Order-α radiative corrections for semileptonic decays of unpolarized baryons

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The theory of the order-α radiative correction to the β decay of baryons is briefly reviewed. An effective procedure for the calculation of the real-photonic part of the order-α correction is outlined. Numerical results are tabulated for the radiative corrections to the branching ratio, the electron-antineutrino correlation parameter, the electron-energy (E_e) spectrum, the final-baryon-energy (E_f) spectrum, and the electron-antineutrino angular-correlation ($\cos\theta_{e\nu}$) distribution, and to the (E_e,E_f) and the $(E_e,\cos\theta_{e\nu})$ two-variable distributions in the cases of the decays $\Sigma^-\rightarrow ne\bar{\nu}$, $\Sigma^-\rightarrow\Lambda e\bar{\nu}$, $\Xi^-\rightarrow\Lambda e\bar{\nu}$, and $\Lambda\rightarrow pe\bar{\nu}$.

I. INTRODUCTION

The progress of high-energy hyperon beams led to a new generation of hyperon β-decay experiments in this decade.[1-4] Now the form factors of the charged weak current are determined with 2–5 % experimental error for several decay modes, approaching the (typically one percent) error of the very precisely measured neutron β-decay parameters. So the Cabibbo theory[5] can now be tested with remarkable precision.[6] This experimental situation revived the problem of the order-α radiative corrections to these decays. Much work was devoted to the calculation of them in relation with the neutron and muon decays already in the 1960s (for a most beautiful review see Ref. 7). Those early works suffered from the severe problem of ultraviolet divergences, and a real breakthrough in solving it was achieved only after the SU(3)⊗SU(2)⊗SU(1) model of the strong, electromagnetic, and weak interactions could be applied.[8,9] Unfortunately, some strong-interaction effects remain uncalculable; therefore, though finite, the order-α radiative corrections to the hyperon semileptonic decays still have a somewhat qualitative character. However, most of the uncertainties can be absorbed into coupling constants (form factors)[10] whereas the variation of the corrections on various kinematical plots can be calculated with reasonable precision. As the order-α radiative correction to the shape of the various distributions is not small, knowledge of the corrections is important in order that the form-factor values which come from different experiments be comparable. The above-mentioned uncalculable effects are not sensitive to the experimental situation, and they are to be considered as theoretical limitation for present tests of the Cabibbo model.

Most of the available calculations of the order-α corrections refer to the electron-energy (E_e) spectrum; some of them refer to the recoil-baryon-energy (E_f) spectrum. More information is contained in the two-variable distributions, and their significance is considerable for the high-statistics experiments. There exist calculations of the radiative corrections for the (E_e,E_f) Dalitz plot,[11,12] and for the $(E_e,\cos\theta_{e\nu})$ distribution,[13-16] where $\theta_{e\nu}$ is the angle between the momenta of the electron and the antineutrino. The numerical results in Refs. 11 and 12 are somewhat special, as they were designed to meet the needs of the experiment of Bourquin *et al.*[2] In this paper we extend that result so that it can be more widely used. The calculations for $(E_e,\cos\theta_{e\nu})$ case in Refs. 13–15 do not take it into account that the antineutrino is not detected. Therefore these results are not suitable for experimental analysis. In Ref. 16 the definition $\cos\theta_{e\nu}=-\mathbf{p}_e\cdot(\mathbf{p}_e+\mathbf{p}_f)/(|\mathbf{p}_e||\mathbf{p}_e+\mathbf{p}_f|)$ is used, which is appropriate for most of the hyperon experiments. In this paper the calculation referring to $\Lambda\rightarrow pe\bar{\nu}$ in Ref. 16 is extended to other semileptonic decay modes. (We also made a minor change in the theoretical input that will be explained later.) The various calculations may differ from each other as the adopted theoretical framework is different, moreover, because, while they accurately contain that part of the radiative correction which is of zeroth order in $\Delta m/m_i\equiv(m_i-m_f)/m_i$, where m_i and m_f denote the mass of the decaying and the final baryon, respectively, the terms of order $\Delta m/m_i$ are usually contained (or neglected) in an *ad hoc* manner. This may cause noticeable differences between the results of the various calculations, because $\Delta m/m_i$ is not too small for hyperon decays. (The authors of Ref. 17 claim that their result for the electron-energy spectrum is complete to first order in $\Delta m/m_i$. This claim is, however, incorrect. First, because they use a bremsstrahlung amplitude, which implies the loss of terms of this order. Second, as they fail to find all the contributions of order $\Delta m/m_i$ even in the case of that amplitude.)

In the present state of the theory of the order-α radiative corrections to semileptonic decays it is not possible to obtain a result which is complete to order $\Delta m/m_i$. This is so because of uncalculable strong-interaction effects in the one-loop diagrams. This can be an excuse for us when we neglect some terms of the bremsstrahlung amplitude which contribute to order $\Delta m/m_i$, although the corrections corresponding to one real photon are calculable to such an order. We found it more important to carry out the calculations with a high numerical precision, so that, if an improvement of the theoretical input is made, its effect can be clearly seen. This goal can only be achieved by using a computer; therefore our results are presented in tabulated form.

We do not add anything to the well-known theory of the order-α radiative corrections to semileptonic decays. Contrary to this, we try to give a summary of that, as we want to help the reader in judging the merits of our calculations. In Sec. II, the status of the one-loop virtual corrections is reviewed, and their contribution is quoted. The one-photon radiation process, which accompany the basic semileptonic decay, are considered in Sec. III. We present the transition amplitude, and give our motivations for this choice. As the calculation of transition probabilities requires the solution of remarkable technical problems, we go into some details of our methods to deal with them. Finally, Sec. IV contains a discussion of our results for unpolarized distributions. Results for the case when hyperon polarization is also detected will be published in a separate paper.

II. THE ONE-LOOP VIRTUAL CORRECTIONS

The problem of the one-loop QED corrections to the muon and neutron β-decay processes is a rather long-standing one. This is so because serious ultraviolet infinities appear when, in order to evaluate the matrix elements of products of interaction currents in the order-α terms, one attempts to use either a naive perturbative approach, or the more sophisticated current algebraic methods, but one preserves the traditional current$\times$current form of the weak interaction. Regularization of the UV integrals can be technically achieved by assuming appropriate momentum-transfer dependence of the hadronic form factors, but the price to be paid for this is, usually, the loss of gauge invariance. Real progress in solving these problems came when the standard SU(3)$\otimes$SU(2)$\otimes$U(1) model of the strong, electromagnetic, and weak interactions was invented. On this theoretical basis the order-α one-loop corrections to a general $i \to f e \bar{\nu}$ beta-decay amplitude were most extensively studied by Sirlin.[8] The result of the rather involved and tedious discussions can be summarized by the short expression

$$\mathcal{M}_\alpha = \frac{\alpha}{2\pi} C \mathcal{M}_0 + \mathcal{M}_\gamma \tag{2.1}$$

for $\mathcal{M}_\alpha$, which is the sum of the one-loop terms in the decay matrix element after neglecting contributions of order $\alpha m^2/M_W^2$, where M_W is the mass of the weak vector boson, and m denotes a typical mass, e.g., the mass m_i of the decaying particle in the process $i \to f e \bar{\nu}$. The second term $\mathcal{M}_\gamma$ in (2.1) is formally the same as the regularized QED correction in the old theory. The first term in (2.1) is entirely a consequence of the new theoretical framework. It is proportional to the zeroth-order matrix element:

$$\mathcal{M}_0 = \sqrt{2}\, G_F [\bar{u}_2 \gamma^\mu (1+\gamma^5) v_1] \langle f | J_W^\mu(0) | i \rangle \ . \tag{2.2}$$

(By u_2 and v_1 we denote the Dirac spinors of the electron and the antineutrino, respectively, normalized as $\bar{u}_2 u_2 = 2m_e$, $\bar{v}_1 v_1 = 0$. For the Dirac γ matrices and for the metric in the scalar products we follow the conventions of Ref. 18.) The coupling constant G_F is equal with the one G_μ observed in muon decay. The content of Eq. (2.1) is that, adopting the renormalization scheme of Ref. 19 which fixes the masses at their physical value and maintains the traditional QED rule for the electric charge, the one-loop electroweak corrections lead to a regularized form of the QED corrections, $\mathcal{M}_\gamma$, and to an almost universal multiple of $\mathcal{M}_0$. An (infinite) multiplier can be absorbed into the weak coupling constant and a (finite) deviation from the universal multiplier remains in the form of

$$C = -\tfrac{3}{2}(\tfrac{1}{2} + \bar{Q}) \ln \cos^2\Theta_W \ , \tag{2.3}$$

where Θ_W is the Weinberg angle and $\bar{Q}$ is the average electric charge of the relevant weak isodoublet ($\bar{Q} = -\frac{1}{2}$ for muon decay, $\bar{Q} = \frac{1}{6}$ for hadron decay).

The term $\mathcal{M}_\gamma$ represents the contribution of three different types of diagrams: (i) A virtual photon is emitted and reabsorbed by the (pointlike) charged lepton in the final state; (ii) a virtual photon is emitted and reabsorbed in the (fWi) weak vertex; (iii) a virtual photon is exchanged between the (fWi) vertex and the charged lepton in the final state.

In the Glashow-Salam-Weinberg model (and in the Feynman gauge) contribution (iii) is finite and reads as

$$\mathcal{M}_\gamma^{(3)} = \sqrt{2}\, G_F \frac{\alpha}{4\pi^3} \int dk\, D_{\mu\nu}(k) \frac{M_W^2}{M_W^2 + k^2} T^{\mu\rho}(k) \bar{u}_2 \frac{2p_{2\nu} - k_\nu - \frac{1}{2}[\gamma_\nu, \not{k}]}{(k - p_2)^2 + m_e^2} \gamma^\rho (1+\gamma^5) v_1 \ , \tag{2.4}$$

where

$$T^{\mu\rho}(k) = \int dx\, e^{-ikx} \langle f | T[J_\gamma^\mu(x) J_W^\rho(0)] | i \rangle \ , \tag{2.5}$$

and $D_{\mu\nu}(k)$ is the photon propagator. p_2 denotes the electron four-momentum, $p_2^2 = -m_e^2$. (The corresponding notations for the antineutrino, the i and f particles will be p_1, p_i and p_f, $p_1^2 = 0$, $p_i^2 = -m_i^2$, $p_f^2 = -m_f^2$, respectively.)

In (2.4) the denominator of the W-boson propagator is approximated by $1/(M_W^2 + k^2)$, neglecting contributions of order $\alpha m_i^2/M_W^2$.

Contributions (i) and (ii) are UV infinite in QED. In the electroweak theory $\mathcal{M}_\gamma$ is finite; as in the case of these contributions,

$$D_{\mu\nu}^{<} = \frac{M_W^2}{k^2 + M_W^2} D_{\mu\nu} \tag{2.6}$$

turns up instead of the photon propagator:

$$\mathcal{M}_\gamma^{(1)}=\mathcal{M}_0\delta Z_{(e)} , \tag{2.7}$$

$$\delta Z_{(e)}=\frac{i\alpha}{8\pi^3}\int dk\, D^{<}_{\mu\nu}(k)\left[\frac{(2p_{2\mu}-k_\mu)(2p_{2\nu}-k_\nu)}{[(k-p_2)^2+m_e^2]^2}+\frac{1}{4i}\frac{1}{m_e^2}\frac{\bar u_2[\gamma_\mu,\not k]\not p_2\not k\gamma_\nu u_2}{[(k-p_2)^2+m_e^2]^2}\right] . \tag{2.8}$$

Finally, contribution (ii) can be summarized as

$$\mathcal{M}_\gamma^{(2)}=-i\sqrt{2}\frac{\alpha}{8\pi^3}G_F\bar u_2\gamma^\mu(1+\gamma^5)v_1\int dk\, D^{<}_{\nu\rho}(k)T^{\mu\nu\rho}(k) , \tag{2.9}$$

where

$$T^{\mu\nu\rho}(k)=\lim_{\bar q\to p_i-p_f}\left[\int dy\, e^{-i\bar q y}\int dx\, e^{-ikx}\langle f|T[J_W^\mu(y)J_\gamma^\nu(x)J_\gamma^\rho(0)]|i\rangle-B^{\mu\nu\rho}\right] . \tag{2.10}$$

The counterterm $B^{\mu\nu\rho}$ is introduced for the appropriate mass renormalization. The notations J_W and J_γ stand for the weak and the electromagnetic current, respectively.

It is necessary to point out that in the electroweak theory contributions similar to $\mathcal{M}_\gamma^{(1)}$ and $\mathcal{M}_\gamma^{(2)}$, but with

$$D^{>}_{\mu\nu}=\frac{k^2}{k^2+M_W^2}D_{\mu\nu} \tag{2.11}$$

instead of $D^{<}_{\mu\nu}$ are also included in the analysis of the UV-divergent graphs. The splitting

$$D_{\mu\nu}=D^{<}_{\mu\nu}+D^{>}_{\mu\nu} \tag{2.12}$$

with the "cutoff mass" M_W is determined by $\mathcal{M}_\gamma^{(3)}$, *a priori* containing $(M_W^2+k^2)^{-1}$, and by the requirement that $\mathcal{M}_\gamma$ be gauge invariant.

In spite of the neat solution for the UV problem in the standard model, the calculation of $\mathcal{M}_\gamma$ remains an unsolved question, because one cannot evaluate $T^{\mu\rho}(k)$ and $T^{\mu\nu\rho}(k)$ for arbitrary k in the case of hadrons. On the basis of general principles, however, one can derive the form of $T^{\mu\rho}$ and $T^{\mu\nu\rho}$ for both the small and the large values of k. The contributions to $\mathcal{M}_\gamma$ corresponding to these limiting forms can be made explicit by writing

$$\mathcal{M}_\gamma=\mathcal{M}_\gamma^{\text{as}}+\mathcal{M}_\gamma^{\text{MI}}+\mathcal{M}_\gamma^{\text{MD}} . \tag{2.13}$$

The first term $\mathcal{M}_\gamma^{\text{as}}$ on the right-hand side of (2.13) represents the large-k contribution of the integrals giving $\mathcal{M}_\gamma$. Similarly to the first term in the expression (2.1) for the $\mathcal{M}_\alpha$, it has the universal form

$$\mathcal{M}_\gamma^{\text{as}}\approx\mathcal{M}_0\frac{3\alpha}{2\pi}(\bar Q+\tfrac{1}{2})\ln\frac{M_W}{M} . \tag{2.14}$$

This is the leading asymptotic dependence of $\mathcal{M}_\gamma$ on the vector-boson mass M_W, related to the next characteristic mass scale M, at which the strong-interaction coupling constant is small enough to justify the substitution of the first term in the short-distance expansion for the T product of the interaction currents. The result (2.14) is gauge invariant.

For small values of k the basic result is that

$$T^{\mu\rho}(k\approx 0)\approx i\frac{p^\mu_{\text{ch}}}{(k\cdot p_{\text{ch}})}\langle f|J_W^\rho(0)|i\rangle , \tag{2.15}$$

where p_{ch} denotes the four-momentum p_i or p_f, depending on whether the particle i or f is charged, respectively. The result (2.15) follows from the generalized Ward identity. Similarly, $T^{\mu\nu\rho}(k)$ is entirely determined for $k\approx 0$, and reduces to

$$T^{\mu\nu\rho}(k\approx 0)\approx-\frac{p^\nu_{\text{ch}}p^\rho_{\text{ch}}}{(k\cdot p_{\text{ch}})^2}\langle f|J_W^\mu(0)|i\rangle . \tag{2.16}$$

By definition, the second term $\mathcal{M}_\gamma^{\text{MI}}$ in (2.13) must fully contain the infrared divergent piece of $\mathcal{M}_\gamma$. This is accomplished by the construction

$$\mathcal{M}_\gamma^{\text{MI}}=\mathcal{M}_\gamma^{\text{MI}(1)}+\mathcal{M}_\gamma^{\text{MI}(2)}+\mathcal{M}_\gamma^{\text{MI}(3)} , \tag{2.17}$$

$$\mathcal{M}_\gamma^{\text{MI}(1)}=\mathcal{M}_\gamma^{(1)} , \tag{2.18}$$

$$\mathcal{M}_\gamma^{\text{MI}(2)}=\mathcal{M}_0\delta Z , \tag{2.19}$$

$$\delta Z=\frac{i\alpha}{8\pi^3}\int dk\, D^{<}_{\mu\nu}(k)\frac{(2p^\mu_{\text{ch}}-k^\mu)(2p^\nu_{\text{ch}}-k^\nu)}{[(k-p_{\text{ch}})^2+m^2_{\text{ch}}]^2} , \tag{2.20}$$

$$\mathcal{M}_\gamma^{\text{MI}(3)}=-i\sqrt{2}\,G_F\frac{\alpha}{4\pi^3}\int dk\, D^{<}_{\mu\nu}(k)\frac{2p'^\mu_{\text{ch}}-k^\mu}{(k-p'_{\text{ch}})^2+m^2_{\text{ch}}}\langle f|J_W^\rho(0)|i\rangle\bar u_2\frac{2p_{2\nu}-k_\nu-\frac{1}{2}[\gamma_\nu,\not k]}{(k-p_2)^2+m_e^2}\gamma^\rho(1+\gamma^5)v_1 . \tag{2.21}$$

Here p'_{ch} is equal with p_i or $-p_f$, depending on whether the initial or final baryon is charged, respectively.

In this construction use of (2.15) and (2.16) is made, and terms of higher order in k are added in order to obtain a gauge-invariant $\mathcal{M}_\gamma^{\rm MI}$, which remains finite, when $M_W\to\infty$. Neglecting terms, which are of order $\alpha m_i^2/M_W^2$, the above $\mathcal{M}_\gamma^{\rm MI}$ coincides with Sirlin's "model independent" expression for the one-loop photonic correction to the amplitude of the β decay of a neutron.[10] This correction is independent of the structure of the weak-current matrix element $\langle f|J_W^\rho(0)|i\rangle$.

The result of the k integrations reads

$$\delta Z_{(e)}=-\frac{\alpha}{2\pi}\left[\frac{1}{2}\ln\frac{M_W}{m_e}-\ln\frac{m_e}{\lambda}+\frac{9}{8}\right], \tag{2.22}$$

$$\delta Z=-\frac{\alpha}{2\pi}\left[\frac{1}{2}\ln\frac{M_W}{m_{\rm ch}}-\ln\frac{m_{\rm ch}}{\lambda}+\frac{3}{4}\right], \tag{2.23}$$

and with the notation $E=-(p_2\cdot p_{\rm ch})/m_{\rm ch}$, $p=\sqrt{E^2-m_e^2}$, $p_+=E+p$,

$$\mathcal{M}_\gamma^{\rm MI(3)}=\frac{\alpha}{2\pi}(-d_0+d_1)\mathcal{M}_0+i\frac{\alpha}{2\pi}d_{11}\sqrt{2}\,G_F\bar u_2\frac{\not p_{\rm ch}}{m_{\rm ch}}\gamma^\mu(1+\gamma^5)v_1\langle f J_W^\mu(0)|i\rangle , \tag{2.24}$$

where

$$d_{11}=\frac{m_e}{p}\ln\frac{p_+}{m_e}, \tag{2.25}$$

$$d_1=\ln\frac{M_W}{m_e}+\frac{1}{2}+\frac{E}{p}\ln\frac{p_+}{m_e}, \tag{2.26}$$

$$d_0=\frac{2E}{p}\ln\frac{p_+}{m_e}\ln\frac{m_e}{\lambda}+\frac{E}{p}\ln^2\frac{p_+}{m_e}+\frac{E}{p}{\rm Sp}\left[\frac{2p}{p_+}\right]+2\frac{E}{m'_{\rm ch}}\left[\ln\frac{p_+}{m_{\rm ch}}-1\right]. \tag{2.27}$$

In (2.27) ${\rm Sp}(x)$ denotes the Spence function; $m'_{\rm ch}$ is equal with m_i or $-m_f$, depending on whether the initial or final baryon is charged, respectively. For neutral decaying particle the so-called Coulomb term π^2E/p must be supplemented to d_1. Equations (2.22)–(2.27) give the "model-independent part" of the one-loop correction including all terms to order $\alpha\Delta m/m_i$.

Lastly the term $\mathcal{M}_\gamma^{\rm MD}$ in (2.13) requires explanation. We call it the "model-dependent part" of the one-loop radiative corrections. (It has to be noted, that in the spirit of Sirlin's original definition the "model dependent part" also incorporates our $\mathcal{M}_\gamma^{\rm as}$, whereas the term $\mathcal{M}_\gamma^{\rm MD}$ is the nonasymptotic piece.) It is the 1–10-GeV^2 region of k^2 in the integrals which determines $\mathcal{M}_\gamma^{\rm MD}$; therefore, in principle, one is confronted here with the full complexity of the structure of hadrons. A detailed as well as reliable calculation of $\mathcal{M}_\gamma^{\rm MD}$ does not exist. The result of estimates (e.g., by dispersion relation methods) is qualitative, rather than quantitative; therefore $\mathcal{M}_\gamma^{\rm MD}$ is not included in our work. Still, the approximation $\mathcal{M}_\gamma\approx M_\gamma^{\rm as}+\mathcal{M}_\gamma^{\rm MI}$ preserves a good deal of information.

In order to give an idea about the approximation $\mathcal{M}_\gamma\approx\mathcal{M}_\gamma^{\rm as}+\mathcal{M}_\gamma^{\rm MI}$ we conclude this section with a few remarks.

(1) The sum of $\mathcal{M}_\gamma^{\rm as}$ and $\mathcal{M}_\gamma^{\rm MI}$ does not imply double counting. On the one hand, the term $\mathcal{M}_\gamma^{\rm as}$ comes from the $k^2>M^2$ region of the k integrals, which define $\mathcal{M}_\gamma$ (Ref. 9). On the other hand, for $\mathcal{M}_\gamma^{\rm MI}$ the region $k^2<m_i^2$ is relevant. This can be seen by inserting a factor

$$1-\frac{M^2}{k^2-2(k\cdot p_{\rm ch})+M^2}$$

into the integrands of (2.8), (2.20), and (2.21). Assuming $m_{\rm ch}^2/M^2\ll 1$, it turns out, that the sum of the three integrals is of the order $m_{\rm ch}^2/M^2$.

This observation indicates the possibility for the following construction. Insert a factor $M^2/[k^2-2(k\cdot p_{\rm ch})+M^2]$ into the integrals giving $\mathcal{M}_\gamma^{\rm MI}$. Then, as if the charged external particle were electromagnetically pointlike, add the "spin terms" to (2.20) and also to (2.21) corresponding to the charged fermion propagator [cf. (2.8)]. Finally, one may even include the terms corresponding to the electromagnetic anomalous magnetic momenta. Apparently, this construction shifts some part of $\mathcal{M}_\gamma^{\rm MD}$ to $\mathcal{M}_\gamma^{\rm MI}$, giving, together with $\mathcal{M}_\gamma^{\rm as}$, another approximation of $\mathcal{M}_\gamma$. Results coming from such a construction were mentioned in Ref. 20. There is no way to tell, whether such a calculation gives better approximation of $\mathcal{M}_\gamma$, though it may contain a lot more details, than Sirlin's $\mathcal{M}_\gamma^{\rm MI}$.

(2) The neglect of $\mathcal{M}_\gamma^{\rm MD}$ limits the use of the radiative corrections. The crucial properties of $\mathcal{M}_\gamma^{\rm MD}$, which make the approximation $\mathcal{M}_\gamma\approx\mathcal{M}_\gamma^{\rm as}+\mathcal{M}_\gamma^{\rm MI}$ acceptable for the present experiments are the following. First, provided that terms of order $\alpha\Delta m/m_i$ are neglected, the form

$$\mathcal{M}_\gamma^{\rm MD}\approx\frac{\alpha}{2\pi}\bar u_f(a\gamma^\mu+b\gamma^\mu\gamma^5)u_i\cdot\bar u_2\gamma^\mu(1+\gamma^5)v_1 \tag{2.28}$$

can be derived, moreover, it can be verified, that a and b are independent of the electron energy and the electron mass.[10] The latter property suggests that the dimensionless constants a and b are ~ 1. In addition to (2.28) $\mathcal{M}_\gamma^{\rm MD}$ contains terms of order $\alpha\Delta m/m_i$, such as terms proportional to $\alpha(m_i-E_f)/m_i$, $\alpha E_e/m_i\ln(E_e/m_i)$, αv_f (where v_f is the velocity of the recoil baryon), but there are no terms of type $\alpha\ln(E_e/m_i)$. Therefore, the approximation $\mathcal{M}_\gamma\sim\mathcal{M}_\gamma^{\rm as}+\mathcal{M}_\gamma^{\rm MI}$ properly includes most of the dependence on the kinematics. Garcia and Juárez pointed out,[21] that, as far as polarizations are not detected, the terms of order $\alpha\Delta m/m_i$ summed up with (2.28) are properly represented by the form

$$\mathcal{M}_\gamma^{\rm MD} \approx \frac{\alpha}{2\pi}[\bar{u}_2\gamma^\mu(1+\gamma^5)v_1]\bar{u}_f\left[a_1\gamma^\mu + a_2\frac{q^\nu}{m_i}\sigma^{\mu\nu} + a_3\frac{q^\mu}{m_i} + \left(b_1\gamma^\mu + b_2\frac{q^\nu}{m_i}\sigma^{\mu\nu} + b_3\frac{q^\mu}{m_i}\right)\gamma^5\right]u_i \quad (q = p_i - p_f) \ . \tag{2.29}$$

The "form factors" a_i, b_i depend, in general, on the kinematical variables. [In particular, dependence of the form $\ln(E_e/m_i)$ may now appear.] An experimental analysis, which takes into account radiative corrections, but neglects $\mathcal{M}_\gamma^{\rm MD}$, results in form-factor values, which include $(\alpha/2\pi)a_i$, $(\alpha/2\pi)b_i$ (strictly speaking, some average values, $\bar{a}_i, \bar{b}_i$ appear).

To summarize, radiative corrections with the approximation $\mathcal{M}_\gamma \approx \mathcal{M}_\gamma^{\rm as} + \mathcal{M}_\gamma^{\rm MI}$ are useful for experimental analysis as far as the aim is to eliminate effects, which are imposed by experimental conditions (acceptance, etc.). The form factors, which are obtained in such a procedure, contain pieces of the order-α correction, however. We expect that these pieces do not exceed 1%, as they come from the $k^2 \sim 1-10\text{-GeV}^2$ region of the loop integrals.

III. THE REAL-PHOTONIC CORRECTION

For well-known reasons the order-α radiative correction to the process $i \to f e \bar{\nu}$ must include the one-photon radiative decay $i \to f e \bar{\nu} \gamma$ as well. The theory of such processes is well established, the transition amplitude is known with remarkable precision. The structure of radiative amplitudes has been studied by several authors. Results, which are directly applicable to semileptonic decays, have been published in Ref. 22.

In general, the matrix element for the emission of a photon consists of two kinds of contributions: photons may arise both from inner bremsstrahlung, i.e., from emission by an ingoing or outgoing particle, and from direct emission, which reflects the internal structure of the weak-interaction vertices in the nonradiative process $i \to f e \bar{\nu}$. In the radiative decay amplitude $\mathcal{M}'$, which can be written as

$$\mathcal{M}' = \mathcal{M}'_l + \mathcal{M}'_h + \mathcal{M}'_d \ , \tag{3.1}$$

the term $\mathcal{M}'_l$ is exactly known. It describes the inner bremsstrahlung coming from the electron in the final state:

$$\mathcal{M}'_l = i(2\pi)^4 \frac{G_F}{\sqrt{2}} e\{\bar{u}_2 \not{\epsilon}^*(k,s)[i(\not{p}_2 + \not{k}) + m_e]^{-1}\gamma^\rho(1+\gamma^5)v_1\}(\bar{u}_f H^\rho u_i) \ . \tag{3.2}$$

The hadronic term $\mathcal{M}'_h$ is known to order $(k)^0$; in our calculation it will be

$$\mathcal{M}'_h = i(2\pi)^4 \frac{G_F}{\sqrt{2}} e\{\bar{u}_f H^\rho(k)[i(\not{p}_i - \not{k}) + m_i]^{-1}\epsilon^*(k,s)u_i\}[\bar{u}_2\gamma^\rho(1+\gamma^5)v_1] \ , \tag{3.3}$$

when the decaying particle is (negatively) charged, and

$$\mathcal{M}'_h = -i(2\pi)^4 \frac{G_F}{\sqrt{2}} e\{\bar{u}_f \not{\epsilon}^*(k,s)[i(\not{p}_f + \not{k}) + m_f]^{-1} H^\rho(k)u_i\}[\bar{u}_2\gamma^\rho(1+\gamma^5)v_1] \ , \tag{3.3'}$$

when it is neutral. $\mathcal{M}'_d$ is a direct emission term, which is needed for a gauge-invariant total amplitude $\mathcal{M}'$. When the decaying particle is charged,

$$\mathcal{M}'_d = -(2\pi)^4 \frac{G_F}{\sqrt{2}} e\epsilon^{*\mu}(k,s)\left(\bar{u}_f \frac{\partial H^\rho}{\partial p_i^\mu} u_i\right)[\bar{u}_2\gamma^\rho(1+\gamma^5)v_1] \ , \tag{3.4}$$

and, for the neutral case,

$$\mathcal{M}'_d = (2\pi)^4 \frac{G_F}{\sqrt{2}} e\epsilon^{*\mu}(k,s)\left(\bar{u}_f \frac{\partial H^\rho}{\partial p_f^\mu} u_i\right)[\bar{u}_2\gamma^\rho(1+\gamma^5)v_1] \ . \tag{3.4'}$$

In expressions (3.3) and (3.3′) the vertex function $H^\rho(k)$ is obtained form the one H^ρ by substituting $p_i - k$ for p_i and $p_f + k$ for p_f, respectively. It implies also terms of order k in $\mathcal{M}'$, which cannot, at present, be fully determined theoretically. They contribute to our numerical results to an extent that is small in comparison with other uncertainties. There are two types of contributions of order $(k)^0$, which we do not include in the hadronic inner bremsstrahlung amplitude $\mathcal{M}'_h$, although they are contained in the general result of Ref. 22. One of them corresponds to the coupling of the photon to the anomalous magnetic momentum of the i and f particles The other one is proportional to the derivative of the invariant form factors with respect to the momentum transfer $q^2 = (p_i - p_f)^2$. These contributions require additional parameters, namely, the anomalous magnetic momenta and the slopes of the form factors. The direct emission term $\mathcal{M}_d$, together with the k dependence of $H^\rho(k)$ in $\mathcal{M}'_h$ could also be neglected without violating gauge invariance and without a noticeable change in the numerical results. Nevertheless, we kept these terms, since, to order $(k)^0$ they assure the manifest equivalence between the form-factor sets f_i, g_i and F_i, G_i in our $\mathcal{M}'$, which is a distinguished property of the zeroth-order amplitude $\mathcal{M}_0$ (Ref. 23). The order-k part of $\mathcal{M}'$ cannot be determined with a rigor, which is comparable with that in the case of the order k^{-1} and $(k)^0$ terms.

The evaluation of transition rates with the matrix element specified above gives rise to remarkable technical problems. There are three sources of them: (i) the large number of terms in the trace calculation; (ii) the infrared problem caused by the low-energy photons; (iii) the strong peaking of the matrix element, when the three-momenta of the photon and the electron are parallel.

We found that an optimum solution to these problems can be achieved by a combined use of analytical and computer methods. We shortly summarize the strategy for the calculation of the (E_e, E_f) Dalitz distribution:

$$\frac{d^2\Gamma'}{dE_e dE_f}=\frac{1}{8m_i}\frac{1}{(2\pi)^6}\int q_{ef}dq_{ef}\int\frac{d^3\mathbf{k}}{2k_0}\sum_{\text{spins}}\left|\frac{L}{2(p_2\cdot k)}+\frac{H_{i,f}}{2(p_{i,f}\cdot k)}+\frac{D_{i,f}}{2(p_{i,f}\cdot k)}\right|^2\delta(p_1^2)\ . \tag{3.5}$$

In the above expression the following notation is introduced:

$$L=(2\pi)^{-4}2(p_2\cdot k)\mathcal{M}'_l,\quad H_i=(2\pi)^{-4}2(p_i\cdot k)\mathcal{M}'_h\ ,$$
$$H_f=(2\pi)^{-4}2(p_f\cdot k)\mathcal{M}'_h\ ,$$
$$D_i=(2\pi)^{-4}2(p_i\cdot k)\mathcal{M}'_d,\quad D_f=(2\pi)^{-4}2(p_f\cdot k)\mathcal{M}'_d\ .$$

Integration over the four-momentum of the antineutrino is performed, so that $p_1=p_i-p_f-p_2-k$ is valid in (3.5). Angular integrations, which are trivial in the rest frame of the decaying particle are also performed. Finally, the variable of integration q_{ef} is defined as $q_{ef}=|\mathbf{p}_2+\mathbf{p}_f|$. Notice that, on the one hand, the distribution (3.5) extends for $m_e\le E_e\le E_{e\max}$, $E_{e\max}=(m_i^2-m_f^2+m_e^2)/(2m_i)$, similarly to the case of nonradiative final states. On the other hand, the region of E_f is wider than that in the nonradiative case, when

$$E_{f\min}\le E_f\le E_{f\max}\ , \tag{3.6}$$

where

$$E_{f\min,\max}=\frac{1}{2}\left[m_i-E_e\mp|\mathbf{p}_e|+\frac{m_f^2}{m_i-E_e\mp|\mathbf{p}_e|}\right]\ . \tag{3.7}$$

In fact, for electron energies $m_e\le E_e\le E'_{e\max}$, where $E'_{e\max}=\frac{1}{2}[m_i-m_f+m_e^2/(m_i-m_f)]$, radiative decay events with E_f in the interval

$$m_f\le E_f\le E_{f\min} \tag{3.8}$$

may also appear. As the integration over q_{ef} must be taken over the interval $q_{ef\min}\le q_{ef}\le q_{ef\max}$, where $q_{ef\min}=||\mathbf{p}_f|-|\mathbf{p}_e||$, $q_{ef\max}=\min\{q_{ef}^0\equiv m_i-E_f-E_e, |\mathbf{p}_f|+|\mathbf{p}_e|\}$; moreover, as the kinematically allowed photon energies extend for $\frac{1}{2}(q_{ef}^0-q_{ef})\le|\mathbf{k}|\le\frac{1}{2}(q_{ef}^0+q_{ef})$, infrared divergence of the photon contribution arises for events of type (3.6) (when $q_{ef}^0\le|\mathbf{p}_f|+|\mathbf{p}_e|$), but not for those of type (3.8) (when $q_{ef}^0\ge|\mathbf{p}_f|+|\mathbf{p}_e|$).

The trace calculations, which result from the spin summations, are more conveniently carried out by means of REDUCE algebraic programs.[24] In the expressions obtained, which consist of scalar products of the four-momenta p_i, p_f, p_2, p_1, and k, substitutions are to be made in order to relate them to E_e, E_f, and the variables of integration q_{ef} and $\mathbf{k}$. The substitutions are as follows:

$$p_i\cdot p_f=(p_i\cdot p_f)^0,\quad p_i\cdot p_2=(p_i\cdot p_2)^0\ ,$$
$$p_i\cdot p_1=(p_i\cdot p_1)^0-p_i\cdot k,\quad p_f\cdot p_2=(p_f\cdot p_2)^0+p_1\cdot k\ ,$$
$$p_f\cdot p_1=(p_f\cdot p_1)^0-p_i\cdot k+p_2\cdot k\ ,$$
$$p_2\cdot p_1=(p_2\cdot p_1)^0-p_2\cdot k-p_1\cdot k\ .$$

The notation $(\ )^0$ refers to quantities which are related to E_e, E_f and to the masses m_i, m_f, m_e in the same manner, as the corresponding scalar products are in the nonradiative case. Namely,

$$-(p_i\cdot p_f)^0=m_iE_f,\quad -(p_i\cdot p_2)^0=m_iE_e\ ,$$
$$-(p_i\cdot p_1)^0=m_iE_\nu^0\ ,$$
$$-(p_f\cdot p_2)^0=m_i(E_{e\max}-E_\nu^0)-m_e^2\ ,$$
$$(p_2\cdot p_1)^0=(p_i\cdot p_2)^0-(p_f\cdot p_2)^0+m_e^2\ ,$$

and we denoted $E_\nu^0\equiv q_{ef}^0=m_i-E_f-E_e$. Only three out of the scalar products $p_i\cdot k$, $p_f\cdot k$, $p_2\cdot k$, and $p_1\cdot k$ are independent. Most practically, $p_f\cdot k$ is eliminated when one deals with the terms $|L|^2$, $|H_i|^2$, $|D_i|^2$, $|LH_i|$, $|LD_i|$, and $|H_iD_i|$ in (3.5). Then the corresponding traces take the form of sums, whose typical terms look like $c_{n_1n_2n_i}(p_1\cdot k)^{n_1}(p_2\cdot k)^{n_2}(p_i\cdot k)^{n_i}$, where the coefficients are dependent on E_e, E_f. In the case of $|H_f|^2$, $|D_f|^2$, and $|H_fD_f|$ it is useful to eliminate $(p_2\cdot k)$, giving terms like $c_{n_1n_fn_i}(p_1\cdot k)^{n_1}(p_f\cdot k)^{n_f}(p_i\cdot k)^{n_i}$. Finally, the terms $|LH_f|, |LD_f|$ can most easily be dealt with, if they are decomposed into the form

$$c_1(p_1\cdot k)+c_2(p_1\cdot k)^2+\sum_{n_1,n_2,n_i}c_{n_1n_2n_i}(p_1\cdot k)^{n_1}(p_2\cdot k)^{n_2}(p_i\cdot k)^{n_i}(p_f\cdot k)+\sum_{n_1,n_f,n_i}c'_{n_1n_fn_i}(p_1\cdot k)^{n_1}(p_f\cdot k)^{n_f}(p_i\cdot k)^{n_i}(p_2\cdot k)\ .$$

TABLE I. Relative correction to the (E_e, E_f) distribution for (a) the $\Sigma^- \rightarrow n e \bar{\nu}$ decay, (b) the $\Sigma^- \rightarrow \Lambda e \bar{\nu}$ decay, (c) the $\Xi^- \rightarrow \Lambda e \bar{\nu}$ decay, and (d) the $\Lambda \rightarrow p e \bar{\nu}$ decay (Coulomb correction is not included).

y	$\mathcal{R}_1^c(x,y)$ or $\mathcal{R}_1'^c(x,y)$ (%)									
					(a)					
0.8067	13.5	4.8	2.6	1.1	−0.0	−1.1	−2.2	−3.5	−5.3	−8.8
0.8044	103	6.5	3.4	1.7	0.4	−0.8	−2.0	−3.5	−5.4	−9.8
0.8021	2.5	8.9	4.0	1.9	0.5	−0.8	−2.1	−3.6	−5.7	
0.7998	1.9	14.9	4.7	2.2	0.6	−0.8	−2.1	−3.7	−6.1	
0.7975	1.5	3.5	5.9	2.5	0.7	−0.8	−2.2	−3.9	−6.8	
0.7951	1.2	2.2	8.4	2.9	0.8	−0.8	−2.4	−4.3	−8.7	
0.7928	0.9	1.5	17.3	3.6	0.9	−0.9	−2.6	−4.8		
0.7905	0.6	1.0	1.9	5.0	1.1	−1.0	−2.9	−5.9		
0.7882	0.4	0.6	1.1	2.9	1.4	−1.2	−3.6			
0.7859	0.2	0.3	0.4	0.8	1.9	−1.9	−6.9			
x	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
y_{min}	0.8043	0.7979	0.7926	0.7885	0.7858	0.7847	0.7855	0.7884	0.7939	0.8024
					(b)					
0.9339	6.8	3.5	2.1	1.1	0.2	−0.7	−1.5	−2.5	−3.8	−6.3
0.9337	7.4	4.1	2.6	1.5	0.6	−0.2	−1.1	−2.2	−3.5	−7.5
0.9335	1.5	4.3	2.8	1.7	0.8	−0.1	−1.0	−2.1	−3.6	
0.9332	1.4	4.3	2.9	1.8	0.9	−0.0	−0.9	−2.1	−4.1	
0.9330	1.3	4.4	2.9	1.8	0.9	0.0	−1.0	−2.3	−5.3	
0.9328	1.2	3.2	2.9	1.9	0.9	−0.0	−1.1	−2.7		
0.9325	1.1	2.4	2.9	1.8	0.9	−0.1	−1.3	−3.5		
0.9323	1.0	2.0	5.6	1.8	0.8	−0.3	−1.8	−6.3		
0.9321	0.8	1.5	3.0	1.6	0.5	−0.7	−2.9			
0.9318	0.5	0.9	1.5	3.5	0.0	−1.9				
x	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
y_{min}	0.9336	0.9329	0.9324	0.9320	0.9318	0.9317	0.9319	0.9323	0.9328	0.9336
					(c)					
0.8558	12.4	4.6	2.5	1.1	0.0	−1.0	−2.1	−3.4	−5.1	−8.5
0.8546	60.5	6.2	3.3	1.7	0.4	−0.8	−1.9	−3.3	−5.3	−10.0
0.8534	2.5	8.1	3.8	1.9	0.5	−0.7	−2.0	−3.5	−5.6	
0.8522	1.9	12.6	4.4	2.1	0.6	−0.8	−2.1	−3.7	−6.2	
0.8510	1.4	42.9	5.3	2.3	0.6	−0.8	−2.3	−4.0	−7.1	
0.8498	1.1	2.2	7.2	2.6	0.6	−0.9	−2.5	−4.4	−10.1	
0.8486	0.8	1.5	13.0	3.1	0.7	−1.0	−2.8	−5.1		
0.8473	0.5	0.9	1.9	4.2	0.8	−.1.2	−3.2	−6.6		
0.8461	0.3	0.5	0.9	7.6	0.9	−1.5	−4.0			
0.8449	0.1	0.2	0.3	0.7	1.3	−2.3	−12.5			
x	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
y_{min}	0.8545	0.8509	0.8481	0.8460	0.8447	0.8443	0.8449	0.8446	0.8495	0.8537
					(d)					
0.8531	11.2	4.2	2.1	0.7	−0.4	−1.5	−2.6	−3.8	−5.5	−8.6
0.8518	27.3	5.7	2.9	1.3	0.1	−1.0	−2.2	−3.5	−5.2	−9.4
0.8505	2.0	7.0	3.5	1.7	0.3	−0.9	−2.1	−3.4	−5.3	
0.8493	1.7	9.1	4.0	1.9	0.5	−0.8	−2.0	−3.5	−5.7	
0.8480	1.5	12.8	4.5	2.2	0.6	−0.7	−2.0	−3.6	−6.5	
0.8467	1.3	2.7	5.3	2.4	0.7	−0.7	−2.1	−3.9	−10.0	
0.8455	1.1	2.0	6.3	2.7	0.8	−0.7	−2.2	−4.5		
0.8442	0.9	1.6	3.3	2.9	0.8	−0.8	−2.6	−6.2		
0.8429	0.6	1.1	2.0	3.2	0.7	−1.1	−3.5			
0.8417	0.4	0.6	1.0	2.0	0.3	−2.2	−11.4			
x	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
y_{min}	0.8516	0.8480	0.8450	0.8428	0.8415	0.8411	0.8417	0.8434	0.8464	0.8509

TABLE II. Relative correction to the $(E_e, \cos\theta_{e\nu})$ distribution for (a) the $\Sigma^- \to ne\bar{\nu}$ decay, (b) the $\Sigma^- \to \Lambda e\bar{\nu}$ decay, (c) the $\Xi^- \to \Lambda e\bar{\nu}$ decay, and (d) the $\Lambda \to pe\bar{\nu}$ decay (Coulomb correction is not included).

$\cos\theta_{e\nu}$	$\mathcal{R}_2^c(x,\cos\theta_{e\nu})$ (%)								
					(a)				
−0.99	21.5	6.3	2.4	0.3	−1.4	−2.9	−4.4	−6.2	−8.7
−0.80	13.5	4.4	1.5	−0.3	−1.7	−3.5	−4.5	−6.3	−8.7
−0.60	11.7	4.1	1.4	−0.3	−1.7	−3.1	−4.5	−6.2	−8.6
−0.40	11.4	4.1	1.5	−0.2	−1.6	−2.9	−4.4	−6.0	−8.4
−0.20	11.8	4.4	1.8	0.1	−1.4	−2.7	−4.1	−5.8	−8.2
0.00	12.5	4.9	2.1	0.4	−1.1	−2.4	−3.8	−5.5	−7.9
0.20	13.8	5.5	2.6	0.8	−0.7	−2.1	−3.5	−5.2	−7.6
0.40	15.8	6.5	3.3	1.4	−0.2	−1.6	−3.0	−4.7	−7.1
0.60	19.0	7.8	4.3	2.1	0.5	−0.9	−2.4	−4.1	−6.5
0.80	24.9	10.2	5.8	3.4	1.6	0.0	−1.5	−3.3	−5.7
0.90	30.6	12.4	7.1	4.4	2.4	0.8	−0.8	−2.6	−5.1
0.95	35.5	14.1	8.2	5.1	3.0	1.3	−0.4	−2.2	−4.7
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					(b)				
−0.99	7.0	2.9	1.1	−0.3	−1.6	−2.5	−3.7	−5.1	−7.2
−0.80	7.6	3.2	1.3	−0.0	−1.2	−2.3	−3.5	−4.9	−6.9
−0.60	8.3	3.5	1.6	0.2	−1.0	−2.1	−3.3	−4.7	−6.7
−0.40	9.2	4.0	1.9	0.5	−0.7	−1.8	−3.1	−4.4	−6.4
−0.20	10.3	4.5	2.3	0.9	−0.3	−1.5	−2.7	−4.1	−6.1
0.00	11.6	5.1	2.8	1.3	0.0	−1.1	−2.3	−3.7	−5.7
0.20	13.4	5.9	3.3	1.7	0.5	−0.7	−1.9	−3.3	−5.3
0.40	15.8	6.9	4.0	2.3	1.0	−0.2	−1.4	−2.9	−4.9
0.60	19.4	8.4	5.0	3.1	1.7	0.4	−0.8	−2.3	−4.3
0.80	25.7	10.8	6.5	4.3	2.7	1.3	−0.0	−1.5	−3.5
0.90	31.3	13.0	7.8	5.2	3.4	1.9	0.5	−1.0	−3.0
0.95	35.8	14.7	8.8	5.8	3.9	2.3	0.9	−0.6	−2.7
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					(c)				
−0.99	18.7	5.3	1.8	−0.1	−1.7	−3.0	−4.4	−6.1	−8.5
−0.80	11.6	3.7	1.1	−0.6	−2.0	−3.3	−4.7	−6.3	−8.6
−0.60	10.2	3.5	1.0	−0.6	−2.0	−3.3	−4.6	−6.2	−8.5
−0.40	10.0	3.6	1.2	−0.4	−1.8	−3.1	−4.5	−6.0	−8.4
−0.20	10.4	3.9	1.5	−0.2	−1.5	−2.8	−4.2	−5.8	−8.1
0.00	11.2	4.4	1.8	0.2	−1.2	−2.5	−3.9	−5.5	−7.8
0.20	12.3	5.0	2.3	0.6	−0.8	−2.1	−3.5	−5.1	−7.5
0.40	14.1	5.9	3.0	1.2	−0.3	−1.7	−3.1	−4.7	−7.0
0.60	17.0	7.1	3.9	1.9	0.4	−1.0	−2.4	−4.1	−6.4
0.80	22.3	9.3	5.4	3.1	1.4	−0.1	−1.6	−3.2	−5.6
0.90	27.4	11.4	6.6	4.1	2.2	0.7	−0.9	−2.6	−5.0
0.95	31.8	13.0	7.6	4.8	2.8	1.2	−0.4	−2.2	−4.6
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					(d)				
−0.99	12.5	4.4	1.6	−0.2	−1.8	−3.1	−4.7	−6.4	−8.9
−0.80	11.9	4.3	1.6	−0.2	−1.7	−2.8	−4.5	−6.3	−8.7
−0.60	11.8	4.4	1.7	−0.0	−1.5	−2.9	−4.4	−6.1	−8.5
−0.40	12.0	4.6	1.9	0.1	−1.3	−2.7	−4.2	−5.9	−8.3
−0.20	12.6	4.9	2.1	0.4	−1.1	−2.5	−3.9	−5.6	−8.0
0.00	13.5	5.3	2.5	0.7	−0.8	−2.2	−3.6	−5.3	−7.7
0.20	14.9	6.0	3.0	1.1	−0.4	−1.8	−3.3	−4.9	−7.3
0.40	16.9	6.9	3.6	1.6	0.0	−1.4	−2.8	−4.5	−6.9
0.60	20.2	8.2	4.5	2.3	0.7	−0.8	−2.3	−4.0	−6.3
0.80	26.2	10.6	6.0	3.5	1.7	0.1	−1.4	−3.2	−5.6

TABLE II. (*Continued*).

$\cos\theta_{e\nu}$	$\mathcal{R}_2^c(x,\cos\theta_{e\nu})$ (%) (d)								
0.90	32.0	12.8	7.3	4.5	2.5	0.8	−0.8	−2.6	−5.0
0.95	36.9	14.6	8.4	5.2	3.0	1.3	−0.4	−2.2	−4.7
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

The various choices are dictated by the denominators $(p_2 \cdot k)^2$, $(p_{i,f} \cdot k)^2$, and $(p_2 \cdot k)(p_{i,f} \cdot k)$ in (3.5). All the integrals can be analytically evaluated. In spite of this the large number of terms makes it more practical to integrate with respect to q_{ef} numerically making use of a computer, unless an infrared-divergent integral is considered. Assuming that this is not the case, e.g., because $n_1 \neq 0$, then all the k integrals can be dealt with as follows. First of all, they are of the form

$$I_{n_i n} = \int \frac{d^3\mathbf{k}}{2k_0} \frac{(p_i \cdot k)^{n_i} (p \cdot k)^n}{(u \cdot k)^2} \delta(p_1^2) , \quad (3.9)$$

where p stands for p_f or p_2. The powers of $(p_1 \cdot k)$ need not be included in the k integration, as $-p_1 \cdot k = \frac{1}{2}[(q_{ef}^0)^2 - q_{ef}^2]$. The four-vector u may stand for p_2, p_i, or p_f, and, in the most general case,

$$u = p_2 + x(ap_{\rm ch} - p_2) , \quad (3.10)$$

where $p_{\rm ch}$ stands for p_i or p_f,

$$a = \frac{1}{p_{\rm ch}^2}[(p_2 \cdot p_{\rm ch}) - \sqrt{(p_2 \cdot p_{\rm ch})^2 - p_2^2 \cdot p_{\rm ch}^2}\,] , \quad (3.11)$$

and x is a Feynman parameter introduced by the familiar relation

$$\frac{1}{(p_2 \cdot k)(p_{\rm ch} \cdot k)} = a \int_0^1 dx \frac{1}{(u \cdot k)^2} . \quad (3.12)$$

The integral (3.9) is Lorentz invariant, so that a transformation $(k_0, \mathbf{k}) \to (k_0^u, \mathbf{k}^u)$ can be made into the "rest system of u," meaning that $(u \cdot k)^2 = -u^2 (k_0^u)^2$, where

$$u^2 = p_2^2 - 2xa\sqrt{(p_2 \cdot p_{\rm ch})^2 - p_2^2 p_{\rm ch}^2} . \quad (3.13)$$

It is convenient to parametrize the integration volume d^3k^u in terms of the magnitude of $\mathbf{k}^u$, $|\mathbf{k}^u|$, that component of $\mathbf{k}^u$, $k_\parallel^u$, which is parallel to the three-vector $(\mathbf{p}_i - \mathbf{p}_f - \mathbf{p}_2)^u$, and of an azimuthal angle ϕ_k, which is measured between those components of $\mathbf{k}^u$ and $\mathbf{p}_i^u$ (or $\mathbf{p}_2^u$), which are orthogonal to $(\mathbf{p}_i - \mathbf{p}_f - \mathbf{p}_2)^u$:

$$d^3\mathbf{k}^\mu = |\mathbf{k}^u| d|\mathbf{k}^u| dk_\parallel^u d\phi_k . \quad (3.14)$$

With this parametrization the integration over $0 \le \phi_k \le 2\pi$ becomes easy, and the Dirac-δ function $\delta(p_1^2)$ simplifies to

$$\delta(p_1^2) = \frac{1}{2q_{ef}^u} \delta(k_\parallel^u - k_{\parallel ef}^u) , \quad (3.15)$$

where

$$q_{ef}^u = |(\mathbf{p}_i - \mathbf{p}_f - \mathbf{p}_2)^u| \quad (3.16)$$

and

$$k_{\parallel ef}^u = \frac{1}{q_{ef}^u}[-(p_1 \cdot k) - (p_i - p_f - p_2)_0^u |\mathbf{k}_u|] . \quad (3.17)$$

Finally, the range of $|\mathbf{k}^u|$ is

$$k_{\rm min}^u \le |\mathbf{k}^u| \le k_{\rm max}^u , \quad (3.18)$$

$$k_{\rm max,min}^u = \tfrac{1}{2}[(p_i - p_f - p_2)_0^u \pm q_{ef}^u] . \quad (3.19)$$

Notice that when $n = n_i = 0$ and $n_1 \neq 0$ one obtains

$$I_{00} = \frac{\pi}{u^2 (p_1 \cdot k)} . \quad (3.20)$$

The Feynman parametrization (3.13) is not needed, when either $n_i \neq 0$ or $n \neq 0$ in (3.9). In such cases the above discussion refers to $u = p_2$ or $u = p_{\rm ch}$. Let us consider, e.g., the case $u = p_2$. Then, in the nominator of (3.9) p stands for p_2 and $(p_2 \cdot k) = -m_e |\mathbf{k}^u|$ by definition. In the rest frame of $u = p_2$ the scalar product $(p_i \cdot k)$ takes the form

$$-(p_i \cdot k) = p_{i0}^u |\mathbf{k}^u| = p_{i\perp}^u k_\perp^u \sin\phi_k - p_{i\parallel}^u k_\parallel^u , \quad (3.21)$$

with

$$p_{i0}^u = m_i \frac{E_e}{m_e} , \quad (3.22)$$

$$p_{i\parallel}^u = \frac{1}{q_{ef}^u} \left[-(p_i \cdot p_1)^0 - (p_i \cdot p_2)^0 \frac{(p_1 \cdot p_2)^0 - (p_1 \cdot k)}{m_e^2} \right] , \quad (3.23)$$

and the transverse components are

TABLE III. Relative correction to the electron energy spectrum in %.

x	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
$\Sigma^- \to ne\bar{\nu}$	48.7	18.3	7.1	3.6	1.4	−0.2	−1.7	−3.2	−5.0	−7.5	−9.7
$\Sigma^- \to \Lambda e\bar{\nu}$	33.5	13.5	5.7	3.0	1.4	0.1	−1.1	−2.4	−3.8	−5.9	−7.7
$\Xi^- \to \Lambda e\bar{\nu}$	42.5	16.3	6.5	3.3	1.3	−0.3	−1.7	−3.2	−4.8	−7.3	−9.3
$\Lambda \to pe\bar{\nu}$	48.4	18.2	7.1	3.5	1.4	−0.3	−1.8	−3.4	−5.2	−7.7	−9.8

TABLE IV. Relative correction to the electron energy spectrum of the $n \to pe\bar{\nu}$ decay in % [upper row: our calculation; lower row: correction with the Sirlin function (Ref. 10)].

x	0.4	0.5	0.6	0.7	0.8	0.9	0.95
$n \to pe\bar{\nu}$	1.874	1.764	1.600	1.390	1.121	0.732	0.408
Sirlin function	1.876	1.766	1.604	1.395	1.127	0.741	0.419

$$k_\perp^u = \sqrt{(\mathbf{k}^u)^2-(k_\parallel^u)^2}\ , \quad (3.24)$$

$$p_{i\perp}^u = \sqrt{(P_{i0}^u)^2-m_i^2-(p_{i\parallel}^u)^2}\ . \quad (3.25)$$

In the actual calculations $(p_i \cdot k)$ and $(p_i \cdot k)^2$ appear, producing a number of terms, which can be dealt with by means of REDUCE algebraic programs. All the integrations over $\mathbf{k}^u$ can easily be done analytically, and the resulting functions of q_{ef} can be inserted by means of the substitution algorithms of REDUCE. Finally, the quantities q_{ef}^u and $(p_i-p_f-p_2)_0^u$ are related to q_{ef} as (in the general case)

$$[(p_i-p_f-p_2)_0^u]^2-(q_{ef}^u)^2=-2(p_1\cdot k)\ , \quad (3.26)$$

$$(p_i-p_f-p_2)_0^u=E_i^u-E_f^u-E_2^u\ , \quad (3.27)$$

$$E_j^u=\gamma\left[E_j-\frac{\mathbf{p}_j\mathbf{u}}{u_0}\right]\quad (j=i,f,2)\ ,$$

$$\gamma=\frac{u_0}{\sqrt{-u^2}},\quad u=(u_0,\mathbf{u}),\quad E_2\equiv E_e\ .$$

The above procedure results in a function of q_{ef}, which is a remarkably long expression. But it is smooth enough and can easily be integrated numerically. Additional complications appear only when $n_1=n_i=n=0$, and the integral with respect to q_{ef} is divergent. For these integrals the standard recipe is to introduce a small photon mass λ, giving the photon energy $k_0^u=\sqrt{(\mathbf{k}^u)^2+\lambda^2}$. We give a representation for the most general infrared-divergent integral in semileptonic processes. All the others are special cases, and can be derived from that. Let us introduce the notation

$$F(q_{ef})=\frac{1}{\sqrt{(p_{\mathrm{ch}}\cdot p_2)^2-m_{\mathrm{ch}}^2m_e^2}}\ln\frac{\sqrt{(p_{\mathrm{ch}}\cdot p_2)^2-m_{\mathrm{ch}}^2m_e^2}-(p_{\mathrm{ch}}\cdot p_2)}{\sqrt{(p_{\mathrm{ch}}\cdot p_2)^2-m_{\mathrm{ch}}^2m_e^2}+(p_{\mathrm{ch}}\cdot p_2)}\ , \quad (3.28)$$

$$-(p_1\cdot k)_{\min}=\tfrac{1}{2}[(q_{ef}^0)^2-(|\mathbf{p}_f|-|\mathbf{p}_2|)^2]\ , \quad (3.29)$$

and

$$-q_{ef}^{u0}=\frac{(u\cdot p_1)^0}{\sqrt{(-u^2)^0}}\ , \quad (3.30)$$

where the definition of u is the same, as in (3.10), and the superscript zero means that the scalar products are to be evaluated at $q_{\mathrm{ef}}=q_{\mathrm{ef}}^0$. Straightforward calculation gives that

$$\int_{||\mathbf{p}_f|-|\mathbf{p}_2||}^{q_{ef}^0}2q_{ef}dq_{ef}\int\frac{d^3k}{2k_0}\frac{1}{(p_2\cdot k)(p_{\mathrm{ch}}\cdot k)}\delta(p_1^2)=\pi\left[F(q_{ef}^0)\left[\ln\frac{q_{ef}^0}{\lambda}+\ln\frac{-(p_1\cdot k)_{\min}}{(q_{ef}^0)^2}\right]+\int_0^1dx\frac{2a^0}{(-u^2)^0}\ln\frac{q_{ef}^0}{q_{ef}^{u0}}\right.$$
$$\left.+\int_{||\mathbf{p}_f|-|\mathbf{p}_2||}^{q_{ef}^0}dq_{ef}\left[\ln\frac{(p_1\cdot k)}{(p_1\cdot k)_{\min}}\right]\frac{d}{dq_{ef}}F(q_{ef})\right]\ . \quad (3.31)$$

In Eqs. (3.28)–(3.31) by p_{ch} we mean $p_{\mathrm{ch}}=p_f$ or $p_{\mathrm{ch}}=p_i$, and $p_{\mathrm{ch}}^2=-m_{\mathrm{ch}}^2$. Equation (3.31) completes the list of inputs, which are needed for the numerical evaluation of the (E_e,E_f) Dalitz distribution for radiative decays of the type $i\to fe\bar{\nu}\gamma$.

Another distribution, which is relevant for experiments, is the one $(E_e,\cos\theta_{e\nu})$, where $\theta_{e\nu}$ is usually called "the angle between the electron and the antineutrino." It

TABLE V. Relative correction to the hadron energy spectrum in % $[E_f(z)=m_f+(E_{f\max}^*-m_f)z$, $E_{f\max}^*=E_{f\max}(E_{e\max})]$.

z	0.1	0.3	0.5	0.7	0.8	0.9	0.95
$\Sigma^-\to ne\bar{\nu}$	0.0	0.0	0.1	0.0	−0.0	−0.2	−0.5
$\Sigma^-\to\Lambda e\bar{\nu}$	0.3	0.3	0.2	0.2	0.1	−0.0	−0.3
$\Xi^-\to\Lambda e\bar{\nu}$	0.2	0.2	0.2	0.1	0.1	−0.1	−0.4
$\Lambda\to pe\bar{\nu}$	−0.2	−0.2	−0.3	−0.3	−0.4	−0.6	−0.9
$n\to pe\bar{\nu}$	1.62	1.60	1.55	1.44	1.30	0.98	0.62

TABLE VI. Relative correction to the $\cos\theta_{e\nu}$ distribution in %.

$\cos\theta_{e\nu}$	−0.99	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	0.9	0.95
$\Sigma^- \to n e \bar{\nu}$	−4.1	−3.6	−3.2	−2.8	−2.3	−1.8	−1.3	−0.6	0.3	1.8	3.1	4.4
$\Sigma^- \to \Lambda e \bar{\nu}$	−2.0	−1.8	−1.5	−1.2	−0.8	−0.4	0.1	0.7	1.5	2.7	3.8	4.7
$\Sigma^- \to n e \bar{\nu}$	−4.0	−3.4	−3.0	−2.6	−2.2	−1.7	−1.2	−0.6	0.3	1.7	3.0	4.1
$\Lambda \to p e \bar{\nu}$	−3.0	−2.6	−2.3	−1.9	−1.5	−1.0	−0.4	0.3	1.2	2.6	3.7	4.6

is important to notice that in hyperon decay experiments $\cos\theta_{e\nu}$ is determined from the relation

$$\cos\theta_{e\nu} = -\frac{\mathbf{p}_e\cdot(\mathbf{p}_e+\mathbf{p}_f)}{|\mathbf{p}_e||\mathbf{p}_e+\mathbf{p}_f|} \tag{3.32}$$

and this is different from the quantity $\mathbf{p}_\nu\cdot\mathbf{p}_e/|\mathbf{p}_\nu|\cdot|\mathbf{p}_e|$ for the radiative events. The $(E_e,\cos\theta_{e\nu})$ distribution with the definition (3.32) can be calculated following the main steps of the calculation in the (E_e,E_f) case. Let us introduce the notation

$$c=\cos\theta_{e\nu}\ , \tag{3.33}$$

$$E_f^* = \sqrt{m_f^2+\mathbf{p}_e^2}\ , \tag{3.34}$$

$$E_f' = \sqrt{m_f^2+\mathbf{p}_e^2-\mathbf{p}_e^2 c^2}\ , \tag{3.35}$$

$$E_f^0 = m_i - E_e - \frac{m_i(E_{e\max}-E_e)}{m_i-E_e+|\mathbf{p}_e|c}\ , \tag{3.36}$$

$$q_{ef}^{\pm} = -|\mathbf{p}_e|c \pm \sqrt{\mathbf{p}_f^2-\mathbf{p}_e^2+\mathbf{p}_e^2c^2}\ , \tag{3.37}$$

$$C(E_e,E_f,q_{ef}) = \frac{1}{2|\mathbf{p}_e|}\left[\frac{\mathbf{p}_f^2-\mathbf{p}_e^2}{q_{ef}} - q_{ef}\right]\ , \tag{3.38}$$

$$c_b = \begin{cases} -1 \ \text{ if } E_e < E'_{e\max}\ , \\ C(E_e,E_{fb},q_{ef}^0(E_e,E_{fb})) \ \text{ if } E_e > E'_{e\max}\ , \end{cases} \tag{3.39}$$

where E_{fb} is that solution of the equation

$$\frac{\partial C}{\partial q_{ef}}(E_e,E_{fb},q_{ef}^0(E_e,E_{fb}))=0\ , \tag{3.40}$$

which is greater than m_f.

We denote by $\omega_{\rm br}(E_e,E_f,q_{ef})$ the bremsstrahlung part of the radiative correction to the (E_e,E_f,q_{ef}) distribution. The bremsstrahlung part of the correction to the (E_e,c) distribution can be expressed by the general $I(y_1,y_2,q)$ integral

$$I(Y_1,y_2,q) = \int_{y_1}^{y_2} dE_f\,\omega_{\rm br}(E_e,E_f,q)\left|\frac{\partial q}{\partial c}\right|\ . \tag{3.41}$$

We have three different cases according to the value of c.

(1) $c>0$

$$W_{\rm br}(E_e,c) = I(E_f^*,E_f^0,q_{ef}^+)\ ; \tag{3.42}$$

(2) $c_b<c<0$

$$W_{\rm br}(E_e,c) = I(E_f'<E_f^0,q_{ef}^+) + I(E_f',E_f^*,q_{ef}^-)\ ; \tag{3.43}$$

(3) $c<c_b$

$$W_{\rm br}(E_e,c) = I(E_f^0,E_f^*,q_{ef}^-)\ . \tag{3.44}$$

The above procedure results in an expression which is particularly suitable for numerical integration by means of computer. Results corresponding to the theoretical input described at the beginning of this section can be obtained with high precision, which far exceeds that of the theoretical input itself.

IV. RESULTS

In the theoretical framework, which is outlined in Secs. II and III we carried out numerical calculations of the radiative correction to various quantities for semileptonic baryon decays. In the course of these calculations the hadronic weak vertex function H^μ was restricted to the expression

$$H^\mu = \begin{bmatrix}\sin\Theta_C\\ \cos\Theta_C\end{bmatrix}\left[\gamma^\mu(f_1-\gamma^5 g_1) - \frac{i}{m_i}\sigma^{\mu\nu}q^\nu f_2\right]\ , \tag{4.1}$$

where Θ_C is the Cabibbo angle, the sine or cosine of which is meant in (4.1) depending on whether we consider a $|\Delta S|=1$ or $|\Delta S|=0$ transition, respectively. The symbols f_1, f_2, and g_1 denote the zero-momentum-transfer value of the form factors. When their numerical value was used in our calculation we accepted the conserved vector-current (CVC) prediction for f_2/f_1, and

TABLE VII. The $A_+^0(ij)$ zeroth-order coefficients of the six form-factor combinations for the decay rate. ($R\approx 0$: $m_i \approx m_f$, $m_e \ll m_i - m_f$.)

	f_1^2	g_1^2	f_1g_1	f_1f_2	f_2g_1	f_2^2
$R\approx 0$	100	300	0	0	0	0
$\Sigma^- \to n e \bar{\nu}$	126.4	372.8	0.0	5.6	0.0	3.3
$\Sigma^- \to \Lambda e \bar{\nu}$	107.2	321.2	0.0	0.4	0.0	0.3
$\Xi^- \to \Lambda e \bar{\nu}$	117.9	351.0	0.0	2.6	0.0	1.6
$\Lambda \to p e \bar{\nu}$	118.4	352.1	0.0	2.8	0.0	1.7
$n \to p e \bar{\nu}$	47.3	142.0	0.0	0.0	0.0	0.0

TABLE VIII. Radiative corrections of the $A_+^0(ij)$ coefficients and the $\mathcal{R}_\sigma$ relative corrections.

	f_1^2	g_1^2	f_1g_1	f_1f_2	f_2g_1	f_2^2	$\mathcal{R}_\sigma$ (%)
$\Sigma^-\to ne\bar{\nu}$	−0.2	−0.5	0.0	−0.0	0.1	−0.0	−0.2
$\Sigma^-\to\Lambda e\bar{\nu}$	0.2	0.6	0.0	0.0	0.0	0.0	0.2
$\Xi^-\to\Lambda e\bar{\nu}$	−0.1	−0.2	0.0	−0.0	0.1	−0.0	−0.1
$\Lambda\to pe\bar{\nu}$	−0.5	−1.4	0.0	−0.0	0.1	−0.0	−0.4
$n\to pe\bar{\nu}$	0.7	2.1	0.0	0.0	0.0	0.0	1.5

for g_1/f_1 we used the result of the WA2 experiment at CERN.[2] These form-factor ratios can be found in Table I of Ref. 12.

In the tables which summarize our results, most of the numbers give "relative correction," that is, an order-α result divided by the appropriate zeroth-order quantity. By this division, on the one hand, we achieve a kind of normalization of the numbers. On the other hand, we get an idea on the size of the order-α contribution in comparison with the lowest-order quantities. For these tentative purposes we also calculated the zeroth-order quantities with (4.1), i.e., with constant form factors. (This should not give the reader the impression that we forgot about the well-known fact, that, in general, the momentum-transfer dependence of the form factors has a significant effect in zeroth order.) In addition, all our results for the $\Lambda\to pe\bar{\nu}$ and the $n\to pe\bar{\nu}$ decays in this paper are presented without the Coulomb correction term.

We carried out the calculation for two types of two-variable distributions: for the (E_e,E_f) Dalitz distribution and for the $(E_e,\cos\theta_{e\nu})$ distribution. In Tables I(a)–I(d) the reader finds the percentage value of the relative correction to the two-variable distribution $d^2\sigma_1^c/dE_e dE_f$:

$$\mathcal{R}_1^c(x,y)=\frac{\mathcal{A}_1^c(x,y)-\mathcal{A}_1^0(x,y)}{\mathcal{A}_1^0(x,y)}\times 100\ , \tag{4.2}$$

where $\mathcal{A}_1^c(x,y)$ is defined as

$$\frac{d^2\sigma_1^c}{dE_e dE_f}=G_F^2 f_1^2\begin{bmatrix}\sin^2\Theta_C\\ \cos^2\Theta_C\end{bmatrix}C_0\mathcal{A}_1^c(x,y)\ . \tag{4.3}$$

The upper label c indicates that radiative correction is included. $\mathcal{A}_1^0(x,y)$ is the counterpart of $\mathcal{A}_1^c(x,y)$ in zeroth order. In (4.2) and (4.3) we use the notations

$$x=\frac{E_e}{E_{e\max}},\quad y=\frac{E_f}{m_i}\ ,$$

$$C_0=\frac{m_i^5R^5}{6000\pi^3},\quad R=\frac{E_{e\max}}{m_i}\ .$$

As it was already mentioned in Sec. III, radiative decay events appear with E_f, which is outside the Dalitz plot $\mathcal{D}_0$ of the zeroth-order decay events. In such cases our tables contain, instead of (4.2), the quantity defined as

$$\mathcal{R}_1'^c(x,y)=\frac{\mathcal{A}_1^c(x,y)}{\bar{\mathcal{A}}_1^0}\times 100\ , \tag{4.2'}$$

where $\bar{\mathcal{A}}_1^0$ is the average value of $\mathcal{A}_1^0(x,y)$ over $\mathcal{D}_0$:

$$\bar{\mathcal{A}}_1^0=\frac{\int_{\mathcal{D}_0}dx\,dy\,\mathcal{A}_1^0(x,y)}{\int_{\mathcal{D}_0}dx\,dy}\ .$$

To help orientation we give the values of y on the lower boundary of $\mathcal{D}_0$ ($y_{\min}$). In certain points our Tables I(a)–I(d) and Table IV of Ref. 12 are directly comparable. The agreement for the lower half of the E_e interval is impressive. A minor difference between the two sets of results for the higher values of E_e is due to the slow convergence of the integration in the case of the calculation published in Ref. 12. (The same problem appears, when the tables giving the one-dimensional E_e spectrum are compared.)

Tables II(a)–II(d) refer to the radiative correction to the distribution $d^2\sigma_2^c/dE_e d(\cos\theta_{c\nu})$. We tabulated the values of

$$\mathcal{R}_2^c(x,\cos\theta_{E\nu})=\frac{\mathcal{A}_2^c(x,\cos\theta_{e\nu})-\mathcal{A}_2^0(x,\cos\theta_{e\nu})}{\mathcal{A}_2^0(x,\cos\theta_{e\nu})}\times 100\ , \tag{4.4}$$

where we define

$$\frac{d^2\sigma_2^c}{dE_e d(\cos\theta_{e\nu})}=G_F^2 f_1^2\begin{bmatrix}\sin^2\Theta_C\\ \cos^2\Theta_C\end{bmatrix}C_0\mathcal{A}_2^c(x,\cos\theta_{e\nu})\ . \tag{4.5}$$

This kind of distribution was already studied in Ref. 16 for the decay $\Lambda\to pe\bar{\nu}$. Table II(d) is included in this paper only, because in Ref. 16 we calculated the zeroth-order distribution with q^2-dependent form factors, and

TABLE IX. The $A_-^0(ij)$ zeroth-order coefficients.

	f_1^2	g_1^2	f_1g_1	f_1f_2	f_2g_1	f_2^2
$R\approx 0$	100	−100	0	0	0	0
$\Sigma^-\to ne\bar{\nu}$	94.0	−205.9	0.0	−3.5	0.0	−2.6
$\Sigma^-\to\Lambda e\bar{\nu}$	99.5	−129.2	0.0	−0.2	0.0	−0.2
$\Xi^-\to\Lambda e\bar{\nu}$	97.1	−172.4	0.0	−1.5	0.0	−1.1
$\Lambda\to pe\bar{\nu}$	97.0	−174.2	0.0	−1.5	0.0	−1.2
$n\to pe\bar{\nu}$	34.5	−34.8	0.0	0.0	0.1	0.0

TABLE X. Radiative corrections of the $A^0_-(ij)$ zeroth-order coefficients.

	f_1^2	g_1^2	f_1g_1	f_2f_2	f_2g_1	f_2^2
$\Sigma^-\to ne\bar{\nu}$	3.2	13.4	−0.7	0.3	−1.4	0.2
$\Sigma^-\to\Lambda e\bar{\nu}$	2.0	8.2	−0.2	0.0	−0.3	0.0
$\Xi^-\to\Lambda e\bar{\nu}$	2.8	11.9	−0.5	0.1	−0.9	0.1
$\Lambda\to pe\bar{\nu}$	4.5	7.9	−0.5	0.1	−0.9	0.1
$n\to pe\bar{\nu}$	1.7	−1.6	0.0	0.0	0.0	0.0

the Coulomb correction was also included there. We emphasize that our definition of $\cos\theta_{e\nu}$ respects the fact that experimentally it is usually determined from the momentum of the electron and the baryon in the final state. This explains that our results are significantly different from other results,[13–15] which refer to the theoretical case, when $\cos\theta_{e\nu}$ is determined from the momentum of the electron and the antineutrino.

In Tables III–VI relative corrections to one-variable distributions, namely, to the E_e and E_f spectra, and the $\cos\theta_{e\nu}$ angular distribution, are given. The small corrections to the E_f spectrum are worth mentioning. It reflects that the $m_e\to 0$ mass singularity disappears when $d^2\sigma_1^c/dE_e dE_f$ is integrated over E_e (Ref. 25).

Our last tables contain completely integrated quantities. Denoting $h_1=1$, $h_2=f_2/f_1$, and $h_3=g_1/f_1$ one can decompose $\mathcal{A}_2^c(x,\cos\theta_{e\nu})$ into the form

$$\mathcal{A}_2^c(x,\cos\theta_{e\nu})=\sum_{i\le j=1}^{3}\mathcal{A}_{ij}^c(x,\cos\theta_{e\nu})h_ih_j\ , \qquad (4.6)$$

and one can define

$$A_+^c(ij)=\int dE_e\int_{-1}^{1}d(\cos\theta_{e\nu})\mathcal{A}_{ij}^c(x,\cos\theta_{e\nu})\ , \qquad (4.7)$$

and

$$A_-^c(ij)=2\int dE_e\int_0^1 d(\cos\theta_{e\nu})[\mathcal{A}_{ij}^c(x,\cos\theta_{e\nu})-\mathcal{A}_{ij}^c(x,-\cos\theta_{e\nu})]\ . \qquad (4.8)$$

Similar definitions can also be made for the zeroth-order case. Our Tables VII–XI contain $A_\pm^0(ij)$, the radiative corrections to them, $A_\pm^c(ij)-A_\pm^0(ij)$, the relative correction to the decay rate σ,

$$\mathcal{R}_\sigma=\frac{\sum_{i\le j=1}^{3}[A_+^c(ij)-A_+^0(ij)]h_ih_j}{\sum_{i\le j=1}^{3}A_+^0(ij)h_ih_j}\times 100\ , \qquad (4.9)$$

and to the electron-antineutrino correlation parameter $\alpha_{e\nu}$,

$$\mathcal{R}_{e\nu}=\frac{\alpha_{e\nu}^c-\alpha_{e\nu}^0}{|\alpha_{e\nu}^0|}\times 100\ , \qquad (4.10)$$

where

$$\alpha_{e\nu}^0=\frac{\sum_{i\le j=1}^{3}A_-^0(ij)h_ih_j}{\sum_{i\le j=1}^{3}A_+^0(ij)h_ih_j}\ ,$$

$$\alpha_{e\nu}^c=\frac{\sum_{i\le j=1}^{3}A_-^c(ij)h_ih_j}{\sum_{i\le j=1}^{3}A_+^c(ij)h_ih_j}\ .$$

In Tables VII and IX we included the limiting case, when $R\approx 0$, that is, when $m_i\approx m_f$, $m_e\ll m_i-m_f$. It is noteworthy that our results for $A_+^0(ij)$ in the case of the neutron decay are remarkably different from those obtained in the limit $R\approx 0$. This contradicts the tables of the classic paper by Linke.[23]

We conclude the paper with some information about tests we made to check the reliability of our calculations. First of all, most of our results were obtained from two independent calculations. The REDUCE programs for the trace calculations were checked by substituting numerical values for the kinematical variables and parameters in the input expressions as well as in the program outputs. The numerical integrations were carried out by two different

TABLE XI. Relative corrections of the $\alpha_{e\nu}^0$ zeroth-order electron-antineutrino correlation parameters.

	$\Sigma^-\to ne\bar{\nu}$	$\Sigma^-\to\Lambda e\bar{\nu}$	$\Xi^-\to\Lambda e\bar{\nu}$	$\Lambda\to pe\bar{\nu}$	$n\to pe\bar{\nu}$
$\alpha_{e\nu}^0$	0.467	−0.403	0.620	0.031	−0.072
$\mathcal{R}_{e\nu}$ (%)	6.3	6.9	4.2	98.3	1.0

methods. In one version of the calculations the DIVON program of Friedman and Wright was used.[26] The other version made use of a multidimensional integration routine developed by one of us (F.G.). A result was accepted when at least the first two digits coincided in both versions.

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