Bunch-length effects in the beam-beam interaction

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The Hamiltonian analysis of the beam-beam interaction is extended, for round beams, by including its finite longitudinal extent. For small synchrotron amplitudes resonance strengths are derived that are smaller than those obtained in the impulse approximation. This is a consequence of averaging over the betatron phase during the collision. Results of simulations that reproduce this feature are also presented. More complete simulations, relevant to storage-ring colliders, argue for bunch lengths comparable to the value of the amplitude function (β) at the interaction point.

I. INTRODUCTION

The luminosity L and total current I of an e^+e^- storage-ring collider are related as

$$L=\frac{I\gamma\xi}{2er_e\beta_0}(1+R_\sigma)\ , \tag{1}$$

where β_0 is the vertical amplitude function, R_σ is the ratio of the minor and major axes of the collision spot, and ξ is the beam-beam strength parameter. (The strength parameter ξ has conventionally been used as a measure of the influence of the beam-beam interaction.) The beam current is utilized effectively by having a "low-β insertion" at the collision point. The bunch length σ_s gives one lower limit to β_0. When they become comparable, luminosity is lost due to an increase in the effective collision area (a geometric effect) and a reduction in ξ from modulation of the betatron phase advance between collisions.[1]

The beam-beam interaction has generally been treated as an impulsive "kick" in calculations and simulations studying the latter effect.[2] This does not reflect the fact that a particle experiences the force spread over a range of betatron phase, of order $2\pi\sigma_s/\beta_0$. Buon[3] does consider the extended nature of the force, but in the linear approximation. However, tune-shift-limiting resonance phenomena appear only if nonlinearities are considered.

In this paper we extend the usual analysis by treating the full, nonlinear, interaction as arising from a longitudinally distributed charge. We restrict ourselves to the weak-strong limit, and use the round-beam model, defined by the potential of Eq. (2) below and corresponding to $R_\sigma=1$ in Eq. (1) above. The calculation of Sec. II provides a framework for interpreting the simulation results of Sec. III. Results of more complete simulations, presented in Sec. IV, offer evidence of an optimal bunch length in the actual operation of a collider.

II. THEORY

This section presents a generalization of the work of Izrailev and Vasserman (Ref. 1). Resonance strengths are calculated in a one-dimensional model in which the strong beam is assumed round, with a Gaussian charge density in the transverse dimension x of rms size σ_x. The beam-beam potential function is then

$$V_R(x)=\int_0^1\frac{d\eta}{\eta}(1-e^{-(x^2/2\sigma_x^2)\eta})\ . \tag{2}$$

The beam size is related to the β function by $\sigma_x^2=\epsilon_0\beta_s$, where ϵ_0 is the equilibrium emittance and β_s the amplitude function of the *strong* beam. The longitudinal distribution is also Gaussian, with rms length σ_s. In the ultrarelativistic limit the electromagnetic field lines are Lorentz contracted into a thin disc, so the longitudinal distribution only serves to reduce the amount of charge producing that potential; the longitudinal forces are negligible.

If the unperturbed betatron motion of a single particle can be described by a Hamiltonian $H_0(x,p_x)$, and it encounters the beam-beam interaction once every turn, then the total Hamiltonian for the particle can be written as

$$H(x,p_x,t)=H_0(x,p_x)+\epsilon\sum_{n=-\infty}^{\infty}V_R(x)\frac{2c}{\sqrt{2\pi\sigma_s^2}}\times\exp\{-2c^2[t-(nT_0+\tau)]^2/\sigma_s^2\}\ . \tag{3}$$

The perturbation parameter is $\epsilon=Nr_e/\gamma$, where N is the number of particles in the strong beam, r_e the classical radius of the electron, c the speed of light, and γ the usual relativistic factor. T_0 is the revolution period. The sum is over all turns, and synchrotron oscillations are introduced parametrically via τ, given by

$$\tau=\frac{\hat{\tau}}{2}\cos(2\pi nQ_s)\ , \tag{4}$$

where $\hat{\tau}$ and Q_s are the synchrotron amplitude and tune, respectively. The synchrotron frequency ω_s is related to the tune by $\omega_s=2\pi Q_s/T_0$. Note that both $\hat{\tau}$ and σ_s have a factor of one-half associated with them that arises from the relative motion of the particle and the strong beam.

Transform the Hamiltonian to action-angle coordinates (I,ψ), defined through[4]

$$x=(2I\beta)^{1/2}\cos[\psi+\chi(t)]\ , \tag{5}$$

where

$$\chi(t)=\int_0^t \frac{c\,d\bar{t}}{\beta(\bar{t})}-\omega_0 Q_x t\ . \tag{6}$$

Q_x is the betatron tune, β is the amplitude function of the particle, and $\omega_0=2\pi/T_0$. For a single ring configuration, where the two betas (β and β_s) are identical, $x^2/2\sigma_x^2$ always equals I/ϵ_0 times the cosine factor, and the transformed Hamiltonian can then be written as

$$\bar{H}(I,\psi,t)=\bar{H}_0(I)+\epsilon\sum_{p=-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega\, C_p(I,\omega)e^{-i\omega t}e^{ip\psi}\ , \tag{7}$$

where

$$C_p(I,\omega)=\frac{1}{(2\pi)^2}\int_0^{2\pi}d\psi\int_0^1\frac{d\eta}{\eta}\int_{-\infty}^{\infty}dt\left[1-\exp\left[-\frac{I\eta}{\epsilon_0}\cos^2[\psi+\chi(t)]\right]\right]e^{-ip\psi}\times\sum_{n=-\infty}^{\infty}\frac{2c}{\sqrt{2\pi\sigma_s^2}}\exp\{-2c^2[t-(nT_0+\tau)]^2/\sigma_s^2\}e^{i\omega t}\ . \tag{8}$$

Performing a change of variables the ψ and t integrals can be factored. If, further, the betatron phase is approximated as varying linearly near the interaction point, so that $\chi(t)\simeq ct/\beta_0-\omega_0 Q_x t$ (modulo nT_0), then the Hamiltonian can be written as

$$\bar{H}(I,\psi,t)=\bar{H}_0(I)+\epsilon\omega_0\sum_{p=-\infty}^{\infty}T_p(I)\sum_{k,m=-\infty}^{\infty}\exp\left[-\frac{\sigma_s^2}{8c^2}\left[m\omega_0-k\omega_s-pQ_x\omega_0+\frac{pc}{\beta_0}\right]^2\right]\times i^k J_k\left[\left[m\omega_0-k\omega_s-pQ_x\omega_0+\frac{pc}{\beta_0}\right]\frac{\hat{\tau}}{2}\right]e^{i(p\psi-m\omega_0 t+k\omega_s t)}\ , \tag{9}$$

where

$$T_p(I)=\frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_0^1\frac{d\eta}{\eta}\left[1-\exp\left[-\frac{I\eta}{\epsilon_0}\cos^2\theta\right]\right]e^{-ip\theta}\ . \tag{10}$$

For motion near a single isolated resonance the phase must be stationary, giving the resonance condition

$$pQ_x+kQ_s=m \tag{11}$$

and the motion can then be represented by a reduced Hamiltonian

$$\bar{H}_{\rm red}=\bar{H}_0+\epsilon\omega_0 F_{00}+\epsilon\omega_0 F_{pk}\ , \tag{12}$$

where

$$F_{pk}=T_p(I)J_k\left[\frac{p}{2}\frac{c\hat{\tau}}{\beta_0}\right]\exp\left[-\frac{1}{2}\left[\frac{p\sigma_s}{2\beta_0}\right]^2\right]\ . \tag{13}$$

In Eq. (12) F_{00} is the average part of the perturbation, and in Eq. (13) phase factors have been dropped. After transforming to resonance coordinates, resonance strengths for small oscillations about the fixed point can be characterized by half-widths in action and frequency spaces, given by[5]

$$\Delta I=2\left[\left|\frac{F_{pk}}{\Lambda_{00}}\right|\right]^{1/2}\ , \tag{14a}$$

$$\Delta\omega=2\epsilon\omega_0\sqrt{|F_{pk}\Lambda_{00}|}\ , \tag{14b}$$

where $\Lambda_{00}=\partial^2F_{00}/\partial I^2$, evaluated at the fixed point.

In the limit $\sigma_s\rightarrow 0$, Eq. (13) reduces to the result of Izrailev and Vasserman for an impulsive beam-beam interaction. If the distributed nature of the interaction is taken into account, resonance strengths are *less* than those for an impulse. Physically this reduction results from averaging over the betatron phase during the collision.

For small synchrotron amplitudes, the Fourier coefficients are reduced by a Gaussian form factor, $\exp(-p^2\sigma_s^2/8\beta_0^2)$. Higher-order betatron resonances have a larger reduction in resonance strength. The form factor is independent of the synchrotron sideband k. The Fourier coefficients for different sidebands vary as $J_k(pc\hat{\tau}/2\beta_0)$. Both the betatron resonance order p and the sideband number k influence the dependence on $\hat{\tau}$, the particle's synchrotron oscillation amplitude.

Several approximations have been made in arriving at Eq. (13). First, the amplitude function was assumed constant near the interaction point; the next term is quadratic near $t=nT_0$. Then $\chi(t)=\arctan(ct/\beta_0)-\omega_0 Q_x t$, and an analytic calculation is not possible. The rapid variation in phase which led to the averaging is reduced, and the Gaussian form factor therefore overestimates the effect of a finite bunch length.

Second, the result is first order in ξ. The strong beam acts as a lens with focal length $f\sim\beta_0/\xi$ for small x/σ_x. Terms of higher order in ξ are necessary when f is comparable to the bunch length. The simulation discussed in the next section does not make these approximations.

III. SIMULATION

The results of Sec. II motivated a multiparticle tracking program to check the main predictions. The simulation had the following features.

(1) It had three spatial dimensions (two transverse and one longitudinal), was weak-strong, and used the round-beam potential of Eq. (2).

(2) The weak beam comprised of 1000 independent test particles that initially had Gaussian distributions in the transverse dimensions, but a fixed longitudinal amplitude.

(3) The amplitude function varied quadratically around the interaction point (IP) for both the strong beam and the test particles.

(4) Chromaticity and dispersion at the interaction point were assumed zero.

(5) The arc transporting particles between collisions was linear. Radiation damping and fluctuations for the whole arc were put in once a turn, and only in the transverse coordinates.[6]

To simulate the distributed nature of the beam-beam interaction, the strong beam was divided longitudinally into several equicharge chunks, and a kick due to each chunk was delivered at its center of charge. Particles were propagated freely between kicks, thus allowing their betatron phase to change. The sensitivity to the number of kicks was investigated (see Fig. 1, cases 1 and 2), and it was found that the number of kicks did not matter provided it was greater than three. Nine kicks were used for the distributed model, while simulations done with one kick are the impulse model.

Tunes were chosen close to various resonance lines (Table I) so that the single isolated resonance approximation would be valid. The resonances chosen for this study were the fourth- and sixth-order betatron resonances and their first two synchrotron sidebands.

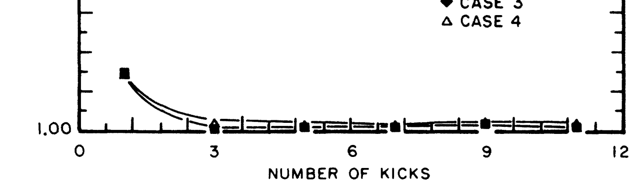

FIG. 1. Emittance against the number of kicks used in the distributed force treatment. Cases 1 and 2 are for the simulation of Sec. III, with no longitudinal radiation, while cases 3 and 4 are for Sec. IV, with longitudinal radiation and a longitudinally distributed weak beam. The cases correspond to (1) $4Q_x - Q_s = 3$ line, $c\hat{\tau}/\beta_0 = 3$, $\sigma_s/\beta_0 = 0.33$, (2) $6Q_x = 2$ line, $c\hat{\tau}/\beta_0 = 1$, $\sigma_s/\beta_0 = 1$, (3) $4Q_x = 3$ line, $\sigma_l/\beta_0 = 0.33$, (4) $6Q_x - Q_s = 2$ line, $\sigma_l/\beta_0 = 1$.

TABLE I. Operating tunes for the different resonances. $Q_s = 0.11$.

Resonance line	Betatron tune ($Q_x = Q_y$)
$4Q_x = 3$	0.745
$4Q_x - Q_s = 3$	0.7725
$4Q_x - 2Q_s = 3$	0.800
$6Q_x = 2$	0.328
$6Q_x - Q_s = 2$	0.347
$6Q_x - 2Q_s = 2$	0.365

The product of the rms sizes in position and angle, $\sigma_x \sigma_{x'}$, calculated at the IP and averaged over the last 1000 turns, was used as a measure of resonance strengths. This measure is related to, but not identical with, the formal definition of the previous section [Eqs. (14)]. It does not equal the emittance when there is resonance structure in phase space; nonetheless in the following discussion it is loosely referred to as the emittance. Results are presented only for the horizontal dimension; the vertical emittance behaved the same. In particular, there was no evidence of any breaking of the symmetry imposed by the potential and the equal tunes. Typical parameters used in the simulations are shown in Table II. The value of 0.035 for the strength parameter, $\xi = Nr_e/4\pi\gamma\epsilon_0$, is chosen as fairly typical of those actually achieved in operating storage rings.

Figure 2 shows plots of emittance against synchrotron amplitude for the six different resonances, in the impulse model. From Eq. (13) the form factor is unity, and the resonance strengths just reflect the Bessel-function structure.

For example, in Fig. 2(a) the $4Q_x = 3$ curve starts off with a maximum strength at zero, and reaches a minimum at $c\hat{\tau}/\beta_0 \approx 1.33$; $J_0(pc\hat{\tau}/2\beta_0)$ has its first zero at $2c\hat{\tau}/\beta_0 = 2.4$. The $4Q_x - Q_s = 3$ and $4Q_x - 2Q_s = 3$ lines have the initial linear and quadratic behavior expected of J_1 and J_2, respectively. At larger amplitudes the $\arctan(x) \approx x$ approximation made in the calculation breaks down, and the behavior will deviate from that predicted by Eq. (13). This is seen in the simulation as a systematic rise in the emittance and a dilution of the expected maxima and minima.

The same general features are seen in Fig. 2(b). How-

TABLE II. Typical parameters used in the simulations.

Energy (E_0)	5.3 GeV
Revolution period (T_0)	2.56 μs
Damping decrement (δ)	1×10^{-3}
Nominal emittance (ϵ_0)	1×10^{-7} m
Beta at the IP (β_0)	3.0 cm
RF freq. (f_{rf})	500 MHz
Synchrotron tune (Q_s)	0.11
Number of particles (N)	1.6×10^{11}
Nominal beam-beam parameter (ξ)	0.035
Number of turns per run	7000

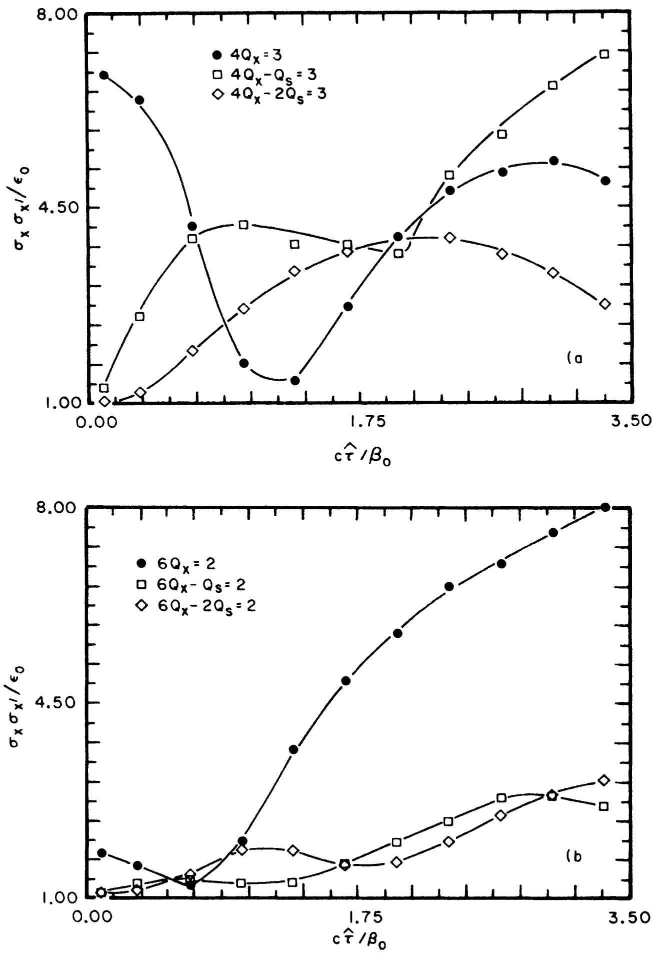

FIG. 2. Emittance as a function of synchrotron amplitude in the impulse model for (a) the fourth-order and (b) the sixth-order betatron resonance, and their first two synchrotron sidebands.

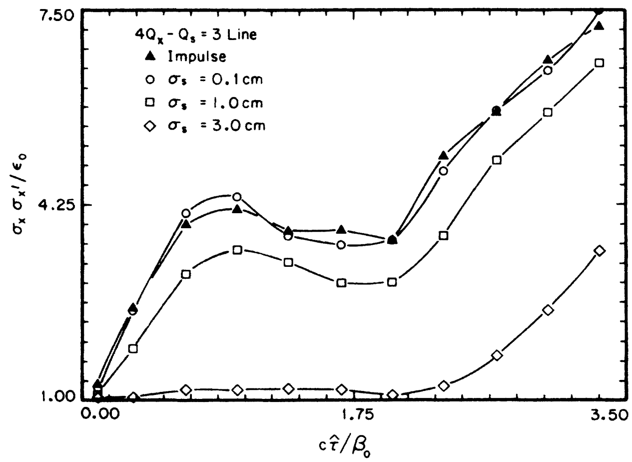

FIG. 3. Emittance as a function of synchrotron amplitude for the $4Q_x-Q_s=3$ resonance, for three different strong-beam bunch lengths, in the distributed model. The impulse-model curve is also shown for comparison.

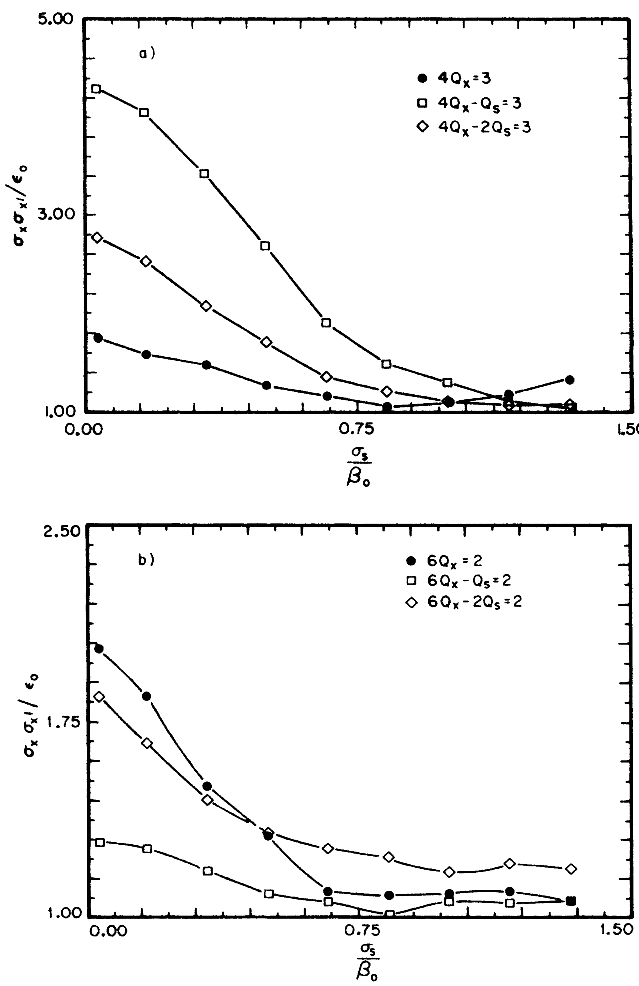

FIG. 4. Emittance as a function of strong beam bunch length in the distributed model for (a) the fourth-order and (b) the sixth-order betatron resonance, and their first two synchrotron sidebands. $c\hat{\tau}/\beta_0=1$.

ever, the minima and maxima are shifted to smaller $c\hat{\tau}/\beta_0$ (relative to the previous figure), reflecting the larger betatron resonance number p. It should be pointed out that a quantitative comparison of the absolute strengths of the resonances is not to be made from these figures, because the emittance blowup depends quite sensitively on the distance of the particles, in tune space, from the resonance.

Figure 3 shows results for the $4Q_x-Q_s=3$ resonance, in the distributed model, for different σ_s/β_0 (corresponding to different magnitudes of the form factor). As expected, the emittance increases as σ_s/β_0 decreases, because of a larger form factor. In the limit $\sigma_s \rightarrow 0$ one smoothly approaches the curve due to the impulse model.

Figure 4 shows, effectively, the form-factor variation as a function of σ_s/β_0 for the different resonances. As predicted, the emittance falls with increasing σ_s/β_0, and the fall is swifter for the sixth-order resonances than for the fourth. A slight rise is noticed in the $4Q_x=3$ curve for larger synchrotron amplitudes. At these amplitudes, as mentioned earlier, the analytic calculation no longer holds. Numerical integration must be taken recourse to, and this does reproduce the observed rise.

IV. DISCUSSION

The results of the previous sections show that there are two different length scales of importance; one set by the strong bunch length σ_s and the other by the test particle's amplitude $\hat{r}$. Further, these have opposite effects on the resonance strength. Larger σ_s leads to greater phase averaging which decreases the emittance blowup, while larger $\hat{r}$ leads to greater depth of modulation, which increases the emittance blowup. In a real collider the test particles have a (nominally Gaussian) longitudinal distribution and, for a single ring configuration, the bunch lengths of the two beams are necessarily equal. This implies that the resonance strength dependence on the bunch length should have a minimum in it.

To test this hypothesis, more complete simulations were performed, in which the weak beam was started with a longitudinal Gaussian distribution of rms length equal to that of the strong beam ($=\sigma_l$, say). Radiation fluctuations and damping were included longitudinally too. Figure 5 shows the emittance as a function of σ_l for two different resonances. The impulse and distributed models are seen to give substantially different bunch length dependence. The latter shows that it is feasible to run colliders with bunch lengths comparable to β_0, without encountering the dynamical degradation in luminosity predicted by the impulse model. (There will, however, still be a reduction in luminosity due to geometric effects.)

Rice[7] has reported experimental results from the Cornell Electron Storage Ring (CESR) that show the luminosity to be maximum at $\sigma_l/\beta_0 \approx 1.1$. This contradicts simulations using the impulse model[2] that predict a decrease in luminosity for $\sigma_l/\beta_0 > 0.7$. The measurements were made with flat beams ($\sigma_x \gg \sigma_y$) and with nonzero horizontal dispersion at the interaction point; they thus represent dynamics different from that considered here. However, the basic phenomenon of betatron phase averaging still occurs, and this could help explain the CESR results.

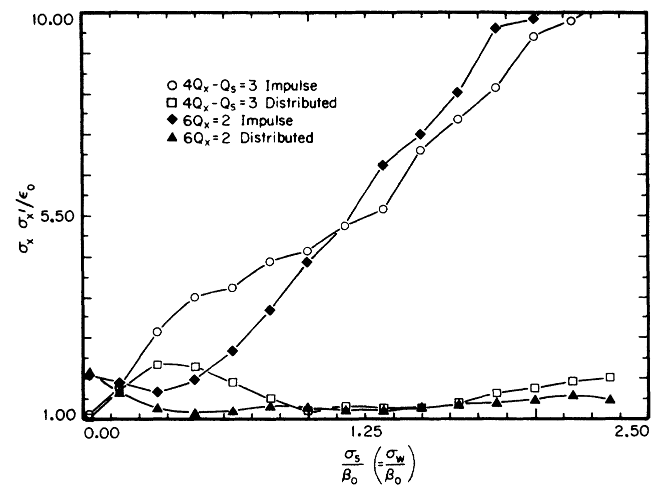

FIG. 5. Emittance as a function of bunch length for the $4Q_x - Q_s = 3$ and $6Q_x = 2$ resonances, in the distributed and impulse models. Here the two beams have equal bunch lengths ($\sigma_s = \sigma_w = \sigma_l$), and radiation has been incorporated longitudinally.

We have been working in the approximation of single isolated resonance and find that the distributed nature of the force reduces resonance strengths. This implies that resonance widths are also smaller than previously calculated. As a result resonance overlap and the consequent stochastic behavior will set in at higher currents than previously estimated.

CONCLUSION

The finite longitudinal extent of the beam-beam interaction results in averaging of the betatron phase over the collision, which mitigates the destructive effects of resonances. As a result, bunch lengths of the order of β_0 are viable in the operation of storage ring colliders.

ACKNOWLEDGMENTS

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