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Gauge-independent bifurcation to the chiral-symmetry-breaking solution of the Dyson-Schwinger equation in continuum QED

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The Dyson-Schwinger equation for the fermion propagator can be effectively solved in the approximation of the small-momentum-transfer vertex function. There exists a critical value of the coupling constant above which the ordinary infrared-divergent solution for massless quantum electrodynamics bifurcates to another, massive solution. With a proper transverse part included in the vertex function, the bifurcation point is gauge independent, the new solution is finite in all gauges, and does not require momentum cutoffs of any kind.

I. INTRODUCTION

Recent numerical results in lattice quantum field theory[1] brought new evidence that quantum electrodynamics may possess a second, chiral-symmetry-breaking phase. A discovery of such phase would greatly enlarge the family of theories that exhibit physically relevant dynamical features such as symmetry breaking, anomalous scaling, etc.

The problem of dynamical chiral-symmetry breaking was also studied using analytical methods. The search for novel solutions of the Dyson-Schwinger (DS) equation for the fermion propagator was initiated by Johnson, Baker, and Willey[2] (BJW). In their method, the infinite hierarchy of DS equations was broken with use of the so-called truncation procedure in which the full vertex of the theory and the gluon propagator were replaced by their respective bare values. In this contribution we shall use a different method of truncation and therefore we shall refer to the BJW method as the constant vertex approximation.

In their pioneering work, Masakawa and Nakajima[3] found that, in the constant vertex expansion, the chiral symmetry of massless electrodynamics is broken. These results were derived in the so-called Landau-like gauge which provides an effective infrared (IR) cutoff by adding a small mass term to the photon propagator. In addition to this cutoff, the standard Pauli-Villars-Rayski regulator had to be employed in order to eliminate the ultraviolet (UV) divergence produced by the photon loop in the truncated DS equation.

Further progress in the study of the symmetry-breaking mechanism directly from the DS equations was hindered by the realization of the fact that the constant vertex procedure was gauge dependent. A generalization of the method beyond the Landau gauge produced off sheet poles and other inconsistencies.[4] The source of troubles can be directly traced to the substitution of the constant vertex in the DS equation. The Ward identity for the vertex function,

$$(p-q)^{\mu}\Gamma_{\mu}(q,p)=S^{-1}(q-p)$$
$$=\not{p}A(p)-\not{q}A(q)-B(p)+B(q)\ , \qquad (1.1)$$

implies that, if $\Gamma_{\mu}\equiv\gamma_{\mu}$, then $A(k)$ must be identically equal to unity. It is a peculiarity of the constant vertex approximation that the function $A(k)$ obtains nontrivial radiative corrections in all but the Landau gauges. Therefore in non-Landau gauges the approximation cannot be properly renormalized. The standard relations between the renormalization constants are a consequence of the Ward identity. If these relations are imposed in a non-Landau gauge, the coupling constant vanishes in the limit of infinite cutoff; to eliminate all radiative corrections and conform to the identity (1.1) which in non-Landau gauges is satisfied only by the free theory solution.

One possible way of restoring the consistency of the approximation is to add supplemental contributions to the vertex function in the DS equation in order to preserve the Ward identity. The method of Delburgo and West,[5,6] inspired by Salam's seasoned gauge technique, is an excellent example of such a procedure.

The Ward identity imposes no constraints on the transverse part of the vertex. Therefore, for the sake of simplicity, the transverse part was typically ignored by virtually all authors. Such simplification jeopardizes consistency, as a transverse component is needed for proper renormalization of the DS equation.[6,7]

Miransky and co-workers[8] have argued (within the constant vertex model with cutoffs) that symmetry breaking can occur in massless fermion theories with the charge parameter above a certain minimal value. In a series of recent papers, Atkinson and Johnson[9] have reexamined this problem restating it mathematically as a bifurcation problem for the DS equation. Symmetry breakdown is then indicated by the existence of a positive critical value of the coupling above which the chirally invariant solution bifurcates to a nontrivial solution that generates a fermion mass term. Atkinson and Johnson examined several regularization cutoff procedures with, and without, a momentum-dependent running coupling constant. They were able to prove the existence of such critical values, but again, like in all previous attempts, the argument explicitly demanded IR and UV momentum cutoffs in the DS equation.

Unfortunately, introduction of external mass scale parameters in the context of the problem of chiral-symmetry breaking is particularly risky since one can never be certain whether the fermion mass is an artifact of a hidden transmutation of the scale parameter or a truly dynamical phenomenon.

The results of Refs. 8 and 9 are also gauge dependent. Gauge covariance may be improved by inclusion of a transverse part of the vertex.[10] The prescription for the transverse part proposed in Ref. 10 assures gauge independence to first order in the coupling constant. This is encouraging, but further expansion in terms of higher powers of the charge parameter would be necessary to eliminate the gauge dependence entirely. Such infinite series expansion is incompatible with a nonperturbative method designed to unravel a phenomenon taking place above a (possibly large) critical value of the coupling.

In our previous paper[11] we discussed the solutions of the DS equation in an expansion scheme based on the observation that the dominant contribution to the propagator equation comes from the vicinity of the photon propagator pole, i.e., the vertex function is dominated by zero-momentum transfer contributions,

$$\Gamma^{\mu}(k,p)\approx\Gamma^{\mu}(p,p) \ . \tag{1.2}$$

The principal advantage of this prescription is that it produces a nontrivial, momentum-dependent longitudinal component of the vertex function. The prescription (1.2) must not be regarded as a definition of the entire vertex function. This would violate Pauli antisymmetry under fermion exchange. The approximation is performed on integrands by effectively replacing photon propagators $D^{\mu\nu}(p^2)$ by $D^{\mu\nu}(p^2)\delta(p^2)$. For the one-loop fermion self-energy diagram this is formally equivalent to (1.2). In more complex graphs different combinations of extremal momenta are routed through the photon lines and the prescription (1.2) does not apply.

In our case the Ward identity implies that

$$\Gamma^{\mu}(p,p)=\partial S^{-1}(p)/\partial p_{\mu}=\partial[\not{p}A(p)-B(p)]/\partial p_{\mu} \ . \tag{1.3}$$

For massive fermions, the prescription (1.3) produces the DS equations which are UV finite in the Landau gauge,[12] and have a structure simple enough to permit their use in the analysis of fermionic bound states and dynamical symmetry breaking.[13]

In Ref. 11 the solutions to the DS equation linearized in the mass function $B(p^2)$ in the propagator were discussed. In the Landau gauge, the exact solution to the equation for the factor $A(p^2)$ of the propagator bifurcates at

$$(e_{\text{crit}}/4\pi)^2=2.39 \ . \tag{1.4}$$

For weaker couplings the boundary conditions for this solution cannot be imposed at zero or infinity, although they can be satisfied at finite values of momenta. Therefore momentum cutoffs need to be imposed in IR and UV regions. This indicates that the propagator remains massless and the symmetry is preserved.

At the bifurcation point an entirely different solution appears. The function $A(p^2)$ corresponding to this solution is very regular and the DS equation does not require any cutoffs. The equation for $B(p^2)$ produces nontrivial solutions which also satisfy their boundary conditions without cutoffs. There is no externally introduced dimensional parameters and therefore the mass pole is dynamically generated.

The scope of Ref. 11 was restricted to the Landau gauge only. In non-Landau gauges, the DS integrals include a logarithmically divergent term.

Conventional renormalization techniques cannot help at this juncture. Our procedure produces a new, essentially nonperturbative solution which defines the first (and hopefully the leading) approximation of the theory with broken symmetry and massive fermions. Contrary to what one encounters in perturbative calculations, now there is no underlying free theory of identical structure, and endowed with a sufficient number of adjustable parameters (mass included) able to absorb the loop divergencies. The only permissible renormalization operations comprise of finite, multiplicative transformations of the coupling constant and wave functions. Such normalizations alone cannot make the model finite nor cure the problem of gauge dependence of the dynamical mass.

There is still one avenue left open. The divergencies and gauge dependence may be eliminated not by counterterms to the existing parameters but by extra terms generated by a transverse part of the vertex function. The transverse part must produce terms of the same structure as the longitudinal part, otherwise it will introduce its own divergencies and produce more gauge-dependent terms. For this reason, the transverse vertex must be linear in $A(p^2)$, $B(p^2)$, and their derivatives.

This paper presents a simple and unique prescription for the proper transverse part which eliminates the divergence producing terms in the integrands of the propagator equations. New equations still depend on the gauge parameter G, but in a much simpler way. A common, G-dependent term appears together with the coupling constant as a multiplier in front of all integrals in the equations for $A(p)$ and $B(p)$. This term can be incorporated in the redefined charge parameter. The bifurcation condition determines the critical value of that redefined coupling and therefore variations of gauge parameter cannot affect the location of fermion's mass pole.

II. THE PROPAGATOR EQUATION IN GENERAL GAUGE

The Dyson-Schwinger equation for the fermion propagator has the form

$$\begin{aligned} S^{-1}(p)&=A(p)\not{p}-B(p)\\ &=\not{p}-m_0-\Sigma(p)\\ &=\not{p}+ie^2\int d\!\!\!{}^-q\ \Gamma_{\mu}(p,q)S(q)\Gamma^0{}_{\nu}D^{\mu\nu}(p-q) \ , \end{aligned} \tag{2.1}$$

where the integration measure $đq \equiv d^4q/(2\pi)^4$, and the ordinary fundamental vertex $\Gamma^0{}_\nu$ of the theory is determined by the bare Lagrangian, e.g., for QED $\Gamma^0{}_\nu = \gamma_\nu$, and the function $\Gamma_\mu(p,q)$ represents the full vertex function. The longitudinal part of Γ^μ is effectively approximated by

$$\Gamma_L{}^\mu(k,p) = \Gamma_L{}^\mu(p,p) = \partial S^{-1}(p)/\partial p_\mu \ . \tag{2.2}$$

In Euclidean space, one can integrate over angular variables to obtain the following equations for the form factors $A(x)$ and $B(x)$:

$$x^2A(y) = x^2 + \tfrac{1}{2}(e/4\pi)^2 \int_0^x dy\, y^2[(G-3)\tau + 2G\xi] + \tfrac{1}{2}(e/4\pi)^2 x^2 \int_x^\infty dy[(G-3)\tau + 2G\xi] \ , \tag{2.3}$$

and

$$xB(x) = \tfrac{1}{2}(3+G)(e/4\pi)^2 \int_0^x dy\, y(y\eta + 2\zeta) + (e/4\pi)^2 x \int_x^\infty dy[2G\eta + (3+G)\zeta] + \tfrac{1}{2}(3-3G)x^2 \int_x^\infty \eta\, dy \ , \tag{2.4}$$

where x and y represent Euclidean momenta squared, and

$$\tau(x) = [x(dA/dx)A + (dB/dx)B]/(xA^2 + B^2) \ , \tag{2.5a}$$

$$\eta(x) = [(dA/dx)B - A(dB/dx)]/(xA^2 + B^2) \ , \tag{2.5b}$$

$$\zeta(x) = AB/(xA^2 + B^2) \ , \tag{2.5c}$$

$$\xi(x) = A^2/(xA^2 + B^2) \ . \tag{2.5d}$$

The chiral-invariant solution bifurcates to a symmetry-breaking solution at the lowest value of the coupling constant for which the linearized (in B) version of (2.3) and (2.4) has a nontrivial solution.[9] Introducing $\beta(x) \equiv \delta B(x)$, we obtain the bifurcation equations in the form

$$A(x) = 1 + (\lambda/2x^2) \int_0^x dy[(G-3)y^2(dA/dy)/A(y) + 2Gy] + (\lambda/2) \int_x^\infty dy[(G-3)(dA/dy)/A(y) + 2G/y] \ , \tag{2.6}$$

and

$$x\beta(x) = (3+G)(\lambda/2) \int_0^x dy\{y[(dA/dy)\beta(y) - A(y)(d\beta/dy)]/A^2(y) + 2\beta/A\} + \lambda x \int_x^\infty dy\{2G[(dA/dy)\beta(y) - A(y)(d\beta/dy)] + (3+G)\beta/(yA)\} + (1-G)(3\lambda/2)x^2 \int_x^\infty dy[(dA/dy)\beta - A(d\beta/dy)]/(yA^2) \ , \tag{2.7}$$

where $\lambda \equiv (e/4\pi)^2$.

In the Landau gauge the divergence structure of the above equations is better than predicted by ordinary power counting. In this gauge the $2G/y$ term in the second integral in (2.6) responsible for logarithmic divergences disappears. The remaining integrands are proportional to the derivatives of A or B. If these derivatives vanish rapidly enough, the integrals in the DS equation are finite and do not require subtractions. Previous work[11] shows that this is indeed the case, provided the coupling exceeds the value (1.4). The IR-divergent solution continues beyond the bifurcation point, but the other solution is IR finite.

This improvement of renormalization properties above the critical coupling results from lucky cancellations which in the Landau gauge eliminate the most divergent terms in the fermion self-energy diagram. This is all that is needed for the purpose of this analysis, but it would be naive to hope for similar improvement in all types of graphs.

In the UV region the mass function oscillates very rapidly:[11]

$$\beta(x) \sim x^\zeta\{c_1 \sin[\omega \ln(x)] + c_2 \cos[\omega \ln(x)]\} \ . \tag{2.8}$$

A similar behavior also characterizes the solutions in the constant vertex approximation,[8] and is a trademark of unstable, tachyonic vacua.[14] We shall see that such oscillations are artifacts of purely longitudinal vertex functions. If an appropriate transverse part is added so that the DS equation for the fermion propagator remains finite in all gauges, the oscillations give way to a regular power behavior.

III. TRANSVERSE PART OF THE VERTEX FUNCTION

The solution described in the previous section is gauge dependent. First, the divergencies disappear in the Landau gauge only. Next, the dependence of the bifurcation equations on the gauge parameter is rather complex. Every integral in these equations appears with a different gauge-dependent multiplier and the location of the mass pole is also gauge dependent.

Our hope is that finiteness and gauge independence of the bifurcated solution can be restored in a single swoop by adding a transverse part of the vertex function. In agreement with Eq. (2.2), we demand that the transverse part of the vertex function has the form

$$\Gamma_T{}^\mu(k,p) \approx \Gamma_T{}^\mu(p,p) = (g^{\mu\nu} - p^\mu p^\nu/p^2)\gamma_\nu \Gamma_T(p) \ . \tag{3.1}$$

The longitudinal part defined by Eq. (2.2) was linear in form factors $A(x)$ and $B(x)$ (and their derivatives). In order to produce cancellations with the contributions from the longitudinal part, the transverse part must be also linear:

$$\Gamma_T(p)=\Gamma_{T1}(p^2)+\not p\,\Gamma_{T2}(p^2)$$
$$=C_1A(p^2)+C_2p^2A'(p^2)$$
$$+\not p[C_3B(p^2)/p^2+C_4B'(p^2)]\ . \tag{3.2}$$

Any other combinations are trivial or excluded for dimensional reasons.

Substituting the above expressions into (2.1), taking traces and integrating over Euclidean angles, we obtain the following additional contributions to the self-energy function $\Sigma(p)$:

$$\tfrac{1}{4}\mathrm{Tr}[\Sigma_T(x)]=\tfrac{3}{4}\lambda(3+G)\int_0^x dy[(y/x)\Gamma_{T1}(y)B(y)+(y^2/x)\Gamma_{T2}(y)A(y)]/D(y)$$
$$+3\lambda\int_x^\infty dy[\Gamma_{T1}(y)B(y)+y\Gamma_{T2}(y)A(y)]/D(y)$$
$$+\tfrac{3}{4}\lambda(G-1)\int_x^\infty dy[(x/y)\Gamma_{T1}(y)B(y)+x\Gamma_{T2}(y)A(y)]/D(y)\ , \tag{3.3}$$

and

$$\tfrac{1}{4}\mathrm{Tr}[\gamma^\alpha\Sigma_T(x)]=\tfrac{3}{4}\lambda(G+1)\int_0^x dy[(y/x)^2\Gamma_{T1}(y)A(y)+(y^2/x)\Gamma_{T2}(y)B(y)]/D(y)$$
$$+\tfrac{3}{4}\lambda(G+1)\int_x^\infty dy[\Gamma_{T1}(y)A(y)+(1/y)\Gamma_{T2}(y)B(y)]/D(y)\ . \tag{3.4}$$

In the above, $D(y)=yA^2(y)+B^2(y)$.

With all additional contributions included, the bifurcation equations are

$$A_{\rm full}(x)=1+(\lambda/x^2)\int_0^x dy\{[G-3C_1(G+1)/4]y+[(G-3)/2-3C_2(G+1)/4]y^2[(dA/dy)/A(y)]\}$$
$$+\lambda\int_x^\infty dy\{[G-3C_1(G+1)/4]/y+[(G-3)/2-3(G+1)C_2/4](dA/dy)/A(y)\}\ . \tag{3.5}$$

and

$$x\beta_{\rm full}(x)=\lambda(G+3)\int_0^x dy(1/x)\{[(2G+1)/(G+1)+\tfrac{3}{4}C_3]\beta(y)/A(y)$$
$$+(\tfrac{1}{2}+\tfrac{3}{4}C_2)y(dA/dy)\beta(y)/A^2(y)+(\tfrac{3}{4}C_4-\tfrac{1}{2})y(d\beta/dy)/A(y)\}$$
$$+\lambda\int_x^\infty dy\{[(3+G)+4G/(G+1)+3C_3]\beta(y)/[yA(y)]+(2G+3C_3)(dA/dy)\beta(y)/A^2(y)$$
$$+(3C_4-2G)(d\beta/dy)/A(y)\}+\lambda\int_x^\infty dy\,x\{(G-1)[G/(G+1)+\tfrac{3}{4}C_3]\beta/[y^2A(y)]$$
$$+\tfrac{3}{2}(1-G)(1-C_2/2)\beta(y)(dA/dy)/[yA^2(y)]$$
$$+(G-1)[\tfrac{3}{2}+\tfrac{3}{4}C_4](d\beta/dy)/[yA(y)]\}\ . \tag{3.6}$$

Only one of the four available constants, C_1, is able to generate a contribution which can eliminate the term contributing to the UV and IR divergences. Finiteness is achieved by setting

$$C_1=4G/(3+3G)\ . \tag{3.7}$$

It is not possible to choose the values of C_2 through C_4 so that the parameter G is completely eliminated from (3.5) and (3.6). Still, the solutions generated at the bifurcation point can be gauge independent even if the bifurcation equations depend on the gauge parameter. This can happen provided all G dependent factors combine in such a way that a single G-dependent term, say λC_G, becomes a common multiplier of all integral terms in (3.5) and (3.6). Then the bifurcation condition determines $C_G\lambda$ rather than λ, and the location of the mass pole is G independent. The proper choice of the three available constants is surprisingly simple:

$$C_3=-C_1=-4G/(3+3G)\ \text{and}\ C_2=-C_4=2\ . \tag{3.8}$$

Then several integrals in (3.6) disappear. The surviving terms produce a common factor of $(G+3)$ and the complete bifurcation equations have the form

$$A_{\rm full}(x)=1-\tfrac{1}{2}\lambda(G+3)\int_0^x dy(1/x^2)A'(y)/A(y)$$
$$+\tfrac{1}{2}\lambda(G+3)\int_x^\infty dy\ A'(y)/A(y)\ , \tag{3.9}$$

and

$$\beta_{\rm full}(x)=\lambda(G+3)\int_0^x dy(1/x)[\beta(y)/A(y)+2yA'(y)\beta(y)/A^2(y)-2y\beta'(y)/A(y)]$$
$$+\lambda(G+3)\int_x^\infty dy[\beta(y)/yA(y)+2A'(y)\beta(y)/A^2(y)-2\beta'(y)/A(y)]\ . \tag{3.10}$$

Up to the multiplicative gauge term, the equation for $A(x)$ is identical to Eq. (2.6) with $G=0$. Therefore, all results for $A(x)$ obtained in the Landau gauge without the transverse part remain valid beyond the Landau gauge, with the transverse vertex included, except that now the effective coupling constant is $\alpha=(3+G)\lambda$, and

$$\alpha_{\rm crit}=(3+G)\lambda_{\rm crit}=(3+G)(e_{\rm crit}/4\pi)^2=7.18\ . \tag{3.11}$$

The equation for β can be converted to a second-order differential equation:

$$[x\beta(x)]''=\alpha\{\beta/[xA(x)]+2A'(x)\beta(x)/A(x)^2 -2\beta'(x)/A(x)\} . \quad (3.12)$$

The boundary conditions for (3.12) are

$$\lim_{x\to 0}[x\beta(x)]=0 \text{ and } \lim_{x\to\infty} d[x\beta(x)]/dx=0 . \quad (3.13)$$

Both boundary conditions can be imposed at $x=0$ after a simulated subtraction is performed at zero momentum. Near $x=0$, both asymptotic solutions to (3.11), behave like x^n, with n determined by

$$n(n-1)+2n[1+\alpha/A(0)]-\alpha/A(0)=0 . \quad (3.14)$$

Table I contains a few chosen values of n_i. Not only $\lim_{x\to 0}[x\beta(x)]=0$, but also $\lim_{x\to 0}\beta(x)=0$. Therefore,

$$x\beta(x)\equiv x\beta(x)-x\beta(0) ,$$

and the integral equation (3.12) has an equivalent form

$$x\beta_{\text{full}}(x)=\alpha\int_0^x dy(1-x/y)[\beta(y)/A(y) +2yA'(y)\beta(y)/A^2(y) -2y\beta'(y)/A(y)] , \quad (3.15)$$

with the boundary conditions

$$\lim_{x\to 0}[x\beta(x)]=0 \text{ and } \lim_{x\to 0} d[x\beta(x)]/dx=0 . \quad (3.16)$$

These conditions are met by each asymptotic solution of the equivalent differential equation (3.13). Moreover, in the UV asymptotic limit, the unwanted oscillatory behavior of the solution is replaced by a regular power behavior. As $x\to\infty$, $A(x)\to 1$, and $A'(x)\to 0$. Therefore, in the asymptotic region,

$$x^2\beta''(x)+2x\beta'(x)=\lambda[\beta(x)-2x\beta'(x)] . \quad (3.17)$$

Near the critical point, the asymptotic solutions are $\beta(x)\sim x^{0.45}$ and $\beta(x)\sim x^{-15.8}$. The magnitudes of the exponents corresponding to higher values of the coupling constant are presented in Table I. The table reveals another desirable characteristic of the new solution. For all values of the coupling constant above the bifurcation point, IR and UV asymptotic types of behavior are such that $B^2(x)$ is suppressed by $xA^2(x)$. This property is particularly important for consistency reasons. In the process of linearization, the $B^2(x)$ terms were disregarded as insignificant compared to $xA^2(x)$.

TABLE I. Power exponents n_i^{IR} and n_i^{UV} of the asymptotic behavior of $\beta(x)$ in the infrared and ultraviolet regions, respectively, for selected values of the coupling parameter $\alpha=(G+3)(e/4\pi)^2$.

α	n_1^{IR}	n_2^{IR}	n_1^{UV}	n_2^{UV}
$\alpha_{\text{crit}}=7.18$	21.1	0.54	-15.8	0.45
7.5	21.7	0.54	-16.5	0.46
10	26.6	0.53	-21.5	0.47
50	107	0.51	$-101.$	0.49
10^2	206	>0.50	$-201.$	<0.50
10^3	2×10^3	>0.50	-2×10^3	<0.50
10^6	2×10^6	>0.50	-2×10^6	<0.50

IV. SUMMARY

Our results confirm that the truncation procedures for the Dyson-Schwinger equations demand that a nontrivial transverse part is included in the vertex function. Without a transverse part, the results are gauge dependent in a most drastic way. Inclusion of a transverse part is instrumental in restoring gauge independence of the solutions, and it helps to absorb the divergences arising from loop integrals.

We began by adopting an approximation for the longitudinal part of the vertex,

$$\Gamma_L{}^{\mu}(k,p)\approx\Gamma_L{}^{\mu}(p,p)=\partial S^{-1}(p)/\partial p_\mu , \quad (4.1)$$

which states that the pole of the photon line adjacent to the vertex dominates all other contributions to the DS propagator equation for the fermion propagator.

The transverse part is fully determined by two requirements. First, we demand that the DS equation is free of UV divergences in all gauges. We also require that the location of the mass pole at the bifurcation point is gauge independent. It turns out that the transverse part has the compact form

$$\Gamma_T{}^{\mu}(k,p)=\Gamma_T{}^{\mu}(p,p) =(g^{\mu\nu}-p^\mu p^\nu/p^2)\gamma_\nu(\not p/p^2) \times\{C_1[\not pA(p^2)-B(p^2)] +C_2p^2[\not pA'(p^2)-B'(p^2)]\} , \quad (4.2)$$

with constants C_1 and C_2 given by (3.8).

The scope of this paper was restricted to the question of the existence of nonstandard solutions to the equations of QED for large values of the coupling constant. The discussion was possible mainly because in the linearized (in the mass term) case, the nonlinear differential equation for $A(p^2)$ is exactly solvable. The analysis was restricted to the linearized system of equations which suffices as a tool for exploring the bifurcation problem. It would be premature to speculate on the physical interpretation of the novel solution. This task had better await the results of (numerical) study of the full, rather than linearized, Dyson-Schwinger equations.

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APPENDIX

Let us demonstrate that in a theory without a transverse part of the vertex function cutoffs are necessary in non-Landau gauges. Equation (2.6) is equivalent to a nonlinear differential equation,

$$d^2A(t)/dt^2+2dA(t)/dt+\lambda(G-3)(dA/dt)/A(t)+2\lambda G=0\ , \quad \text{(A1)}$$

where

$$x\equiv x_s e^t,\quad t\equiv\ln(x/x_s)\ . \quad \text{(A2)}$$

The results presented here are x_s independent and therefore we shall arbitrarily substitute $x_s=1$.

The boundary conditions for (A1) are

$$\lim_{t\to-\infty} e^{2t}(dA/dt)=0\ , \quad \text{(A3)}$$

$$\lim_{t\to+\infty} [A(t)+\tfrac{1}{2}(dA/dt)]=1\ . \quad \text{(A4)}$$

We define

$$X(t)=A(t),\quad Y(t)=(dA/dt)\ . \quad \text{(A5)}$$

Then

$$(dY/dt)\equiv P(X,Y)=-(2+\alpha/X)Y-\xi\ , \quad \text{(A6)}$$

$$(dX/dt)\equiv Q(X,Y)=Y\ , \quad \text{(A7)}$$

$$dY(X)/dX\equiv F(X,Y)=P(X,Y)/Q(X,Y)=-2-\alpha/X-\xi/Y\ , \quad \text{(A8)}$$

where $\alpha=(3-G)\lambda$ and $\xi=2G\lambda$.

The function $F(X,Y)$ is single valued and continuous everywhere except at $X=Y=0$, and it satisfies the Lipshitz condition $|F(X,Y)-F(X,Y_0)|\le\text{const}|Y_0-Y|$ in any interval that does not include $Y=0$; hence, Eq. (2.10) satisfies the Cauchy criterion[15] for uniqueness and therefore it possesses integral curves passing through all finite points except $Y(t)\equiv(dA/dt)=0$, $X=0$. For $\xi\neq0$, the isoclines of the equation, $(dY/dX)=-2-\alpha/X-\xi/Y=f=\text{const}$, pass through the origin for any value of f. Hence the origin is a node. Since there is no other finite singular points, at least one end of any integral curve approaches $|X|=\infty$. For $G\neq0$, the boundary conditions cannot be met at this point.

Indeed, as $|X|\to\infty$, the asymptotic equation has the form $dY/dX=-2-\xi/Y$. Therefore $X\simeq(Y_0-Y)/2+(\beta/4)\ln[(2Y+\xi)/(2Y_0+\xi)]$. The condition for $|X|\to\infty$ requires that $Y\simeq-2X$ or $Y\simeq-\xi/2$. In the former case, the limit $|X|\to\infty$ is achieved as $t\to-\infty$. Then the solution behaves as $X(t)\sim e^{-2t}$, and the IR boundary condition (A3) is violated. If the same limit is achieved at $t\to+\infty$, the solution behaves as $X(t)\sim Y_0t+Ce^{-2t}$ and this contradicts the UV condition.

If $Y\simeq-\xi/2$, then $X\sim-\xi t/2+\phi(t)$, with ϕ such that in the asymptotic region $d\phi/dt\to0$ and $\phi\to0$. Solving for ϕ we find that $\phi\sim-(Y_0/2+\xi/4)e^{-2t}$. This means that the above limits can be reached only for $t\to+\infty$, but then $(X+Y/2)\to\infty$ and the condition (A4) is again violated.

The only way to satisfy both boundary conditions is to impose them at finite points rather than at zero and infinity. This is equivalent to introducing cutoffs in the original equation for A.

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