Field identifications in coset conformal theories from projection matrices

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We demonstrate the usefulness of projection matrices for finite subalgebras $\bar{h}\subset\bar{g}$ and their affine counterparts $\hat{h}\subset\hat{g}$ in finding field identifications (and selection rules) in coset conformal field theories.

Coset conformal field theories[1] may include all two-dimensional rational conformal field theories.[2] For every finite Lie subalgebra $\bar{h}\subset\bar{g}$, one can construct a two-dimensional conformal field theory. [We assume $\bar{h}$ is a maximal subalgebra of $\bar{g}$, otherwise the coset theory factors into $(\bar{g}/\bar{k})\otimes(\bar{k}/\bar{h})$ theories, where $\bar{k}\subset\bar{g}$ is maximal.] If G, H are the (covering) Lie groups whose algebras are $\bar{g}$, $\bar{h}$, the embedding $\bar{h}\subset\bar{g}$ will quite generally specify a relation between the centers $B(G),B(H)$ of the two groups. One of their consequences in the coset conformal field theory is a selection rule saying that certain primary fields do not occur.[2–5]

Let $\hat{g},\hat{h}$ denote the Kač-Moody algebras that are the central extensions of the loop algebras of $\bar{g},\bar{h}$, respectively. Then the finite subalgebra $\bar{h}\subset\bar{g}$ with index of embedding e induces an affine subalgebra $\hat{h}^{ek}\subset\hat{g}^{k}$, where the superscripts are the levels (see, for example, Ref. 6).

There exist automorphisms of $\hat{g}(\hat{h})$ which are not themselves elements of $\hat{g}(\hat{h})$ and are therefore called outer automorphisms.[7] The outer automorphisms of $\hat{g}$ permute the fundamental weights ω^{μ} [$\mu=0,1,\cdots R$; $R=\mathrm{rank}(\bar{g})$] in such a way as to leave the Dynkin diagram of $\hat{g}$ invariant. Similarly, outer automorphisms of $\hat{h}$ permute the fundamental weights ω^{α} [$\alpha=0,1,\cdots,r$; $r=\mathrm{rank}(\bar{h})$] of $\hat{h}$.

The group of outer automorphisms of $\hat{g}$, $O(\hat{g})$, is isomorphic to the center $B(G)$. Relations between the centers $B(G)$ and $B(H)$ are therefore accompanied by relations between the outer-automorphism groups $O(\hat{g})$ and $O(\hat{h})$. One consequence of these outer-automorphism relations is that certain fields in the coset conformal theory built from $\bar{h}\subset\bar{g}$ must be identified.[2,4,5,8]

In this paper we show how the projection matrices specifying the finite Lie subalgebra $\bar{h}\subset\bar{g}$ and its induced affine subalgebra $\hat{h}\subset\hat{g}$ may be used to find the aforementioned selection rules and field identifications in a coset conformal field theory. We must note that previous treatments have used implicitly the projection matrices for the finite subalgebras, at least for particular examples.[2–5] Our treatment makes its use explicit and therefore general. We also introduce here the affine projection matrix as a useful tool in this context.

Let $\Lambda=\sum_{\mu=0}^{R}\Lambda_{\mu}\omega^{\mu}$ $(\lambda=\sum_{\alpha=0}^{r}\lambda_{\alpha}\omega^{\alpha})$ with $0\le\Lambda_{\mu}\in\mathbb{Z}$ $(0\le\lambda_{\alpha}\in\mathbb{Z})$ be an affine weight of $\hat{g}(\hat{h})$. Also let $\bar{\Lambda}=\sum_{m=1}^{R}\Lambda_{m}\underline{\omega}^{m}$ $(\bar{\lambda}=\sum_{a=1}^{r}\lambda_{a}\omega^{a})$ be the $\bar{g}(\bar{h})$ weight that is the finite restriction of $\Lambda(\lambda)$. Then the isomorphism $O(\hat{g})\simeq B(G)$ may be described in the following manner. If we denote an outer automorphism by $\underline{A}\in O(\hat{g})$, there exists a corresponding element of the center $\underline{\alpha}\in B(G)$ whose eigenvalue on a $\bar{g}$ representation with highest weight $\bar{\Lambda}$ is $\exp[2\pi i(\underline{A}\,\underline{\omega}^{0}|\bar{\Lambda})]$. [$(\Lambda|\Lambda')$ and $(\bar{\Lambda},\bar{\Lambda}')$ are dot products of weights Λ,Λ' and $\bar{\Lambda},\bar{\Lambda}'$ determined by the Killing forms of $\hat{g}$ and $\bar{g}$, respectively, and normalized so that a long simple root satisfies $(\alpha|\alpha)\equiv|\alpha|^{2}=2$.] The element $\underline{\alpha}\in B(G)$ also acts diagonally on representations of $\hat{g}$ with the same eigenvalues. For a representation with highest weight Λ, we have (symbolically)

$$\underline{\alpha}\cdot\Lambda=\Lambda\cdot\exp[2\pi i(\underline{A}\,\underline{\omega}^{0}|\Lambda)]\ , \tag{1}$$

where we have used

$$(\underline{A}\,\underline{\omega}^{0}|\bar{\Lambda})=(\underline{A}\,\underline{\omega}^{0}|\Lambda)\ . \tag{2}$$

Similarly, for $A\in O(\hat{h})$, there exists $\alpha\in B(H)$ such that

$$\alpha\cdot\lambda=\lambda\cdot\exp[2\pi i(A\,\omega^{0}|\lambda)] \tag{3}$$

for all highest-weight representations λ of $\hat{h}$.

Because of the form of the eigenvalues (1) and (3), to get relations between the centers of H and G we examine the relation between weights of $\bar{g}$ and $\bar{h}$. The embedding $\bar{h}\subset\bar{g}$ is specified by the projection that takes weights of $\bar{g}$ onto weights of $\bar{h}$. Denoting a weight $\bar{\Lambda}$ of $\bar{g}$ by a column vector $\bar{\Lambda}=(\Lambda_{1}\Lambda_{2}\cdots\Lambda_{R})^{T}$, we can construct a so-called projection matrix[9] $\bar{F}$ such that $\bar{\Lambda}$ is projected onto the weight $\bar{F}\bar{\Lambda}$ of $\bar{h}$. $\bar{F}$ is an $r\times R$ matrix with integer entries greater than or equal to zero. $\bar{F}\bar{\Lambda}$ is a column vector whose r entries are the coefficients of the fundamental weights ω^{a} of $\bar{h}$.

If we let $\bar{F}$ act on all the weights $\{\bar{\Lambda}'\}$ in a representation with highest weight $\bar{\Lambda}$, the weights $\{\bar{F}\bar{\Lambda}'\}$ of $\bar{h}$ will fill out several representations with highest weights $\bar{\lambda}_{i}$. This can be denoted symbolically by

$$\bar{\Lambda}\rightarrow\sum_{i}\bar{\lambda}_{i}\ , \tag{4}$$

and is known as a branching rule. Two embeddings with distinct projection matrices $\bar{F}$ are said to be equivalent when their branching rules are identical. Thus there are, in general, more than one valid projection matrices for the "same" embedding. This will be useful later on.

Because of (1,2,3), $\underline{\alpha}\in B(G)$ and $\alpha\in B(H)$ are identified if and only if

$$(\underline{A\omega}^0|\bar{\Lambda})=(A\omega^0|\bar{F}\bar{\Lambda}) \bmod 1 \tag{5}$$

for all $\bar{\Lambda}$. So with a projection matrix $\bar{F}$, it is straightforward to find relations between the centers of G and H.

To find consequences of the center relations (5), we study characters. Let $\underline{\chi}^{\Lambda}(\tau)$ $[\chi^{\lambda}(\tau)]$ denote the (specialized) character of the $\hat{g}$ $[\hat{h}]$ representation with highest weight Λ $[\lambda]$. The characters of the coset theory are the branching functions $b_{\lambda}^{\Lambda}(\tau)$ (Ref. 3) of the subalgebra $\hat{g}\supset\hat{h}$, defined by

$$\underline{\chi}^{\Lambda}(\tau)=\chi^{\lambda}(\tau)b_{\lambda}^{\Lambda}(\tau) \ . \tag{6}$$

The corresponding coset fields are labeled by two highest weights (Λ,λ). In matrix notation (6) is

$$\underline{\chi}=\chi\cdot b \ . \tag{7}$$

Note that

$$\begin{aligned}(A\omega^0|\lambda+\beta)&=(A\omega^0|\lambda) \bmod 1 \ ,\\ (\underline{A\omega}^0|\Lambda+\underline{\beta})&=(\underline{A\omega}^0|\Lambda) \bmod 1\end{aligned} \tag{8}$$

for any roots $\beta,\underline{\beta}$ of $\bar{h},\bar{g}$. Suppose $\bar{\Lambda}$ is the finite restriction of a weight Λ in the $\hat{g}$ representation with highest weight Λ', and $\bar{F}\bar{\Lambda}$ is the restriction of a weight in the $\hat{h}$ representation with highest weight λ. Then (8) means the center relation (5) implies

$$(\underline{A\omega}^0|\Lambda')=(A\omega^0|\lambda) \bmod 1 \ . \tag{9}$$

The representation with highest weight Λ' will branch only to those representations of $\hat{h}$ with highest weights λ obeying (9). This means only those primary fields (Λ',λ) obeying (9) appear in the coset conformal theory.[2-5]

This selection rule can be expressed using the characters in the following way:

$$\exp[2\pi i(A\omega^0|\lambda)]b_{\lambda}^{\Lambda}\exp[2\pi i(\underline{A\omega}^0|\Lambda)]=b_{\lambda}^{\Lambda} \tag{10}$$

or in matrix notation,

$$\alpha b\underline{\alpha}=b \ . \tag{11}$$

The phases introduced in (11) by $\underline{\alpha}\in B(G)$ and $\alpha\in B(H)$ must cancel, or else the element b_{Λ}^{λ} of b must vanish, implying that the corresponding primary field does not appear.

Equation (11) also requires that certain fields in the coset theory be identified. To see this, consider how the characters transform under the modular transformation $S(\tau\to-1/\tau)$. If

$$\underline{\chi}(-1/\tau)=\underline{\chi}(\tau)\underline{S}, \quad \chi(-1/\tau)=\chi(\tau)S \tag{12}$$

then from (7) we have

$$b(-1/\tau)=S^{\dagger}b(\tau)\underline{S} \ . \tag{13}$$

Now in the space of characters of a Kač-Moody algebra $\hat{g}$, it is the modular transformation S which diagonalizes an outer automorphism $\underline{A}$ (Ref. 10):

$$\underline{S}^{\dagger}\underline{AS}=\underline{\alpha} \ , \tag{14}$$

thereby manifesting the isomorphism $O(\hat{g})\simeq B(G)$. A similar relation holds for $\hat{h}$:

$$S^{\dagger}AS=\alpha \ , \tag{15}$$

where $A\in O(\hat{h})$, $\alpha\in B(H)$. Applying (13)–(15) to (11) then yields

$$Ab\underline{A}=b \ . \tag{16}$$

The characters of the fields $(\underline{A}\Lambda, A\lambda)$ and (Λ,λ) are identical, and so they must be identified:

$$(\underline{A}\Lambda, A\lambda)\simeq(\Lambda,\lambda) \ . \tag{17}$$

Thus field identifications are a consequence of relations between the centers of G and H that may be easily found via (5) using a projection matrix $\bar{F}$.

Of course, the field identifications (17) are simply consequences of the relations between outer automorphisms of $\hat{g}$ and $\hat{h}\subset\hat{g}$. One should not have to introduce characters to find them. In the following we will discuss how they may be discovered in a manner as direct as relations between centers are found.

To do this we study projection matrices $\hat{F}$ for the affine subalgebra $\hat{h}^{ek}\subset\hat{g}^{k}$ (Refs. 11 and 12). Since affine Kač-Moody algebras $\hat{g},\hat{h}$ have the fundamental weights $\underline{\omega}^0,\omega^0$ as well as those of the finite algebras $\bar{g},\bar{h}$, the matrix $\hat{F}$ is a $(r+1)\times(R+1)$-dimensional matrix. (Here we assume both $\bar{g}$ and $\bar{h}$ are simple. Generalization is straightforward.) An affine weight $\Lambda(\lambda)$ is written as a column vector $[\Lambda_0\Lambda_1\cdots\Lambda_R]^T$ $([\lambda_0\lambda_1\cdots\lambda_r]^T)$. Then the weight Λ of $\hat{g}$ is projected onto the weight $\hat{F}\Lambda$ of $\hat{h}$.

One way to construct a projection matrix $\hat{F}$ for $\hat{g}^{k}\supset\hat{h}^{ek}$ is to demand that the finite parts of affine weights be projected according to a valid matrix $\bar{F}$ for $\bar{g}\supset\bar{h}$. Denoting the elements of $\hat{F}$ by $\hat{F}^{\mu}_{\alpha}$, that is

$$\hat{F}\underline{\omega}^{\mu}=\omega^{\alpha}\hat{F}^{\mu}_{\alpha} \ , \tag{18}$$

this specifies all elements $\alpha\neq0$. The remaining elements are determined by requiring that a level-k weight Λ of $\hat{g}$ be projected onto a level-ek weight of $\hat{h}$. The level of a $\hat{g}(\hat{h})$ weight $\Lambda(\lambda)$ is $\Lambda_{\mu}\underline{k}^{V\mu}(\lambda_{\alpha}k^{V\alpha})$ where $\underline{k}^{V\mu}(k^{V\alpha})$ are the co-marks of $\hat{g}(\hat{h})$. So we demand

$$k^{V\alpha}\hat{F}^{\mu}_{\alpha}\Lambda_{\mu}=\epsilon k \tag{19}$$

if $\Lambda_{\mu}k^{V\mu}=k$. Taking $k=\underline{k}^{V\nu}$ and $\Lambda_{\mu}=\delta^{\nu}_{\mu}$ gives

$$k^{V\alpha}\hat{F}^{\nu}_{\alpha}=e\underline{k}^{V\nu} \ , \tag{20}$$

or in matrix notation

$$(k^{V})^{T}\hat{F}=e(\underline{k}^{V})^{T} \ , \tag{21}$$

completing the determination of $\hat{F}$ from $\bar{F}$. Note in particular that

$$\hat{F}^{0}_{\alpha}=e\delta^{0}_{\alpha} \ . \tag{22}$$

An affine projection matrix manifests a relation between $\underline{A}\in O(\hat{g})$ and $A\in O(\hat{h})$ if the following is true

$$A\hat{F}\underline{A}=\hat{F}' , \tag{23}$$

where $\hat{F}'$ is another valid projection matrix. In (23) A and $\underline{A}$ are the matrices which permute the rows and columns, respectively, of $\hat{F}$ in the manner prescribed by the corresponding outer automorphisms. [Note that these matrices are in general of dimension smaller than those of Eqs. (14)–(16).] Relations of the type (23) with $\hat{F}'=\hat{F}$ were found in Refs. 11 and 12.

Unfortunately, we have no general test for a valid affine projection matrix. We can only check those that are built from a finite matrix $\bar{F}$ in the manner just described. The test is then simply the requirements of the matrix $\bar{F}$ that is a submatrix of $\hat{F}$. A sufficient requirement[9] is that the matrix $\bar{F}$ produce the correct branching rule for the second smallest (i.e., not the scalar) irreducible representation of $\bar{g}$ into representations of $\bar{h}$.

This means we must restrict the $\hat{F}'$ in (23) to those satisfying (22). This restricts us to a subset among the pairs $A,\underline{A}$ satisfying (23) in the general sense. Our ignorance concerning affine projection matrices therefore makes the center relations (5) easier to verify.

However, quite often there are matrices $\hat{F}$ which manifest outer automorphism relations in an obvious way (see Refs. 11 and 12). Furthermore, in all cases we have checked, there is a sufficient number of different $\bar{F}$'s such that a complete set of relations may be derived from (23). At the very least, even with the technical restriction (22) imposed on $\hat{F}'$, the relations (23) provide checks on the center relations.

There is even a case when the relations (23) are the only ones that may be simply verified. Suppose we drop for the moment the restriction that $\bar{h}$ is a maximal subalgebra of $\bar{g}$ (as mentioned before), and suppose $\bar{h}\otimes\bar{h}'\subset\bar{g}$ is maximal. Suppose further there is a center relation for this maximal subalgebra of the form

$$(\underline{A}\omega^0|\Lambda)=(A\omega^0|\bar{F}\bar{\Lambda})+(A'\omega^{0'}|\bar{F}\bar{\Lambda}) \bmod 1 , \tag{24}$$

where $A',\omega^{0'}$ are an outer automorphism and the 0th fundamental weight of $\hat{h}'$. Then if we consider the nonmaximal embedding $\bar{h}\subset\bar{g}$, we do not have

$$(\underline{A}\omega^0|\Lambda)=(A\omega^0|\bar{F}\bar{\Lambda}) \bmod 1 \tag{25}$$

even though $\underline{A}$ and A should be identified. On the other hand, a relation of the type (23) will exist, at least subject to the restrictions discussed above.

The following examples should clarify our general discussion.

Example 1. $G=SO(7)$, $H=SU(4)$.

Our first example is the subalgebra $so(7)\supset su(4)$, with index of embedding $e=1$. This is an example of a regular maximal subalgebra, i.e., it can be understood by deleting a node from the extended Dynkin diagram of so(7) (see, for example, Ref. 13). The node omitted is the one representing the short simple root of so(7), so that the long roots and the negative of the highest root are projected onto the simple roots of su(4). So the finite subalgebra projection matrix is

$$\bar{F}=\begin{bmatrix}0&1&1\\1&0&0\\0&1&0\end{bmatrix} . \tag{26}$$

The affine matrix built from (26) by the method discussed above is

$$\hat{F}=\begin{bmatrix}1&0&0&0\\0&0&1&1\\0&1&0&0\\0&0&1&0\end{bmatrix} . \tag{27}$$

A sufficient check of the validity of $\bar{F}$ is that it reproduce the branching rule

$$(100)^T\to(010)^T+(000)^T . \tag{28}$$

We can check (28) by letting $\hat{F}$ act on the states having the minimum L_0 eigenvalue in the $\widehat{so}(7)$ representation with highest weight $[0100]^T$, since these states transform under SO(7) as the representation with highest weight $(100)^T$. They are

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}1\\0\\1\\-2\end{bmatrix}\begin{bmatrix}2\\1\\-1\\0\end{bmatrix}\begin{bmatrix}2\\-1\\0\\0\end{bmatrix} \tag{29}$$

and (28) is easily verified.

Now $\widehat{so}(7)$ has outer automorphism group $\mathbb{Z}_2$, generated by $\underline{\alpha}$, acting in the following way on a weight Λ:

$$\underline{\alpha}[\Lambda_0\Lambda_1\Lambda_2\Lambda_3]^T=[\Lambda_1\Lambda_0\Lambda_2\Lambda_3]^T . \tag{30}$$

The $\mathbb{Z}_4$ outer automorphism group of $\widehat{su}(4)$ is generated by a, with action

$$a[\lambda_0\lambda_1\lambda_2\lambda_3]^T=[\lambda_3\lambda_0\lambda_1\lambda_2]^T . \tag{31}$$

It is simple to verify

$$\begin{aligned}(\underline{\alpha}\omega^0|\Lambda)&=(\underline{\omega}^1|\Lambda)=\tfrac{1}{2}\Lambda_3 \bmod 1 ,\\(a^2\omega^0|\bar{F}\bar{\Lambda})&=(\omega^2|\bar{F}\bar{\Lambda})=\tfrac{1}{2}\Lambda_3 \bmod 1 ,\end{aligned} \tag{32}$$

implying the following relation, of the form (5), between the centers of SO(7) and SU(4):

$$(\underline{\alpha}\omega^0|\Lambda)=(a^2\omega^0|\bar{F}\bar{\Lambda}) \bmod 1 , \tag{33}$$

for all Λ. This relation implies that any field (Λ',λ) appearing in the coset conformal theory, labeled by highest weights Λ',λ of representations of $\hat{g},\hat{h}$, respectively, must satisfy the selection rule

$$(\underline{\alpha}\omega^0|\Lambda')=(a^2\omega^0|\lambda) \bmod 1 . \tag{34}$$

The corresponding relation of the type (23) between the outer automorphism groups is also easily verified. We have

$$a^2\hat{F}_{\underline{\alpha}}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&1&1\end{bmatrix}\equiv\hat{F}' \tag{35}$$

and this matrix acting on the weights (29) reproduces the correct branching rule (28). Fields of the coset so(7)/su(4) theory must therefore be identified as follows:

$$(\alpha\Lambda', a^2\lambda) \simeq (\Lambda',\lambda) \ . \tag{36}$$

Example 2. $G = H \otimes H$.

Our second example is the diagonal embedding $\bar{h} \subset \bar{h} \oplus \bar{h}$. A weight of $\hat{h} \oplus \hat{h}$ may be denoted $[\lambda,\lambda']^T$, where λ is a weight of the first $\hat{h}$ and λ' of the second. If we demand that a pure $\hat{h}$ weight $[\lambda,0]^T$ or $[0,\lambda]^T$ is projected onto the same weight $[\lambda]^T$ of the diagonal subalgebra, we get

$$\hat{F}^{\mu}_{\alpha} = \hat{F}^{c\alpha'}_{\alpha} = \delta^{\alpha'}_{\alpha} \ , \tag{37}$$

where $c = 1,2$ specify the two summands of $\hat{h} \oplus \hat{h}$ and α, α' denote the fundamental weights of $\hat{h}$. So a weight $[\lambda,\lambda']^T$ of $\hat{h} \oplus \hat{h}$ is projected onto the weight $[\lambda+\lambda']^T$ of $\hat{h}$:

New consider any outer automorphism A of $\hat{h}$. The corresponding automorphism of $\hat{g} = \hat{h} \oplus \hat{h}$ is $\underline{A} = A \otimes A$. Since

$$(\underline{A}\omega^0 | \Lambda) = (A\omega^0|\lambda) + (A\omega^0|\lambda') \tag{38}$$

and

$$(A\omega^0|\bar{F}\Lambda) = (A\omega^0|\lambda+\lambda') \tag{39}$$

we have a center relation of the type (5) for all A. If $\Lambda' = [\rho,\sigma]^T$ and $\lambda = [\zeta]^T$ are highest weights of $\hat{h} \oplus \hat{h}$ and $\hat{h}$ representations, respectively, then only those fields obeying the selection rule (9) may occur in the diagonal coset theory. In this example, it means $\bar{F}\bar{\Lambda}' - \bar{\lambda} = \bar{\rho} + \bar{\sigma} - \bar{\zeta}$ must lie in the root lattice of $\bar{h}$ (Refs. 3 and 4).

Equation (23) also holds obviously, with $\hat{F}' = \hat{F}$:

$$A\hat{F}(A \otimes A) = \hat{F} \ . \tag{40}$$

Therefore the field $([\rho,\sigma]^T,[\zeta]^T)$ is identified with $([A\rho, A\sigma]^T,[A\zeta]^T)$.

Example 3. $G = \mathrm{SU}(6)$, $H = \mathrm{SU}(2) \otimes \mathrm{SU}(3)$.

The last example illustrates that quite nontrivial relations exist between the centers of G and H. It also shows the limitations imposed by the technical restriction (22) on the relations (23) that can be found.

The embedding $\widehat{su}(p)^{kq} \times \widehat{su}(q)^{kp} \subset \widehat{su}(pq)^k$ with $k=1$ was studied in Ref. 11. In this example we will not restrict k, but set $p=2$ and $q=3$, just for the sake of simplicity. The following is a valid projection matrix[11]:

$$\hat{F} = \left[\begin{array}{cccccc} 3 & 2 & 3 & 2 & 3 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right] \ . \tag{41}$$

Let A_6, A_2, A_3 be the generators of the outer automorphism groups $O(\widehat{su}(6))$, $O(\widehat{su}(2))$, $O(\widehat{su}(3))$, respectively, so that $(A_i)^i = 1$, $i = 6,2,3$. Then this projection matrix immediately gives

$$(1 \otimes A_3)\hat{F}(A_6)^2 = \hat{F} \ . \tag{42}$$

On the other hand, with $\bar{F}$ the finite projection matrix contained in (41), we have the following center relation

$$(A_6\omega_6^0|\bar{\Lambda}) = (A_2\omega_2^0 + (A_3)^2\omega_3^0|\bar{F}\bar{\Lambda}) \bmod 1 \ , \tag{43}$$

valid for all $\bar{\Lambda}$, where ω_i^0 is the 0th fundamental weight of $\widehat{su}(i)$. The resulting selection rule for coset fields (Λ',λ) is

$$(A_6\omega_6^0|\Lambda') = (A_2\omega_2^0 + (A_3)^2\omega_3^0|\lambda) \bmod 1 \ . \tag{44}$$

The nontrivial center relation (43) cannot be verified in the form of

$$[A_2 \otimes (A_3)^2]\hat{F}A_6 = \hat{F}' \ , \tag{45}$$

since $\hat{F}'$ in this last equation is not built from a valid finite projection matrix $\bar{F}'$; i.e., it does not satisfy (22).

However, another affine projection matrix[11]

$$\hat{F} = \left[\begin{array}{cccccc} 3 & 2 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ \hline 2 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{46}$$

manifests

$$(A_2 \otimes 1)\hat{F}(A_6)^3 = \hat{F} \ . \tag{47}$$

Together Eqs. (42) and (47) verify, albeit indirectly, the identification of A_6 with the product of A_2 and $(A_3)^2$. Consequently, the identifications

$$(A_6\Lambda', A_2(A_3)^2) \simeq (\Lambda',\lambda) \tag{48}$$

must be made.

Before concluding, let us note that field identifications are properties of the coset describing a particular conformal field theory, not necessarily of the field theory itself. For example, the Ising model may be described by the subalgebra $\widehat{su}(2)^1 \oplus \widehat{su}(2)^1 \supset su(2)^2$. Since $O(\widehat{su}(2)) = \mathbb{Z}_2$, there is a nontrivial identification of fields due to an outer automorphism relation of the type discussed in Example 2. However, the Ising model is also described by another diagonal subalgebra: $\hat{E}_8^1 \oplus \hat{E}_8^1 \supset \hat{E}_8^2$. Since $\hat{E}_8$ has no outer automorphisms, this coset has no such identifications. So we may conclude that field identifications are not intrinsic to the Ising model. (The authors wish to thank D. Lewellen for this observation.)

We must also emphasize that we have chosen to label coset theories by their algebras $\bar{h} \subset \bar{g}$ and therefore by their coverings groups H, G. There are in general many theories possible for each subalgebra, and some of these may be more precisely labeled by nonsimply-connected groups $\tilde{G} = G/Z(G)$, $\tilde{H} = H/Z(H)$ with $Z(G) \subset B(G)$, $Z(H) \subset B(H)$ (Ref. 2). When $Z(H)$ and $Z(G)$ have elements in common, the corresponding outer automorphisms are identified. Then pairs of highest weights (Λ,λ) that are invariant under these automorphisms will correspond to more than one primary field.[2] The simplest example of this phenomenon occurs for strings on nonsimply-connected group manifolds.[14,2] However, it is not clear that all theories admit such a description in terms of nonsimply-connected groups. (A counterexample may be the F_4 theory recently found by Schellekens and Yankielowicz.[15]) So we stick to the general labeling by the subalgebra, keeping in mind that a multiplicity greater than one for (Λ,λ) denotes different operators in the theory.

In summary, we have pointed out the importance of projection matrices for embeddings $\hat{h}^{ek}\subset\hat{g}^k$ in the coset conformal field theories on which they are based. Relations between the centers of G and H can be easily identified, and they imply selection rules excluding certain pairs of highest weights (Λ,λ) as possible primary fields in the coset theory. Relations between outer automorphism groups are also easily found from affine projection matrices, and result in identification of fields $(\Lambda,\lambda)\simeq(\underline{A}\Lambda, A\lambda)$ in the coset theory.

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