# Symmetry and internal time on the superspace of asymptotically flat geometries

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(Received 12 June 1989)

A difficulty with the canonical approach to quantum gravity, leading to attempts at "third quantization," is the absence of symmetry vectors on the superspace of three-metrics: vector fields that generate transformations of superspace leaving the action invariant. We show that on the superspace of asymptotically flat three-metrics, such symmetry vectors exist. They correspond to diffeomorphisms of each three-geometry that behave asymptotically as elements of the symmetry group at spatial infinity. The conserved momentum associated with a symmetry vector has a conjugate variable which can be regarded as an internal time coordinate of an isolated system. In particular, for asymptotic translations, a corresponding internal time is a center-of-mass coordinate. An appendix considers the natural contravariant and covariant metrics on superspace. Because natural contravariant metrics are not everywhere invertible, the associated covariant metrics are not everywhere defined.

## I. INTRODUCTION

In the canonical approach to quantum gravity, the configuration space is taken to be the "superspace" of three-geometries on a fixed manifold $M$. The action looks formally like that of a relativistic particle moving in this infinite-dimensional superspace with a potential proportional to the Ricci scalar.[1] For a relativistic particle, a consistent one-particle quantization is possible only if the spacetime metric has a conformal Killing vector whose action on the potential is also a conformal rescaling, but, as Kuchař[2] has shown, there is no such symmetry vector on the superspace of three-geometries on a compact spatial manifold. On the superspace of asymptotically flat geometries, however, spatial diffeomorphisms that are nontrivial at spatial infinity are generated by symmetry vectors that leave the dynamics of the theory invariant.

In the case of a relativistic particle with a timelike Killing vector $\xi^a$, one identifies time with the parameter $\lambda$ along each integral curve $\lambda \rightarrow c_\lambda(p)$ of $\xi^a$ through a point $p$ of an initial hypersurface. Here one would like to identify as an internal time the value of the parameter along the analogous paths in superspace. For the symmetry of superspace that corresponds to a spatial translation of each three-geometry, the corresponding internal time is an asymptotically defined center-of-mass coordinate: An external observer can, in effect, use as a clock the position of the observed system. For a superspace symmetry that corresponds to spatial rotations, one expects the corresponding time to be an angular variable, like the orientation of a clock's hand, but it is not clear whether natural conditions for asymptotic flatness near spatial infinity allow a well-defined angular orientation of an isolated system.

## II. SYMMETRY VECTORS ON SUPERSPACE

### A. Symmetries at spatial infinity

Initial data for the gravitational field on a three-manifold $M$ can be regarded as a pair $(g_{ab}, \pi^{ab})$, where $g_{ab}$ is a positive-definite metric on $M$ and $\pi^{ab}$ a symmetric tensor density, satisfying the momentum and Hamiltonian constraints,

$$\mathcal{P}_a := -2\nabla_b \pi_a{}^b \tag{2.1}$$

and

$$\mathcal{H} := G_{abcd}\pi^{ab}\pi^{cd} - g^{1/2}R = 0 , \tag{2.2}$$

where

$$G_{abcd} = \tfrac{1}{2} g^{-1/2}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}) .$$

Asymptotic flatness at spatial infinity has been recently discussed in terms of initial data $(g_{ab}, \pi^{ab})$ by Ashtekar and Magnon[3] and by Beig and Ó Murchadha.[4] To pick out the Lie algebra of the Poincaré group as the symmetry algebra at spatial infinity, it suffices to require the existence of an asymptotically flat metric $\delta_{ab}$ defined outside of a compact region of $M$, for which the pair $(g_{ab}, \pi^{ab})$ satisfies the Regge-Teitelboim[5] conditions

$$\text{(i)}\quad g_{ab} = \delta_{ab} + r^{-1}h_{ab} + r^{-2}j_{ab} + o_{ab} ,$$

where $r$ is a natural radial coordinate for the flat metric $\delta$, $h_{ab}$ is a function only of the corresponding angular coordinates (a smooth tensor on the unit two-sphere) that is even under parity,

$$h_{ab} = Ph_{ab} ,$$

$j_{ab}$ is also a tensor on the two-sphere, and where $o_{ab}$

 

satisfies

$$o_{ab}=o(r^{-1}), \quad \partial_c o_{ab}=o(r^{-2}) ,$$

$$\text{(ii)} \quad \pi^{ab}=r^{-2}\rho^{ab}+\mathcal{O}^{ab} .$$

Here $\rho^{ab}$ is a smooth tensor density on the two-sphere that is odd under parity,

$$\rho^{ab}=-P\rho^{ab}$$

and

$$\mathcal{O}^{ab}=o(r^{-2}), \quad \partial_c\mathcal{O}^{ab}=o(r^{-3}) .$$

The derivative operator $\partial_c$ is the covariant derivative with respect to the metric $\delta_{ab}$.

We shall suppose that the operators $\mathbf{g}_{ab}$ and $\boldsymbol{\pi}^{ab}$ in the canonical approach to quantum gravity satisfy the analogous conditions: that in the Schrödinger framework, with a state vector regarded as a functional of three-metrics, $\Psi=\Psi(g)$, $\Psi$ has support on three-metrics satisfying (i), while $\boldsymbol{\pi}^{ab}(x)\Psi|_{g_{ab}}$ satisfies (ii). The operator form

$$\nabla_b \boldsymbol{\pi}^b{}_a(x)\Psi=0 , \tag{2.1'}$$

of the momentum constraint (2.1), implies that state vectors $\Psi$ are invariant under diffeomorphisms that are trivial at infinity and which are in the component of the identity. The class of diffeomorphisms that preserves condition (i) is broader, including, in addition, diffeomorphisms that are trivial at infinity but not in the component of the identity, and diffeomorphisms which at infinity are elements of the three-dimensional Euclidean group, generated by translations and rotations of the flat metric $\delta$. These latter symmetry transformations are generated by vector fields having the asymptotic form

$$\xi^a=\zeta^a+o(r^0) , \tag{2.3}$$

where $\zeta^a$ is a Killing vector of the flat metric $\delta$ and $\partial_b\xi_a=O(r^{-2})$. For translations, $\zeta^a$ is a constant vector field with respect to the connection $\partial$ of $\delta$, and for rotations $\zeta^a$ has the form $\epsilon^a{}_{bc}n^b x^c$, with $n^a$ a unit constant vector field of $\delta$ and $x^a=r\delta^{ac}\partial_c r$ a radial vector field.

Corresponding momentum and spatial angular momentum operators are defined on functionals $\Psi(g)$ satisfying the constraint (2.1′) by

$$\mathbf{P}_\xi=2\int_{\sigma_\infty} dS_a \xi^b \boldsymbol{\pi}^a{}_b=\int d\mathbf{V}\,\mathcal{L}_\xi \mathbf{g}_{ab}\boldsymbol{\pi}^{ab} . \tag{2.4}$$

Then we have

$$\mathbf{P}_\xi\Psi(g)=\frac{d}{d\lambda}\Psi(\chi_\lambda g) , \tag{2.5}$$

where $\chi_\lambda$ is the family of diffeomorphisms generated by $\xi$.

Let $\mathcal{M}$ be the space of asymptotically flat metrics on $M$, and let $D_0$ denote the component of the identity of the set of diffeomorphisms, $\chi$, that are trivial at infinity: the geodesic distance from $\chi(x)$ to $x$ is $o(r(x)^0)$. (Equivalently, in a Cartesian chart $\{x^i\}$ for the flat metric $\delta$, $\lim_{r\to\infty}[\chi^i(x)-x^i]=0$, for $\chi\in D_0$.) The momentum constraint may then be regarded, in the Schrödinger picture, as a requirement that state vectors be functionals on the superspace

$$\mathcal{S}=\mathcal{M}/D_0 \tag{2.6}$$

of asymptotically flat three-geometries.

In addition to the spatial translations and rotations whose generators are given by Eq. (2.4), we shall need to consider time translations and boosts. In the 3+1 formalism, these are generated by operators of the form[4]

$$H_N=\int dV\,N(x)\mathcal{H} +2\oint_\infty dS_a \mathbf{g}^{a[b}\mathbf{g}^{c]d}[N\partial_c\mathbf{g}_{bd}+\partial_c N(\mathbf{g}_{bd}-\delta_{bd})] . \tag{2.7}$$

Here $\mathcal{H}$ is the Hamiltonian density of Eq. (2.2), while $N(x)$ has the form

$$N(x)=1+o(r^0) \tag{2.8}$$

for time translations and

$$N(x)=n_a x^a+o(r^0) \tag{2.8'}$$

for boosts, where, as above, $n^a$ and $x^a$ are, respectively, constant and radial vector fields of $\delta_{ab}$. When the Hamiltonian constraint

$$\mathcal{H}(x)\Psi=0$$

is satisfied, only the surface term survives.

### B. Symmetry vectors on superspace

Let $D$ be the set of diffeomorphisms of the form $\exp(\lambda\xi)$, where $\xi$ has the form (2.3) of an asymptotic symmetry, corresponding to a generator of the Euclidean group at spatial infinity. The group $\hat{E}=D/D_0$ is then isomorphic either to the Euclidean group or to its covering group, depending on whether or not diffeomorphisms corresponding to a $2\pi$ rotation at infinity are deformable to the identity on $M$ (Ref. 6). Given a family of diffeomorphisms $\chi_\lambda\in D$, one has a family of symmetries $\hat{\chi}_\lambda:=[\chi_\lambda]\in\hat{E}$, and through each $[g]\in\mathcal{S}$ a path of three-geometries $[g_\lambda]=\hat{\chi}_\lambda[g]=[\chi_\lambda g]$. A transformation $\chi\in D$ is what Kuchař calls a "conditional symmetry" on the space of three-metrics: it is generated by a dynamical variable $P_\xi$ that is linear and homogeneous in the momentum $\pi^{ab}$ and has weakly vanishing Poisson brackets with Hamiltonian and momentum constraints. That is, if we rewrite the constraints (2.1), (2.2) in the equivalent smeared form $P_\eta=0=H_N$ for all $\eta^a$ and $N$ of compact support, the Poisson brackets have the form

$$\{P_\xi,P_\eta\}=P_{\mathcal{L}_\xi\eta}=0 , \tag{2.9}$$

$$\{P_\xi,H_N\}=H_{\mathcal{L}_\xi N}=0 . \tag{2.10}$$

The fact that the symmetry algebra at spatial infinity is the Poincaré algebra also implies the vanishing of the Poisson brackets $\{P_\xi,H_N\}$ for $N=1+o(r^0)$, $\partial_a N=o(r^{-1})$; that is, time translation commutes with spatial rotations and translations. Thus the transformations $\hat{\chi}$ are symmetries of superspace, and the vector fields tangent to the paths $\hat{\chi}_\lambda$ are the corresponding symmetry vectors. It is precisely these symmetries that fail to exist for the superspace of metrics on a compact manifold.

## III. INTERNAL TIMES

### A. An asymptotically defined center of mass

Historically, one measured time by observing the orientation and position of Earth relative to its environment—to the Sun and more distant stars. Internal times associated with the symmetry vectors on superspace have a similar character: they measure the position and orientation of an isolated system relative to the surrounding Universe. The system's position can be represented by a center-of-mass vector, which is well defined when the values of the Poincaré generators are well defined. An assignment of angular variables is not yet clear: we do not know whether one is entitled to expect a well-defined orientation of an asymptotically flat system that could be expressed in terms of the asymptotic metric.

Recall that in flat space one can write as follows the center of mass of a system with conserved momentum and angular momentum: Let $P^\alpha$ be the total momentum, and $J_{\alpha\beta}$ the four-dimensional angular momentum tensor—the values of the generators of rotations and boosts about some origin 0; and let $S$ be a spacelike hypersurface through 0 with unit future-pointing normal $t^\alpha$. Then the connecting vector in $S$ from 0 to the center of mass is given by

$$X_\alpha = M^{-1}(\delta_\alpha^\beta - u_\alpha t^\beta / u\cdot t) J_{\beta\gamma} u^\gamma \ , \tag{3.1}$$

where $P_\alpha P^\alpha = -M^2$ and $u^\alpha = P^\alpha / M$. Equation (3.1) follows from the fact that the angular momentum tensor about the center of mass satisfies the relation

$$J^{\text{c.m.}}_{\alpha\beta} u^\beta = 0 \ , \tag{3.2}$$

together with the transformation law

$$J^{\text{c.m.}}_{\alpha\beta} = J_{\alpha\beta} - (R_\alpha P_\beta - R_\beta P_\alpha) \tag{3.3}$$

for angular momentum under a change of origin by a displacement $R_\alpha$.

For an asymptotically flat spacetime, the quantities appearing on the right-hand side (RHS) of Eq. (3.1) are all defined in terms of the values of the Poincaré generators (and the observer's four-velocity $t^\alpha$), and we may therefore adopt Eq. (3.1) as the definition of a center-of-mass vector. Because $X^\alpha$ lies in $S$, we may regard it as a three-vector $X^a$; let $P_a$ similarly denote the spatial momentum, the projection of $P_\alpha$ on $S$. The Poisson brackets of the Poincaré generators then imply that $X^a$ and $P_a$ are conjugate variables

$$\{X^a, P_b\} = \delta^a_b \ , \tag{3.4}$$

and, of course, that

$$\dot{X}^a := \{X^a, H\} = P^a / H \ , \tag{3.5a}$$

$$\dot{P}^a := \{P^a, H\} = 0 \ , \tag{3.5b}$$

where $H = P_\alpha t^\alpha$.

Thus the variable $P^a$, regarded as a vector field on the superspace of three-geometries, generates time translations of superspace, if one adopts the conjugate variable $X^a$ as the internal time. On spacetimes with vanishing mass, $X^a$ is not well defined; and on spacetimes with $P^a=0$, the definition is not useful—the clock has stopped. As long as the spacetime has a nonzero mass, however, there is no loss of generality in describing an isolated system from the standpoint of an observer who sees a nonzero three-momentum $P^a$ and who can therefore use the position of the system itself as a natural time.

In the case of asymptotic rotations, one would similarly like to choose as internal times angular coordinates that describe the orientation of an isolated system relative to its surroundings. We believe that it is possible to define such coordinates for a restricted class of asymptotically flat spacetimes that have to $O(r^{-2})$ the form of a boosted Schwarzschild solution. But the class of all asymptotically flat spacetimes for which the asymptotic symmetry group can be uniquely restricted to the Poincaré group may well be too large to allow one uniquely to define a system's orientation.

### B. Comments on an inner product

When one quantizes a scalar field on Minkowski space or on a stationary spacetime, one uses both the conserved symplectic product

$$(\phi,\psi) = \frac{1}{i}\int (\phi^* \nabla^\alpha \psi - \psi \nabla^\alpha \phi^*) \nabla_\alpha t (-g)^{1/2} d^3x \tag{3.6}$$

and a time-translation symmetry vector $t^\alpha$ to define a positive-definite inner product, the Klein-Gordon product, as the restriction of ( , ) to the subspace of solutions with a positive frequency relative to the symmetry vector $t^\alpha$. Equivalently, one can define the inner product on the space of real solutions to the scalar wave equation by writing

$$\langle \phi, \psi \rangle := -i(\phi, J\psi) \ , \tag{3.7}$$

where $J$ is the complex structure given by

$$J\phi = i(\phi^+ - \phi^-) \ , \tag{3.8}$$

with $\phi^+$ and $\phi^-$ the positive- and negative-frequency parts of $\phi$, respectively.

In the canonical approach to quantum gravity, one can regard the Wheeler-DeWitt equation, smeared with a lapse function $N$ as a Klein-Gordon equation with the potential for a particle moving on an infinite-dimensional, curved space (the space of three-metrics), with a metric $G_N$ given in its contravariant form by

$$G_N(\pi,\pi) = \int d^3x \, N G_{abcd} \pi^{ab} \pi^{cd} \ . \tag{3.9}$$

The smeared Wheeler-DeWitt equation has the form

$$[G_N(\pi,\pi) + \mathbf{V}_N]\Psi = 0 \ , \tag{3.10}$$

where

$$\mathbf{V}_N = -\int d\mathbf{V} \, N\mathbf{R} \ ; \tag{3.11}$$

if $\nabla N$ is nowhere vanishing, Eq. (3.10) together with the diffeomorphism invariance of $\Psi$, implies that the Wheeler-DeWitt equation is satisfied everywhere (i.e., smeared with arbitrary lapse), assuming closure of the quantum constraints. If one could make sense of the Wheeler-DeWitt equation with a factor ordering for which $G_N(\pi,\pi)$ was the covariant Laplacian of the metric $G_N$, then the symplectic product

$$(\Phi,\Psi)=\frac{1}{i}\int_\Sigma dg\int_M dx\left[\Phi^*\frac{\delta}{\delta g_{cd}(x)}\Psi-\Psi\frac{\delta}{\delta g_{cd}(x)}\Phi^*\right]NG_{abcd}t^{cd}(x) \tag{3.12}$$

would be conserved. Here $\Sigma$ is a hypersurface of codimension one, $t^{cd}$ is its normal, $dg$ the measure on $\Sigma$, and $\Phi$ and $\Psi$ square-integrable solutions to the Wheeler-DeWitt equation, with respect to the measure $dg$.

To turn the symplectic product on the space of metrics $\mathcal{M}$ into a positive-definite inner product on superspace $\mathcal{S}=\mathcal{M}/D_0$, is now straightforward. One must first make the metric $G_N$ on $\mathcal{M}$ into a metric on $\mathcal{S}$; and then one must use a symmetry vector on $\mathcal{S}$ to restrict the symplectic product to a space of positive-frequency solutions (i.e., to define a complex structure). In the form (3.9), the metric is not well defined on superspace, because it is not diffeomorphism invariant, at least not unless the lapse $N$ is constant. One can avoid the difficulty simply by choosing a gauge for the three-metrics—a cross section of the bundle $\mathcal{M}$ over $\mathcal{S}$ (unless the topology is Euclidean, one can only choose a local gauge). Alternatively, the metric $G_N$ can be written in a diffeomorphism-invariant form as follows. Let $\chi$ be a diffeomorphism and denote by $\chi T^{a\cdots b}_{c\cdots d}$ the result of dragging a tensor field by $\chi$. If $\chi^{*a}{}_b$ is the differential map (the Jacobian of $\chi$),

$$\chi T^{a\cdots b}_{c\cdots d}[\chi(x)] =\chi^{*a}{}_m\cdots\chi^{*b}{}_n(\chi^{*-1})^r_c\cdots(\chi^{*-1})^s_d T^{m\cdots n}_{r\cdots s}(p)\ . \tag{3.13}$$

A contravariant metric is a bilinear map from covectors to $\mathbb{R}$; a covector at a point $[g]$ of $\mathcal{S}$ can be written as the gradient of a function $\Phi[g]$ on $\mathcal{S}$. One may regard $\Phi[g]$ as a function $\Phi(g)$ on $\mathcal{M}$ for which $\Phi(\chi g)=\Phi(g)$, all $\chi\in D_0$, and a covector $(\delta/\delta[g])\Phi[g]$ on $\mathcal{S}$ may then be regarded as a covector $p^{ab}=[\delta/\delta g_{ab}(x)]\Phi(g)$ on $\mathcal{M}$ for which $p^{ab}(\chi g)=\chi p^{ab}(g)$. Then, if the lapse $N$ is constant, or if it is a function on $\mathcal{M}$ satisfying

$$N(\chi g)[\chi(x)]=N(g)(x)\ , \tag{3.14}$$

the smeared Wheeler-DeWitt equation (3.10) will be diffeomorphism invariant.[7] [This makes the LHS of Eq. (2.10) vanish strongly. Then $P_\xi$ will be a symmetry (not a conditional symmetry) of the Wheeler-DeWitt equation.]

Finally, we observe that there are unique symmetry vectors on superspace corresponding to asymptotic translations and rotations. On $\mathcal{M}$, the diffeomorphism corresponding to an asymptotic symmetry is ambiguous by precisely the freedom—composition by a diffeomorphism in $D_0$—that is removed in passing to $\mathcal{S}$. Thus to each generator $\xi^a$ of the asymptotic Euclidean group corresponds to a unique vector field $\tau[g]$ on $\mathcal{S}$. Again we may identify $\tau$ with a vector field $\tau_{ab}(g)=\mathcal{L}_\xi g_{ab}$ on $\mathcal{M}$ by choosing a field $\xi^a(g)(x)$ for which $\chi\xi^a(g)=\xi^a(\chi g)$. We may decompose the space of solutions to the Wheeler-DeWitt equation into positive- and negative-frequency parts with respect to $\tau$ and thereby obtain a positive-definite inner product (3.12), if one can choose a surface $\Sigma$ of codimension one that is orthogonal to $\tau$. As we shall see, this is in fact possible, because one can use the freedom in choosing the vector field $\tau_{ab}$ on $\mathcal{M}$ to make $\tau_{ab}$ hypersurface orthogonal. One can explicitly define a hypersurface-orthogonal vector field $\tau_{ab}$ as the gradient of a scalar on $\mathcal{M}$. Let

$$\Phi_R=\int dx\, g^{1/2}(g^{ab}+\delta^{ab})\mathcal{L}_{\xi_R}\delta_{ab}\ ,$$

where $\xi^a_R$ is a vector field that has asymptotically the form of a translation: for example,

$$\xi^a_R=z^a\theta(r-R)\ ,$$

with $z^a$ a constant unit vector with respect to the metric $\delta_{ab}$ and $\theta$ the step function. Then, writing

$$\tau^{ab}_R(x)=\frac{\delta}{\delta g_{ab}(x)}\Phi_R\ ,$$

we have

$$\tau^{ab}_R(x)=G^{abcd}(x)\mathcal{L}_{\xi_R}g_{cd}+O(R^{-1})\ .$$

A vector may be regarded as a linear map from a space $\{\Psi\}$ of scalars to itself; here we may take $\{\Psi\}$ to be a space of scalars on $\mathcal{M}$ satisfying asymptotic condition (ii). The desired hypersurface-orthogonal vector field $\tau^{ab}$ on $\mathcal{M}$ may now be defined by

$$\tau^{ab}(x)=\lim_{R\to\infty}\tau_R{}^{ab}=\frac{\delta}{\delta g_{ab}(x)}\Phi_\infty\ ,$$

where

$$\Phi_\infty=\lim_{R\to\infty}\Phi_R=\int d\Omega\left[j\frac{z}{r}-j^{ab}r_a z_b\right]\ ,$$

where $j^{ab}$ is the coefficient of the $r^{-2}$ part of the metric $g_{ab}$ [as given by asymptotic condition (i) that defines $\mathcal{M}$]. As required, $\tau^{ab}$ is a translational symmetry vector on $\mathcal{M}$. For metrics with the asymptotic form of a Schwarzschild metric of mass $M$ translated by a distance $a$, we have $\Phi_\infty=(32\pi/3)Ma$. Note that, at least when the superspace metric is invertible, one can show that the translational symmetry vector on $\mathcal{S}$ is also hypersurface orthogonal.

In the case of a free particle, the symmetry vector on which the decomposition into positive- and negative-frequency solutions is based is timelike. While the existence of a positive-definite inner product is unrelated to this fact, one does need a timelike vector to make the free particle's kinetic energy positive. It is not clear that one needs a positive-definite kinetic part of the corresponding super Hamiltonian in quantum gravity; at least in a theory of pure quantum gravity, there is no mechanism for a transition to negative eigenstates of the super Hamiltonian. But it is natural to ask whether the symmetry vectors on the space $\mathcal{M}$ of metrics or on the superspace $\mathcal{S}$ are analogously timelike with respect to a "supermetric" $G_N$ on $\mathcal{M}$ or its projection to $\mathcal{S}$. The question has several subtleties, which we mention briefly and discuss in more detail in the Appendix. First, there is a different supermetric $G_N$ for each choice of lapse $N[g](x)$, and the norm of the symmetry vector depends on $N$. Second, although two vector fields on the three-space $M$ that agree asymptotically are equivalent in their action on superspace, they in general have different norms with respect to a given supermetric $G_N$. It turns out that for each metric $g_{ab}$, one can always extend an asymptotic translation to a vector field $\xi^a(x)$ on the interior of the three-space $M$ in such a way that the corresponding vector field on $\mathcal{M}, h_{ab}(g)=\mathcal{L}_\xi g_{ab}$, has a negative norm:

$$\int dx\, N^{-1} G^{abcd} h_{ab} h_{cd} < 0 \ ,$$

where

$$G^{abcd} = g^{1/2}(g^{a(c}g^{d)b} - g^{ab}g^{cd}) \ . \tag{3.15}$$

A different choice of $\xi$, however, can change the sign of the norm. Finally, if one tries to overcome the ambiguity in the choice of $\xi$ by computing the norm of the vector field $\tau$ on *superspace* $\mathcal{S}$, rather than on the space of metrics $\mathcal{M}$, one finds that a covariant metric on superspace does not always exist; and when it does exist, one must solve an elliptic equation in order to obtain the norm of a vector on superspace. In particular, the covariant metric does not exist when the space is asymptotically Schwarzschild, with $N$ given by the Schwarzschild lapse. For more general spacetimes, one could numerically compute the norm of $\tau$ on superspace, but we have not yet done so.

The physical meaning of our division into positive and negative frequencies is apparent. By choosing as time an asymptotic center-of-mass coordinate $X$, we require the asymptotic observer to be in a frame for which the system moves in the positive $X$ direction.[8] That is, with $\tau$ chosen to be the generator of spatial translations along, say, the $x$ axis of an asymptotic observer, the positive-frequency solutions are built from eigenfunctions of $P_x$ with $P_x > 0$.

From an asymptotically flat superspace, the internal times inherit their formal virtues: They correspond to dynamical symmetries and obey the clocklike dynamics of Eqs. (3.5a) and (3.5b). If one were to represent an isolated system more realistically as a subgeometry of a larger spacetime, one's internal times would only approximate these features. Even in the asymptotically flat context, however, the practical virtures of the internal times depend on the extent to which they approximate classical clocks. Since $X$ and $P_x$ are conjugate variables, if the state vector $\Psi$ is to have a nontrivial evolution with respect to the time $X$ in a representation in which $X$ is diagonal, it cannot be an eigenfunction of $P_x$. Thus for a system of atomic scale, the time $X$ is blurred. If, however, the system is massive, the state vector $\Psi$ can be peaked sharply about some value of $P$; that is, $\Delta P/\langle P\rangle$ can be small, and, at the same time have a nontrivial $X$ dependence, with $\Psi(X+\Delta X)$ substantially different from $\Psi(X)$ when $\Delta X > (\Delta P)^{-1}$. Only in this circumstance will the system have the character of a clock; and only for such a massive system is it a sensible idealization to restrict the Hilbert space to eigenvalues $P_x > 0$.

### C. Internal and external time

While we have been concerned with identifying internal times to label asymptotically flat spacetimes in quantum gravity, it is, of course, possible to define an external time $t$ by writing

$$i\partial_t \Psi(g,t) = \mathbf{H}_N \Psi(g,t) \ , \tag{3.15$'$}$$

where the lapse $N$ approaches 1 at spatial infinity. [Equation (3.15) appears in Dirac's first paper on quantum gravity,[9] and a variant of this approach is implicit in the path-integral framework discussed by Teitelboim.[10]] Because its "conjugate momentum" $(1/i)\partial_t$ appears linearly in Eq. (3.15), the external time $t$ is formally inequivalent to the internal times that we consider, and whose conjugate momenta appear quadratically in the Wheeler-DeWitt equation. The external time is formally similar to the past volume that has been advocated for cosmological time by Sorkin[11] and by Unruh and Wald.[12] The internal times, on the other hand, are related to clocks constructed from the three-metric of a compact universe.[13] That is, if one regards an asymptotically flat space as an idealization of an isolated system embedded in a larger surrounding universe, then the internal times are approximately present in the quantum theory of the larger spacetime. Each internal time of the asymptotically flat space is a limit of operators corresponding to the position or orientation of the (approximately defined) isolated system.

For an asymptotically flat space in an eigenstate of mometum, since the variable $X$ cannot serve as a clock, one can use the external time $t$. Similarly for an asymptotically flat space in an eigenstate of mass $M$, the external time cannot serve as a clock, but one can use an internal time. This case is analogous to a closed universe, which one can regard as being in an eigenstate of zero mass.[14] For a massive, asymptotically flat space, with $M$, $P_x$, and $X$ peaked about classical values, the external time is redundant. At each time $t_0$, the functional $\Psi([g],t_0)$ already contains the time evolution of $\Psi$. That is, the value of $\Psi([g],t)$ for metrics with $X \approx X_0$ can be expressed in terms of $\Psi([g],t_0)$ for metrics with $X \approx X - \langle \mathbf{P}_x/\mathbf{M}\rangle(t-t_0)$:

$$\Psi([g,X=X_0],t) \approx \Psi([g,X=X_0-\langle \mathbf{P}_x/\mathbf{M}\rangle(t-t_0)],t_0) \ .$$

To summarize, in a physical situation in which $\langle \mathbf{P}^a\rangle$ (or $\langle \mathbf{J}^a\rangle$) are nonzero, one already has an internal time, and introducing an additional, external time seems unnatural.

### ACKNOWLEDGMENTS

We are indebted for helpful conversations and correspondence to A. Ashtekar, K. Kuchař, N. Ó Murchadha, R. Sorkin and R. Wald. This work was supported in part by NSF Grant No. PHY8603173.

## APPENDIX; THE CONTRAVARIANT AND COVARIANT METRICS OF SUPERSPACE

Let $\pi:\mathcal{M}\rightarrow\mathcal{S}$ be the projection mapping a metric $g_{ab}$ on a three-manifold $M$ to the equivalence class $[g]$ of metrics, where two metrics are equivalent if related by an asymptotically trivial diffeomorphism in the component of the identity. We shall adopt the abstract index notation, using indices $A,B,\ldots$ for tensors on the (infinite-dimensional) manifold $\mathcal{M}$, and indices $\alpha,\beta,\ldots$ for tensors on $\mathcal{S}$. One makes $\mathcal{S}$ into a manifold as well by taking $\mathcal{M}$ to be the space of generic metrics in the sense of metrics without Killing vectors;[15] we shall assume that this has been done and that the tensor bundles are Hilbert bundles (e.g., by requiring that metrics on the three-manifold $M$ belong to a weighted Sobolev space[16]). A contravariant vector $h^A(g)$ at $g_{ab}\in\mathcal{M}$ may be defined as the tangent to a path $\lambda\rightarrow g_{ab}(\lambda,x)$ of metrics on $M$. That is, $h^A(g)$ is a symmetric tensor field on the three-manifold $M$:

$$h^A=h_{ab}(x)=\left.\frac{d}{d\lambda}g_{ab}(\lambda,x)\right|_{\lambda=0} . \tag{A1}$$

A covariant vector $p_A$ at $g_{ab}\in\mathcal{M}$ may similarly be identified with a tensor density $p^{ab}(x)$. If $\Psi$ is a function on $\mathcal{M}$ (i.e., a functional of three-metrics), its gradient (functional derivative) is a tensor density on $\mathcal{M}$, having at a point $g_{ab}$ the value

$$p_A=\nabla_A\Psi|_g=\frac{\delta\Psi}{\delta g_{ab}(x)}=p^{ab}(x) . \tag{A2}$$

Let $\pi^\alpha_A:T\mathcal{M}\rightarrow T\mathcal{S}$ be the differential map (Jacobian map $\pi_*$), and denote by $G^{AB}$ the contravariant metric on $\mathcal{M}$ associated with some lapse function $N[g](x)$. Then, at a point $g_{ab}\in\mathcal{M}$, the covector $p_A$ has the norm

$$G^{AB}p_Ap_B=\int dx\, N(x)G_{abcd}(x)p^{ab}(x)p^{cd}(x) . \tag{A3}$$

The map $\pi$ projects (drags) the contravariant metric $G^{AB}$ on $\mathcal{M}$ to a contravariant metric

$$\hat{G}^{\alpha\beta}=\pi^\alpha_A\pi^\beta_B G^{AB} \tag{A4}$$

on $\mathcal{S}$. If $\psi$ is a function on $\mathcal{S}$, the covector $\nabla_\alpha\psi$ on $\mathcal{S}$ has the norm

$$\begin{aligned}\hat{G}^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi&=G^{AB}\nabla_A\Psi\nabla_B\Psi\\&=\int dx\, NG_{abcd}\frac{\delta\Psi}{\delta g_{ab}}\frac{\delta\Psi}{\delta g_{cd}} ,\end{aligned} \tag{A5}$$

where $\Psi=\psi\cdot\pi$ is a gauge-invariant function of three-metrics.

While the superspace $\mathcal{S}$ thereby inherits from $\mathcal{M}$ a contravariant metric, the metric can be degenerate, because the metric on $\mathcal{M}$ has an indefinite signature. In order to find the norm of a vector field $\tau^\alpha$ on $\mathcal{S}$, one needs a covariant metric $\hat{G}_{\alpha\beta}$, the inverse of $\hat{G}^{\alpha\beta}$; and $\hat{G}_{\alpha\beta}$ exists if and only if $\hat{G}^{\alpha\beta}$ is nondegenerate. We shall see that there are at least some points of $\mathcal{S}$ where $\hat{G}^{\alpha\beta}$ is nondegenerate, but it is not yet clear whether it is nondegenerate at generic points. There is a straightforward criterion for nondegeneracy, which can be expressed geometrically as the existence of a horizontal projection on $\mathcal{M}$ and algebraically as the existence of solutions to an elliptic equation:

*Definition.* A vector $v^A$ on $\mathcal{M}$ is *vertical* iff it has the form $v^A=\nabla_{(a}\zeta_{b)}$, where $\zeta_a$ vanishes at infinity; that is, a vertical vector is the tangent to a path of gauge-related metrics. A vector $h^A$ is *horizontal* iff $G_{AB}h^Av^B=0$ for all vertical vectors $v^A$; equivalently, $h^A$ is horizontal iff $h_A:=G_{AB}h^B=\nabla_A\Psi$, for some gauge-invariant function $\Psi$.

Then $h^A=h_{ab}(x)$ is horizontal:

$$\begin{aligned}&\Longleftrightarrow\int dx\, N^{-1}G^{abcd}h_{ab}\nabla_c\zeta_d=0,\quad\forall\zeta_d ,\\&\Longleftrightarrow\nabla_b(N^{-1}G^{abcd}h_{cd})=0 ,\end{aligned} \tag{A6}$$

where $G^{abcd}$ is given by Eq. (3.15).

*Proposition* 1. If $\hat{G}^{\alpha\beta}$ is nondengenerate, one can define a horizontal projection on $\mathcal{M}$ in terms of the covariant metric $\hat{G}_{\alpha\beta}$.

*Proof.* Suppose $\hat{G}^{\alpha\beta}$ is nondegenerate and let $u^A$ be a vector at a point $g_{ab}\in\mathcal{M}$. Define a projection operator $H^A_B$ to the horizontal subspace of the tangent space at $g$ by writing

$$H^A_C=G^{AB}\pi^\beta_B\hat{G}_{\beta\gamma}\pi^\gamma_C . \tag{A7}$$

Then $\tilde{u}^A:=H^A_Cu^C$ is horizontal because, for vertical vectors, $v^A,v^A\pi^\alpha_A=0\Longrightarrow v_A\tilde{u}^A=0$. The vectors $\tilde{u}^A$ and $u^A$ differ by a vertical vector because $\pi^\alpha_A\tilde{u}^A=\pi^\alpha_Au^A$. Finally, $H^A_B$ is a projection operator:

$$\begin{aligned}H^A_CH^C_B&=(G^{AD}\pi^\delta_D\hat{G}_{\delta\gamma}\pi^\gamma_C)(G^{CE}\pi^\epsilon_E\hat{G}_{\epsilon\beta}\pi^\beta_B)\\&=G^{AD}\pi^\delta_D\hat{G}_{\delta\gamma}\hat{G}^{\gamma\epsilon}\hat{G}_{\epsilon\beta}\pi^\beta_B\\&=G^{AD}\pi^\delta_D\hat{G}_{\delta\beta}\pi^\beta_B=H^A_B \qquad\Box .\end{aligned}$$

*Corollary.* When $\hat{G}_{\alpha\beta}$ exists, its pullback to $\mathcal{M}$ is the horizontal projection of $G_{AB}$, $\pi^\alpha_A\hat{G}_{\alpha\beta}\pi^\beta_B=H_{AB}:=H^C_AG_{CD}H^D_B$. That is, the norm of a vector on $\mathcal{S}$ is the norm of its horizontal lift to $\mathcal{M}$.

*Proof.* It suffices to check that

$$h^AH_{AB}h^B=h^A\pi^\alpha_A\hat{G}_{\alpha\beta}\pi^\beta_Bh^B \tag{A8}$$

for all horizontal $h^A$, because $\pi^\alpha_A\hat{G}_{\alpha\beta}\pi^\beta_Bv^B=H_{AB}v^B=0$ for $v^A$ vertical. Since horizontal vectors have the form $h^A=\pi^{A\alpha}\nabla_\alpha\phi$, where $\pi^{A\alpha}=G^{AB}\pi^\alpha_B$ and $\phi$ is a function on $\mathcal{S}$, Eq. (A8) is equivalent to

$$\pi^{A\alpha}H_{AB}\pi^{B\beta}=\pi^{A\alpha}\pi^\gamma_A\hat{G}_{\gamma\delta}\pi^\delta_B\pi^{B\beta} ,$$

and this relation follows from the defining equation (A7) for $H^A_B$. $\Box$

Using the corollary above, it is not difficult to see that, for constant lapse, $\hat{G}_{\alpha\beta}$ is well defined at points $[g]$ for which the eigenvalues of the Ricci tensor, $R^a_b$, on a compact manifold $M$ are nonpositive. (The eigenvalues of $R^a_b$ depend only on the equivalence class $[g]$ of $g_{ab}$.) We need to show that no vertical vector $\nabla_{(a}\zeta_{b)}(x)$ at $g_{ab}\in\mathcal{M}$ is horizontal, that the equation

$$\nabla_b(G^{abcd}\nabla_c\xi_d)=0 \tag{A9}$$

has no solutions which vanish at infinity. After commuting covariant derivatives, Eq. (A9) can be written in the form

$$L^a{}_b\xi^b := \nabla_b(\nabla^b\xi^a-\nabla^a\xi^b)+2R^a{}_b\xi^b=0 \ . \tag{A10}$$

When the eigenvalues of $R^a{}_b$ are nonpositive, the operator $L^a{}_b$ is negative:

$$\int \xi_a L^a{}_b\xi^b g^{1/2}dx = 2\int(-\nabla_{[a}\xi_{b]}\nabla^{[a}\xi^{b]} + R^a{}_b\xi_a\xi^b)g^{1/2}dx < 0 \ , \tag{A11}$$

for $\xi^a \neq 0$. Hence ker $L=0$ [i.e., Eq. (A9) has no nonzero solution], and the corollary implies that $\hat{G}^{\alpha\beta}$ is invertible.

*Proposition 2.* $G^{\alpha\beta}$ *is degenerate at* $[g] \Longleftrightarrow$ *some* vector $v^A$ at $g$ is both vertical and horizontal.

*Proof.* Suppose $G^{\alpha\beta}$ is degenerate at $[g]$. Then $G^{\alpha\beta}\nabla_\beta\phi|_{[g]}=0$ for some function $\phi$ defined in a neighborhood of $[g]$. Let $v^A=G^{AB}\nabla_B(\phi\circ\pi)$. Then $v^A$ is horizontal by construction. On the other hand, $v^A$ will be vertical if $v^A\nabla_A\Psi=0$ for all $\Psi$ of the form $\Psi=\psi\circ\pi$, with $\psi$ a function on $\mathcal{S}$. But

$$v^A\nabla_A\Psi = v_A G^{AB}\nabla_B\Psi = \nabla_\alpha\phi\pi^\alpha_A G^{AB}\pi^\beta_B\nabla_\beta\psi = \nabla_\alpha\phi G^{\alpha\beta}\nabla_\beta\psi = 0 \ .$$

To prove the converse, suppose that some vector $v^A$ is vertical and horizontal. Since $v^A$ is horizontal, $v_A=\nabla_A(\phi\circ\pi)=\pi^\alpha_A\nabla_\alpha\phi$, some function $\phi$ on $\mathcal{S}$. Then, for any $\psi$ on $\mathcal{S}$, $\nabla_\alpha\phi G^{\alpha\beta}\nabla_\beta\psi=\nabla_\alpha\phi\pi^\alpha_A G^{AB}\pi^\beta_B\nabla_\beta\psi = v^A\nabla_A(\psi\circ\pi)=0$, where the last equality holds because $v^A$ is vertical. Thus $G^{\alpha\beta}\nabla_\beta\phi=0$ and $G^{\alpha\beta}$ is degenerate. □

In particular, when ker $L=0$, one can find as follows the norm $\hat{G}_{\alpha\beta}\gamma^\alpha\gamma^\beta$ of a vector $\gamma^\alpha$ on $\mathcal{S}$. A vector at $[g]\in\mathcal{S}$ is the tangent, $(d/d\lambda)[g_{ab}(\lambda)]|_{\lambda=0}$ to a path $[g_{ab}(\lambda)]$ in $\mathcal{S}$. Write $\gamma_{ab}=(d/d\lambda)g_{ab}(\lambda)$. If $\chi_\lambda$ is a family of diffeomorphisms in $D_0$ generated by a vector field $\xi^a$, then

$$\frac{d}{d\lambda}[g_{ab}(\lambda)] = \frac{d}{d\lambda}[\chi_\lambda g_{ab}(\lambda)] = (\gamma_{ab}+\mathcal{L}_\xi g_{ab}) \ . \tag{A12}$$

A lift to $g_{ab}\in\mathcal{M}$ of a tangent vector $\gamma^\alpha\in\mathcal{S}$ is then any of the gauge-related tensor fields $\gamma_{ab}+\mathcal{L}_\xi g_{ab}$. If ker $L=0$, there is a unique horizontal lift, a unique solution $\xi^a$ to the equation,

$$\nabla_b[G^{abcd}(\gamma_{cd}+\mathcal{L}_\xi g_{cd})]=0 \ ,$$

or, equivalently,

$$L^a_b\xi^b = -\nabla_b(\gamma^{ab}-g^{ab}\gamma) \ . \tag{A13}$$

Then, writing $\tilde{\gamma}_{ab}=\gamma_{ab}+\mathcal{L}_\xi g_{ab}$ to denote the horizontal lift of $[\gamma_{ab}]$,

$$\hat{G}_{\alpha\beta}\gamma^\alpha\gamma^\beta = \int dx\, G^{abcd}\tilde{\gamma}_{ab}\tilde{\gamma}_{cd} \ . \tag{A14}$$

If the space has a finite flat region, any vector field of the form $\xi^a=\nabla^a f$, with $\nabla f$ vanishing at infinity, is in ker $L$, and $\hat{G}^{\alpha\beta}$ is therefore not invertible. Moreover, when the lapse and the Ricci tensor of $g_{ab}$ satisfy the relation

$$R_{ab}=N^{-1}\nabla_a\nabla_b N, \quad R=0 \ , \tag{A15}$$

on any finite region $U\subset M$, then $\hat{G}^{\alpha\beta}$ is again not invertible. In this case, any vector field of the form $\xi^a=N^2\nabla^a\phi$, with support on $U$ is in ker $L$, because $L^a{}_b\xi^b$ has the form

$$\nabla_b(N^{-1}G^{abcd}\nabla_c\xi_d) = N^{-2}\nabla_b\{N^3[\nabla^b(N^{-2}\xi^a)-\nabla^b(N^{-2}\xi^a)]\}=0 \ .$$

In particular, Eq. (A15) holds for the vacuum Schwarzschild geometry, and thus $\hat{G}^{\alpha\beta}$ is not invertible for a three-geometry and choice of lapse that agree, for $r$ greater than some radius $R$, with vacuum Schwarzschild. There is a wide class of geometries for which $\hat{G}^{\alpha\beta}$ is invertible, but it is not yet clear to us whether it is invertible at almost all points of $\mathcal{S}$.

Note that at those points $[g]\in\mathcal{S}$ where the covariant metric, $\hat{G}_N$, exists (where $\hat{G}^{\alpha\beta}_N$ is invertible), the vector field $\tau$ associated with an asymptotic translation has norm *independent of* **N**.

The propositions and discussion above are valid for the space of metrics on closed three-manifolds if one simply omits the reference to the behavior of quantities at spatial infinity. We have checked for $M=S^3$ and $N=1$, that $\hat{G}^{\alpha\beta}$ is invertible at the constant curvature metric on $M$.

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