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Electromagnetic fields in Khan-Penrose spacetime

T. M. Helliwell
Department of Physics, Harvey Mudd College, Claremont, California 91711

D. A. Konkowski
Department of Mathematics, U.S. Naval Academy, Annapolis, Maryland 21402
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The behavior of test electromagnetic waves on the Khan-Penrose colliding gravitational-wave spacetime is used to probe the nature of the quasiregular singularities present. It is argued that the divergence of stress-energy scalars for most wave modes makes these singularities unstable, converting them into scalar curvature singularities. However, a special subset of modes does not lead to divergence of stress-energy scalars at the singularities. In the presence of such modes the singularities should remain quasiregular in an exact back-reaction calculation, as confirmed in the colliding gravitational- and electromagnetic-wave spacetime of Chandrasekhar and Xanthopoulos.

I. INTRODUCTION

The nature of spacetime singularities remains an interesting problem in classical general relativity.[1] Ellis and Schmidt have classified singularities in maximal, four-dimensional spacetimes into three basic types: scalar curvature, nonscalar curvature, and quasiregular.[2] The obstacle which bars the embedding of singular spacetimes into larger nonsingular spacetimes is obvious for those with scalar curvature singularities, where physical quantities such as energy density and tidal forces diverge for all observers who encounter the singularity. The physical significance of the other two types of singularity is less obvious. In the case of a nonscalar curvature singularity some, but not all, observers feel infinite tidal forces as they approach the singularity. It is still more curious that for a quasiregular singularity *no* observers see physical quantities diverge, even though their world lines end at the singularity in a finite proper time.

Quasiregular singularities are the mildest type of true singularity, and they are also the least well understood.[2] By definition a singular point q is a C^k (or C^{k-}) *quasiregular* singularity ($k \geq 0$) if all components or derivatives of the Riemann tensor $R_{abcd;e_1 \cdots e_k}$ evaluated in an orthonormal (ON) frame parallel propagated (PP) along an incomplete geodesic ending at q are C^0 (or C^{0-}). In other words, the Riemann tensor components and derivatives tend to finite limits (or are bounded) in every PPON frame. On the other hand, a singular point q is a C^k (or C^{k-}) *curvature* singularity if some component or derivative is not bounded in this way. If all scalars in g_{ab}, the antisymmetric tensor η_{abcd}, and $R_{abcd;e_1 \cdots e_k}$, nevertheless tend to a finite limit (or are bounded), the singularity is *nonscalar,* but if any scalar is unbounded, the point q is a *scalar* curvature singularity.

Quasiregular singularities have been found in "Taub-NUT (Newman-Unti-Tamburino)-type" cosmologies,[3] cosmic-string models,[4–6] and in colliding plane-wave spacetimes.[7,8] One suspects from their strange properties that although they occur in exact solutions of Einstein's equations they may be unstable, so that the addition of generic matter or fields to quasiregular spacetimes may convert these mild singularities into a stronger form. We have previously studied the stability of singularities in Taub-NUT-type cosmologies using test scalar and electromagnetic fields.[3,9] We conjectured that if one introduces a test field whose stress-energy tensor evaluated in a PPON frame mimics the behavior of the Riemann tensor components which indicate a particular type of singularity (quasiregular, nonscalar curvature, or scalar curvature), then a complete nonlinear back-reaction calculation would show that this type of singularity actually occurs. For example, if a scalar quantity such as $T_{\mu\nu}T^{\mu\nu}$ constructed from a test field's stress-energy tensor diverges as a quasiregular singularity is approached, the conjecture is that a scalar curvature singularity will actually develop if the field is allowed to influence the geometry. Evidence for this conjecture was presented from a few known exact solutions; the evidence also showed that most test-field wave modes do in fact mimic scalar curvature singularities, but that very special wave modes can mimic nonscalar or quasiregular singularities.[9] Therefore, if generic fields are added to Taub-NUT-type cosmology, one expects that their quasiregular singularities will be converted into scalar curvature singularities.

In this paper we extend our conjecture to include the quasiregular singularities in the Khan-Penrose colliding impulsive plane gravitational-wave spacetime,[10] by examining the behavior of test electromagnetic fields. We show that the behavior of fields and their stress-energy tensors are similar to their behavior in Taub-NUT-type spacetimes. We show also that the exact solution of the Einstein-Maxwell equations found by Chandrasekhar and Xanthopoulos,[11] which describes the collision between two plane impulsive gravitational waves, each supporting an electromagnetic shock wave, is in effect an exact back-reaction solution corresponding to a restricted class of the electromagnetic fields we consider here. Finally, we show that the effect of these restricted fields on the quasiregular singularities of the Khan-Penrose spacetime is consistent with our conjecture. Tensor components in coordinate frames and in PPON frames are represented throughout by greek and latin indices, respectively.

II. TEST ELECTROMAGNETIC FIELDS AND STRESS TENSORS

In Khan-Penrose spacetime, δ-function gravitational waves propagate into an initially flat region. In the two-dimensional slice shown in Fig. I, the planes $u=0$ and $v=0$ are δ-function waves. Regions I, II, and III are flat with double-null metrics[10]:

$$\text{I:}\quad ds^2=-2du\,dv+dx^2+dy^2\ , \tag{1a}$$

$$\text{II:}\quad ds^2=-2du\,dv+(1+u)^2dx^2+(1-u)^2dy^2\ , \tag{1b}$$

$$\text{III:}\quad ds^2=-2du\,dv+(1+v)^2dx^2+(1-v)^2dy^2\ . \tag{1c}$$

Region IV, which is in the absolute future of the collision, is curved with the metric

$$\text{IV:}\quad ds^2=\frac{-2t^2du\,dv}{rw(uv+rw)^2}+t^2\left[\frac{r+v}{r-v}\right]\left[\frac{w+u}{w-u}\right]dx^2+t^2\left[\frac{r-v}{r+v}\right]\left[\frac{w-u}{w+u}\right]dy^2\ , \tag{1d}$$

where $r=(1-u^2)^{1/2}$, $w=(1-v^2)^{1/2}$, and $t=(1-u^2-v^2)^{1/2}$. A three-dimensional picture due to Penrose is given in Matzner and Tipler.[12]

Khan and Penrose[10] showed through an analysis of the Green's function for the solution that a curvature singularity forms to the future of the collision at $u^2+v^2=1$. No curvature singularity, however, forms at the points $u=0$, $v=1$, or $v=0$, $u=1$ or on the surfaces $u=1$ and $v=1$. These are quasiregular singularities.[7,8]

To study the stability of these quasiregular singularities, we will consider the behavior of an electromagnetic test field. Consider region II (one of the two regions bounded by a quasiregular singularity). The field equations for sourceless electromagnetic waves in terms of the vector potential are

$$A^{\mu;\nu}{}_{;\nu}=0\ . \tag{2}$$

It is straightforward to solve Eq. (1) in region II in the Lorentz gauge, $A^{\mu}{}_{;\mu}=0$. The vector potential is

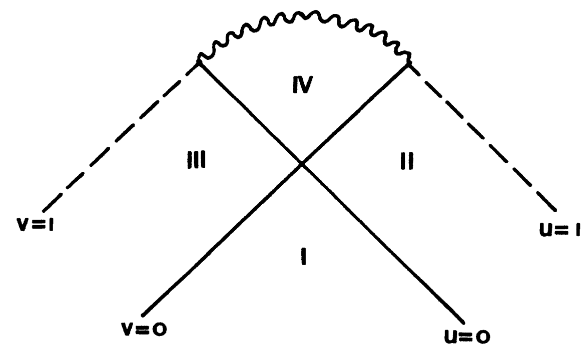

FIG. 1. Colliding impulsive gravitational waves. The surfaces $u=0$ and $v=0$ are δ-function plane gravitational waves. The wavy line is a scalar curvature singularity and the dashed lines are quasiregular singularities.

$$\begin{aligned}
A^0(u,v)&=\frac{a(v)}{(1-u^2)^{1/2}}+f_0(u)\ ,\\
A^1(u,v)&=\frac{b(v)u}{(1-u^2)^{3/2}}+\frac{v}{(1-u^2)^{1/2}}\int\frac{f_0(u)(1+u^2)}{(1-u^2)^{3/2}}du+f_1(u)\ ,\\
A^2(u,v)&=\frac{c(v)}{(1+u)^{3/2}(1-u)^{1/2}}+f_2(u)\ ,\\
A^3(u,v)&=\frac{d(v)}{(1-u)^{3/2}(1+u)^{1/2}}+f_3(u)\ ,
\end{aligned} \tag{3}$$

where

$$b(v)=\int a(v)dv$$

and

$$f_0''-\frac{3u}{1-u^2}f_0'-\frac{1+u^2}{(1-u^2)^2}f_0=0\ .$$

The other functions are unconstrained.

The electromagnetic field tensor $F_{\mu\nu}=A_{\nu;\mu}-A_{\mu;\nu}$. If we assume for simplicity that $f_0=0$, then

$$F_{\mu\nu}=\begin{bmatrix}0 & F\\ -F^T & 0\end{bmatrix},$$

where

$$F=\begin{bmatrix}\dfrac{c(v)}{(1-u)^{3/2}(1+u)^{1/2}}+2(1+u)f_2+(1+u)^2f_2' & \dfrac{-d(v)}{(1+u)^{3/2}(1-u)^{1/2}}-2(1-u)f_3+(1-u)^2f_3'\\ c'(v)\dfrac{(1+u)^{1/2}}{(1-u)^{1/2}} & d'(v)\dfrac{(1-u)^{1/2}}{(1+u)^{1/2}}\end{bmatrix}. \tag{4}$$

The stress-energy tensor is

$$T_{\mu\nu}=(1/4\pi)(F_{\mu\alpha}F_{\nu}{}^{\alpha}-\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})\ ;$$

that is

$$T_{\mu\nu}=\begin{bmatrix} T_{00} & 0 & 0 & 0 \\ 0 & T_{11} & 0 & 0 \\ 0 & 0 & T_{22} & T_{23} \\ 0 & 0 & T_{32} & T_{33} \end{bmatrix},$$

where

$$T_{00}=\frac{1}{4\pi}\left[(1+u)^{-2}\left[\frac{c(v)}{(1-u)^{3/2}(1+u)^{1/2}}+2(1+u)f_2+(1+u)^2f_2'\right]^2 + (1-u)^{-2}\left[\frac{-d(v)}{(1+u)^{3/2}(1-u)^{1/2}}-2(1-u)f_3+(1-u)^2f_3'\right]^2\right],$$

$$T_{11}=\frac{1}{4\pi}\left[\frac{[c'(v)]^2}{(1-u)(1+u)}+\frac{[d'(v)]^2}{(1-u)(1+u)}\right],$$

$$T_{22}=\frac{1}{4\pi}\frac{(1+u)^{1/2}}{(1-u)^{1/2}}\left[-c'(v)\left[\frac{c(v)}{(1-u)^{3/2}(1+u)^{1/2}}+2(1+u)f_2+(1+u)^2f_2'\right] + d'(v)\left[\frac{-d(v)}{(1-u)^{3/2}(1+u)^{1/2}}-2(1+u)f_3+(1+u)(1-u)f_3'\right]\right], \tag{5}$$

$$T_{23}=T_{32}=\frac{-1}{4\pi}\left[\frac{(1-u)^{1/2}}{(1+u)^{1/2}}d'(v)\left[\frac{c(v)}{(1-u)^{3/2}(1+u)^{1/2}}+2(1+u)f_2+(1+u)^2f_2'\right] + \frac{(1+u)^{1/2}}{(1-u)^{1/2}}c'(v)\left[\frac{-d(v)}{(1+u)^{3/2}(1-u)^{1/2}}-2(1-u)f_3+(1-u)^2f_3'\right]\right],$$

$$T_{33}=\frac{1}{4\pi}\frac{(1-u)^{1/2}}{(1+u)^{1/2}}\left[c'(v)\left[\frac{c(v)}{(1+u)^{3/2}(1-u)^{1/2}}+2(1-u)f_2+(1+u)(1-u)f_2'\right] - d'(v)\left[\frac{-d(v)}{(1+u)^{3/2}(1-u)^{1/2}}-2(1-u)f_3+(1-u)^2f_3'\right]\right].$$

Thus, all components of the stress-energy tensor except T_{33} generally diverge as u approaches 1; that is, $T_{\mu\nu}$ generally diverges at the quasiregular singularity. In particular,

$$T_{\mu\nu}\sim\begin{bmatrix} O((1-u)^{-5/2}) & 0 & 0 & 0 \\ 0 & O((1-u)^{-1}) & 0 & 0 \\ 0 & 0 & O((1-u)^{-2}) & O((1-u)^{-1}) \\ 0 & 0 & O((1-u)^{-1}) & O(1) \end{bmatrix}.$$

As a check on the calculation, it can be shown that $T_{\mu}{}^{\nu}{}_{;\nu}=0$ and $T_{\mu}{}^{\mu}=0$ as is required for an electromagnetic field. The scalar $T_{\mu\nu}T^{\mu\nu}$ is not, however, zero. A straightforward calculation gives

$$T_{\mu\nu}T^{\mu\nu}=\frac{[c'(v)]^2+[d'(v)]^2}{4\pi^2}\left[\frac{c^2+d^2}{(1-u)^4(1+u)^4}+\frac{4(cf_2+df_3)}{(1+u)^{5/2}(1-u)^{3/2}} + \frac{2cf_3'}{(1+u)^{3/2}(1-u)^{5/2}}-\frac{2df_3'}{(1+u)^{5/2}(1-u)^{3/2}}+\frac{4(f_2{}^2+f_3{}^2)}{(1+u)(1-u)} + \frac{4f_2f_2'}{1-u}-\frac{4f_3f_3'}{1+u}+f_2'^2\frac{1+u}{1-u}+f_3'^2\frac{1-u}{1+u}\right]. \tag{6}$$

Notice that $T_{\mu\nu}T^{\mu\nu}$ is finite as $u\to 1$ if (1) c and d are constants and/or (2) the quantity in large parentheses is finite. Actually, condition (2) requires $c=d=0$ and f_2 and f_3 chosen in a very special way.

Assuming that c and d are constant (so $T_{\mu\nu}T^{\mu\nu}$ is finite), we can calculate the stress-energy tensor in a PPON frame. Then

$$T_{ab}=e_a{}^{\mu}e_b{}^{\nu}T_{\mu\nu},$$

where

$$e_0{}^{\mu}=\begin{bmatrix} a^1 \\ \dfrac{1}{2a^1}\left[1+\dfrac{(a^2)^2}{(1+u)^2}+\dfrac{(a^3)^2}{(1-u)^2}\right] \\ a^2(1+u)^{-2} \\ a^3(1-u)^{-2}\end{bmatrix},$$

$$e_1{}^{\mu}=\begin{bmatrix} 0 \\ \dfrac{a^2}{a^1}(1+u)^{-1} \\ (1+u)^{-1} \\ 0\end{bmatrix},\quad e_2{}^{\mu}=\begin{bmatrix} 0 \\ \dfrac{a^3}{a^1}(1-u)^{-1} \\ 0 \\ (1-u)^{-1}\end{bmatrix}, \tag{7}$$

$$e_3{}^{\mu}=\begin{bmatrix} a^1 \\ \dfrac{1}{2a^1}\left[\dfrac{(a^2)^2}{(1+u)^2}+\dfrac{(a^3)^2}{(1-u)^2}-1\right] \\ a^2(1-u)^{-2} \\ a^3(1-u)^{-2}\end{bmatrix}$$

are the PPON frame vectors, where a^1, a^2, a^3 are constants. Since $c=$const and $d=$const only T_{00} is nonzero, and

$$T_{ab}=(a^1)^2 T_{00}\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix}, \tag{8}$$

where T_{00} is evaluated in the coordinate frame. A parallel propagated frame which approaches the $u=1$ singularity cannot have $a^1=0$, so $T_{ab}\rightarrow\infty$ as $u\rightarrow 1$ unless $c=d=0$ and f_3 is chosen to make T_{00} finite.

Both $T_{\mu\nu}T^{\mu\nu}$ and T_{ab} diverge as $u\rightarrow 1$ for generic electromagnetic test fields. The Khan-Penrose spacetime plus generic fields therefore "mimics" the behavior of a spacetime which reacts to the presence of the fields by converting the quasiregular singularity into a scalar curvature singularity. Although no back-reaction calculation has been carried out in the generic case, we expect that these fields will convert the quasiregular singularities at $u=1$ in region II and at $v=1$ in region III into scalar curvature singularities. There is no proof that this conversion takes place. However, in the few cases where a back-reaction calculation on a quasisingular spacetime has been made, the mimicking of scalar curvature, nonscalar curvature, and quasiregular singularities by the behavior of test fields is a completely reliable guide.[9]

III. AN EXACT BACK-REACTION SOLUTION

It is interesting that in the special case $a=b=c=d=f_0=f_1=0$ of the general fields of Eq. (3), an exact back-reaction calculation has already been carried out by Chandrasekhar and Xanthopoulos.[11] Their spacetime features the collision between two plane impulsive gravitational waves, each supporting an electromagnetic shock wave. In our region II, their fields depend only upon u and not v, corresponding to waves moving in the v direction toward the boundary with the interaction region IV. These are also waves for which $T_{\mu\nu}T^{\mu\nu}$ and T_{ab} are finite as $u\rightarrow 1$, as shown by Eqs. (5), (6), and (8); they therefore mimic the behavior of a quasiregular singularity. Consistency with our conjecture then requires that the Chandrasekhar-Xanthopoulos spacetime retain the quasiregular singularities exhibited by the Khan-Penrose spacetime at $u=1$ in region II and at $v=1$ in region III. And, in fact, this is the case.

Chandrasekhar and Xanthopoulos introduce electromagnetic potentials A and B, and find the metric to be

$$ds^2=-4X^2[(dx^0)^2-(dx^3)^2]+X^2(1+u)^2(dx^1)^2 + \frac{1}{X^2}(1-u)^2(dx^2)^2 \tag{9}$$

in our region II, if the gravitational waves are parallel polarized, as they are in the Khan-Penrose spacetime. Here $X=a+b(1-u)^2$, where a and b are constants. In the limit of weak electromagnetic fields (i.e., as $A,B\rightarrow 0$) they find $a\rightarrow 1$ and $b\rightarrow 0$, so the metric reduces to that of Khan-Penrose if also $x^0=(1/2\sqrt{2})(u+v)$, $x^3=(1/2\sqrt{2})(u-v)$. Transformations from their tetrad frame to the Khan-Penrose metric show that in terms of the potentials of Eq. (3), all of these potentials are zero except for f_2 and f_3, which are related to their A and B by $A_{,u}=[(1+u)^2 f_2(u)]_{,u}$ and $B_{,u}=[(1-u)/(1-u)][(1-u)^2 f_3(u)]_{,u}$. The condition that f_3 be chosen to keep T_{00} finite in Eq. (5) is equivalent to the requirement that $B_{,u}$ remain finite as $u\rightarrow 1$.

We must now verify that the singularities at $u=1$ in region II and at $v=1$ in region III in the exact Chandrasekhar-Xanthopoulos solution are in fact quasiregular. For the metric

$$ds^2=-2du\,dv+X^2(u)(1+u)^2dx^2+X^{-2}(u)(1-u)^2dy^2$$

the timelike geodesic equations are $\dot{u}=a^1$, $X^2(u)(1+u)^2\dot{x}=a^2$,

$$X^{-2}(u)(1-u)^2\dot{y}=a^3,$$

$$\dot{v}=\frac{1}{2a^1}\left[1+\frac{(a^2)^2}{X^2(1+u)^2}+\frac{(a^3)^2}{X^{-2}(1-u)^2}\right]$$

in region II, where a^1, a^2, a^3 are constants and $\dot{u}\equiv du/ds$, etc. A PPON frame with frame vectors obeying the orthonormality condition $e_a{}^{\mu}e_{b\mu}=\eta_{ab}=\mathrm{diag}(-1,1,1,1)$ and parallel propagation condition $e_a{}^{\mu}{}_{;\nu}e_0{}^{\nu}=0$ is then found to be the same as those of Eq. (7), except that every factor of $(1+u)$ in Eq. (7) should be replaced by $X(u)(1+u)$, and every factor of $(1-u)$ should be replaced by $X^{-1}(u)(1-u)$.

In the coordinate frame, the only nonzero components of the Riemann tensor are

$$R^{v}{}_{xux}=X^2(u)(1+u)^2R^{x}{}_{uux} =\tfrac{1}{2}[X^2(1-u)^2]_{,uu} -\frac{\{[X^2(1+u)^2]_{,u}\}^2}{4X^2(1+u)^2}, \tag{10a}$$

$$R^{v}{}_{yuy}=X^{-2}(u)(1-u)^2R^{y}{}_{uuy}$$
$$=\tfrac{1}{2}[X^{-2}(1-u)^2]_{,uu}-\frac{\{[X^{-2}(1-u)^2]_{,u}\}^2}{4X^{-2}(1-u)^2}\;, \tag{10b}$$

and others obtained from the symmetries of $R_{\mu\nu\lambda\sigma}$. All components are finite as $u\rightarrow 1$. However, to discern the singularity type, it is necessary to evaluate R_{abcd} in PPON frames. Nonzero components of $R_{abcd}=e_{a\mu}e_{b}{}^{\nu}e_{c}{}^{\lambda}e_{d}{}^{\sigma}R^{\mu}{}_{\nu\lambda\sigma}$ are

$$R_{0101}=-R_{0113}=R_{1313}=-(a^1)^2R^{x}{}_{uux}\;, \tag{11a}$$

$$R_{0202}=-R_{0223}=R_{2323}=-(a^1)^2R^{y}{}_{uuy}\;, \tag{11b}$$

and others obtained by symmetries of the tensor. All components converge and are continuous as $u\rightarrow 1$; in fact,

$$R_{0101}\rightarrow -2b(a^1)^2/a \quad\text{and}\quad R_{0202}\rightarrow 6b(a^1)^2/a$$

for all PPON frames approaching $u=1$, where a and b are the constants in $X(u)$ and $a^1=\dot{u}$ for the geodesic. The $u=1$ singularity in region II (and similarly the $v=1$ singularity in region III) are therefore quasiregular as expected.

The agreement between the behavior of electromagnetic test fields on the Khan-Penrose colliding-wave spacetime with the singularity structure of the exact spacetime of Chandrasekhar and Xanthopoulos supports the general usefulness of the mimicking conjecture we had originally introduced for Taub-NUT-type cosmologies. In every case studied so far, the addition of test-field modes of given symmetry to a background quasisingular spacetime accurately predicts the nature of the singularities produced when an exact back-reaction calculation is carried out.

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