS

The Bethe-Salpeter amplitude for the bound state under consideration can be found from the equation

$$(k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)\psi(k_1,k_2) = -im^2\frac{\lambda}{\pi}\int d^2l_1\int d^2l_2\delta^2(k_1+k_2-l_1-l_2)\times\psi(l_1,l_2)\ , \quad (1)$$

which is the ladder approximation to the Bethe-Salpeter equation in the particular limit of the model field theory considered by MS. Factoring out the overall momentum-conservation factor

$$\psi(k_1,k_2)=\delta^2(P-k_1-k_2)\phi_P(k_1)\ , \quad (2)$$

we obtain the equation

$$\phi_P(k_1)=-im^2\frac{\lambda}{\pi}\frac{1}{k_1^2-m^2+i\epsilon}\frac{1}{(P-k_1)^2-m^2+i\epsilon}\times\int d^2l_1\phi_P(l_1)\ . \quad (3)$$

We introduce now the light-front variables $k^+=k^0+k^1, k^-=k^0-k^1$, and define the light-front wave function[3]

$$f_P(k^+)=\int_{-\infty}^{\infty}dk^-\phi_P(k)\ . \quad (4)$$

Upon collecting the residues of poles under the k_1^- integral in Eq. (3), we find that the function $f_P(k^+)$ satisfies the Weinberg equation

$$f_P(k^+)=-m^2\lambda\frac{1}{k^+(P^+-k^+)}\frac{\theta(k^+)\theta(P^+-k^+)}{M^2/P^+-m^2/k^+-m^2/(P^+-k^+)}\int_{-\infty}^{\infty}dl^+f_P(l^+)\ , \quad (5)$$

which is solved by MS. Note that Eq. (3) implies that the Bethe-Salpeter vertex function is a constant. Thus, the light-front model considered by MS corresponds to the constant bound-state vertex, such as in a covariant perturbation theory considered in Ref. 4. The analytic solution given by MS is nothing else than the result of projecting two free Feynman propagators on the light-front plane. However, mass m of the bound state is expressed by the constituent masses, coupling constant, and the intermediate-boson mass, according to the Weinberg equation, Eq. (5).

Thus we have the corresponding Bethe-Salpeter wave function at hand.

VI. MANDELSTAM FORM FACTOR

The bound-state form factor can be calculated from the Mandelstam formula,[2] illustrated in Fig. 1(a). Under the assumption that only constituent 1 carries one unit of charge, the bound-state current is defined by

$$J^\mu(x_1)=\langle P+q|j^\mu(x_1)|P\rangle\ , \quad (6)$$

where $j^\mu(x_1)$ is the electromagnetic current of the charged constituent, and one obtains

$$J^\mu(0)=(2P+q)^\mu F(Q^2)\ , \quad (7)$$

where $Q^2=-q^2\geq 0$, and the form factor $F(Q^2)$ equals

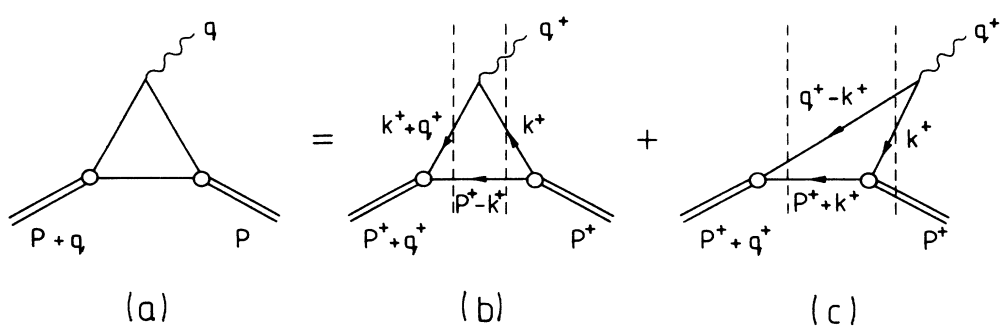

FIG. 1. Covariant Mandelstam diagram (a) for the bound-state current represented as the sum of light-front triangle diagram (b) and light-front pair-creation diagram (c).

$$F(Q^2)=N\int_0^1 du_1\int_0^1 du_2\frac{u_1(1-u_1)}{[m^2-M^2u_1(1-u_1)+Q^2u_2(1-u_2)u_1^2]^2}\ . \tag{8}$$

Here N is the normalization constant such that $F(0)=1$. Some details of the calculation are given in the Appendix. We plot this form factor together with the MS results in Figs. 2(a) and 2(b).

The mean-square radius, calculated from Eq. (8) according to the definition

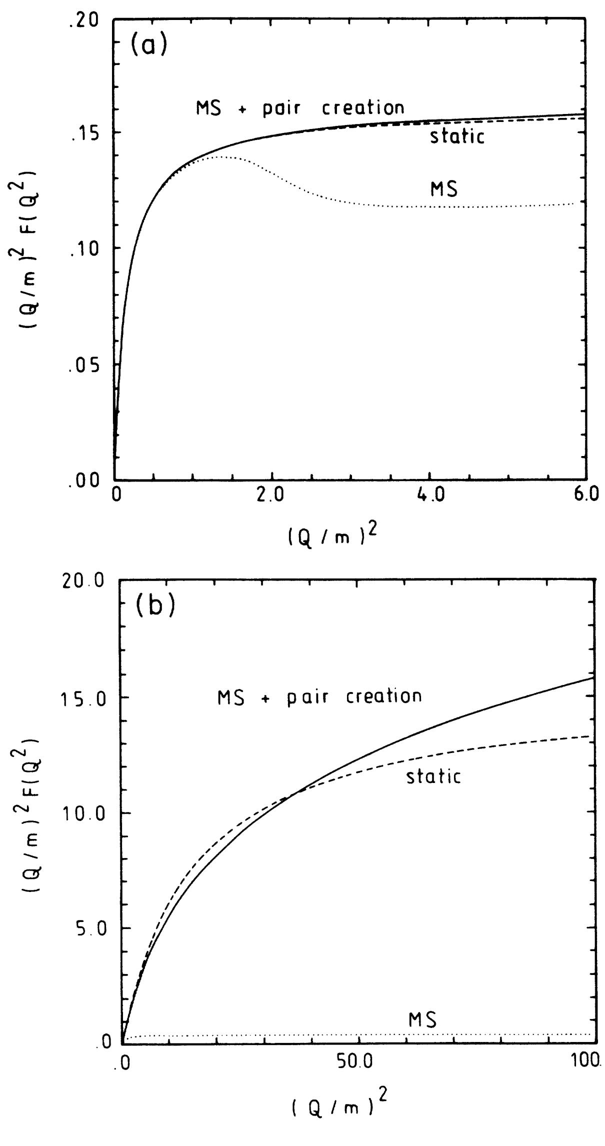

FIG. 2. Electromagnetic form factor $(Q/m)^2F(Q^2)$ for weakly bound system with $\tilde{\beta}=0.01$, where $\tilde{\beta}=[(2m)^2-M^2]/(2m)^2$ is the measure of binding energy (mass defect). Solid line represents the Mandelstam form factor, the dotted line is the contribution from the light-front triangle diagram (b) of Fig. 1, and dashed line represents the static approximation as defined in Ref. 1. The difference between the solid and dotted lines is contributed by the pair-creation diagram. (b) the same as in (a), but for strongly bound system with $\tilde{\beta}=0.96$.

$$\langle r^2\rangle=-2\frac{d}{dQ^2}F(Q^2)\Big|_{Q^2=0}\ , \tag{9}$$

decreases monotonically to the minimal value

$$\langle r^2\rangle_{\min}=1/(5m^2)\ , \tag{10}$$

when the binding energy B increases from zero to its maximal value $B=2m$. This is shown in Fig. 3. The strange behavior visible in the MS form factor is absent in the form factor calculated from the Mandelstam formula.

V. PAIR CREATION IN 1+1 DIMENSIONS

The bound-state triangle diagram from Fig. 1(a) can be alternatively calculated upon introducing the light-front variables k^+,k^- in the explicit expression (A3) for the bound-state current $J^\mu(0)$, and summing residues of poles in the variable k^-. Some details of this calculation are given in the Appendix. The remaining integral over k^+ is the sum of two terms written in detail in the Appendix and represented by diagrams (b) and (c) in Fig. 1. The first term is exactly the MS expression. The second term is the pair-creation diagram. The MS result supplemented by the pair-creation diagram equals the Mandelstam formula, Eq. (8).

We wish to stress two things here.

(i) The light-front technique offers a way to represent the full Mandelstam amplitude in 1+1 dimensions in terms of only two diagrams of clear physical meaning, which can be understood in terms of the light-front Fock-space decomposition of the bound state. The equal

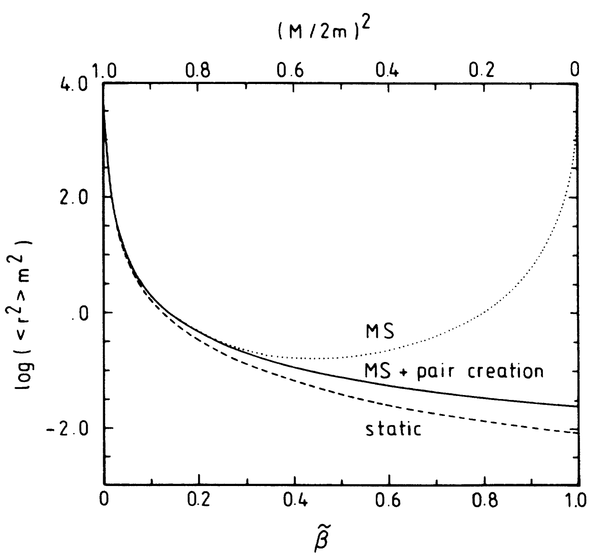

FIG. 3. The bound-state size $\langle r^2\rangle$, in units of $1/m^2$, for different values of the binding parameter $\tilde{\beta}$. The notation and the meaning of the curves as in Fig. 2.

time technique would result in six diagrams of very complicated structure,[4] which cannot be interpreted in terms of a fixed number of constituents and a single bound state.

(ii) In $1+1$ dimensions the pair-creation diagram cannot be eliminated for nonzero momentum transfers. If $q^2=q^+q^-$ differs from zero, we necessarily have q^+ different from zero, say, greater than zero. The pair-creation diagram must then be included in $1+1$ dimensions, since the external current brings a positive longitudinal momentum and can create particles with positive longitudinal momenta. The importance of the pair-creation diagram is illustrated in Figs. 2(a) and 2(b). One clearly sees that the peculiar structure of from factors found by MS disappears when the pair-creation diagram is included. Moreover, a naive static approximation works definitely better than the MS contribution only.

The pair-creation diagram does not contribute to the normalization of the form factor, since it vanishes like $(q^+)^2$ for vanishing q^+. However, from the elasticity condition $(P+q)^2=P^2=M^2$, it follows that $(q^+)^2$ is proportional to q^2 for small q^2. Therefore, the pair-creation diagram contributes to the bound-state radius. It is this contribution that cancels the unphysical growth of the MS radius.

If more than one spatial dimension is available, the pair-creation diagram can be eliminated upon choosing a frame of reference in which $q^+=0$, $q^2=-q_\perp^2$, and the single light-front triangle diagram (b) of Fig. 1 yields the full covariant result, free from unphysical effects.[4] However, in $1+3$ dimensions the bound-state equation with quartic interaction becomes ill defined.

Finally, we point out that it is the pair-creation diagram that dominates the behavior of the form factor for large momentum transfers in $1+1$ dimensions. Indeed, the light-front triangle diagram as well as static approximation yield the asymptotic falloff of the form factor of the type Q^{-2}, as it is discussed in detail in MS, whereas the triangle diagram yields a falloff of the type $Q^{-2}\ln(Q^2)$.

VI. CONCLUSION

The peculiar effects observed in the light-front MS formula for the bound-state form factor disappear once the pair-creation currents are included. Thus completed, the light-front formulation of the $(1+1)$-dimensional MS bound-state theory is equivalent to the Bethe-Salpeter-Mandelstam formulation. The model offers unique opportunities for calculating many processes involving bound states. The fully covariant description of the interacting bound states can be directly interpreted in terms of the light-front Fock-space decomposition. The model is strongly limited in the number of dimensions. However, an analytic study of deep-inelastic structure, screening, intermittency, many off-shell effects and relation to the ϕ^4 field theory seems to be very attractive.

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APPENDIX

The Mandelstam formula for the bound-state current is

$$J^\mu(x_1)=\int d^2x_2\int d^2y_2\bar{\psi}(x_1,x_2)\Delta^{-1}(x_2-y_2)\times j^\mu(x_1)\psi(x_1,y_2)\ , \tag{A1}$$

where

$$j^\mu(x_1)=i\frac{\overleftrightarrow{\partial}}{\partial x_{1\mu}}\ , \tag{A2}$$

Δ is the Feynman propagator of a scalar field, and ψ is the Bethe-Salpeter wave function of Eq. (1), now in the position representation. In our case Eq. (A1) yields

$$J^\mu(0)=N\frac{i}{\pi}\int d^2k\frac{1}{(k+q)^2-m^2+i\varepsilon}(2k+q)^\mu\times\frac{1}{k^2-m^2+i\varepsilon}\frac{1}{(P-k)^2-m^2+i\varepsilon}\ . \tag{A3}$$

Combining denominators one obtains Eq. (8) in a standard way. Alternatively, integrating Eq. (A3) over k^- by residua one obtains

$$J^+(0)=(2P+q)^+F(Q^2)\ , \tag{A4}$$

where

$$F(Q^2)=\frac{N}{2+\alpha}(I_1+I_2) \tag{A5}$$

and I_1 and I_2 correspond to diagrams (b) and (c) of Fig. 1, respectively. One has explicitly

$$I_1=\int_0^{P^+}dk^+\frac{2k^++q^+}{k^+(k^++q^+)(P^+-k^+)}\frac{1}{(P^-+q^-)-\left[q^-+\dfrac{m^2}{P^+-k^+}+\dfrac{m^2}{k^+}\right]}\frac{1}{(P^-+q^-)-\left[\dfrac{m^2}{P^+-k^+}+\dfrac{m^2}{k^++q^+}\right]}$$
$$=\int_0^1 dx\frac{2x+\alpha}{x(x+\alpha)(1-x)}\frac{1}{M^2-\dfrac{m^2}{1-x}-\dfrac{m^2}{x}}\frac{1}{M^2-Q^2/\alpha-\dfrac{m^2}{1-x}-\dfrac{m^2}{x+\alpha}}\ , \tag{A6}$$

where

$$\alpha=q^+/P^+=\frac{1}{2}\frac{Q^2}{M^2}(1+\sqrt{1+4M^2/Q^2}) \tag{A7}$$

and

$$I_2=\int_0^{q^+} dk^+ \frac{q^+-2k^+}{k^+(q^+-k^+)(P^++k^+)} \frac{1}{(P^-+q^-)-\left[P^-+\dfrac{m^2}{q^+-k^+}+\dfrac{m^2}{k^+}\right]} \frac{1}{(P^-+q^-)-\left[\dfrac{m^2}{P^++k^+}+\dfrac{m^2}{q^+-k^+}\right]}$$

$$=\int_0^{\alpha} dx \frac{2x-\alpha}{x(\alpha-x)(1+x)} \frac{1}{Q^2/\alpha+\dfrac{m^2}{\alpha-x}+\dfrac{m^2}{x}} \frac{1}{M^2-Q^2/\alpha-\dfrac{m^2}{1+x}-\dfrac{m^2}{\alpha-x}} . \tag{A8}$$

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