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Multipole expansion of conserved tensor currents and the number of form factors

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We develop a formalism for defining and studying the form factors which describe the coupling of a spin-J particle through a conserved tensor current to a spin-S boson. For $S=2$, these form factors are the single-graviton gravitational form factors. We investigate the behavior of the various form factors under C, P, T, and CPT and classify them accordingly. We show that under CPT invariance all the multipoles of the mass distribution are the same for a particle and its antiparticle. We also show that in a quantum theory of gravity the coupling of a particle to the graviton is in general different from that of its antiparticle. Only when P and T are both valid symmetries of the underlying theory do a particle and its antiparticle couple identically to the graviton. The number of form factors for arbitrary J and S is given. We show that a massive Majorana particle (a CPT self-conjugate particle) coupling to an odd-spin boson possesses only anapole moments. Massless Majorana particles with spin$\neq\frac{1}{2}$ have no single-photon electromagnetic form factors while the ones with spin$\neq\frac{1}{2}$, 1, or $\frac{3}{2}$ have no interactions with a spin-3 boson; this applies to the graviton ($J=2$, $S=1,3$) and to the massless gravitino ($J=\frac{3}{2}$, $S=1$). Our results also apply to extended objects (nuclei, . . .) in the low-energy limit.

I. INTRODUCTION

In this paper we shall develop a multipole expansion for classifying and counting the dynamical form factors which describe the interaction of a spin-J particle with a virtual spin-S boson as illustrated in Fig. 1. To be more specific, we shall always assume that the spin-S boson is emitted by the particle through a conserved, totally symmetric, and Hermitian tensor current operator $T_{\mu_1\cdots\mu_S}$. The generalization to a nonconserved current is straightforward. Our considerations rely solely on Lorentz invariance with no specification of the dynamics. The physically interesting cases of gravity and electromagnetism arise for $S=2$ and $S=1$, respectively. For $S=1$ the form factors in question are the single-photon electromagnetic form factors of a spin-J particle. It is well known that the electromagnetic structure of a spin-$\frac{1}{2}$ particle is determined by four form factors: the charge, magnetic dipole, electric dipole, and the anapole (also called the charge radius). It turns out[1] that a massive CPT eigenstate particle of spin $\frac{1}{2}$, such as the massive Majorana neutrino, can have at most one form factor determining its interaction with a single photon, this form factor being the anapole. This result has been generalized[2,3] to the case of a CPT self-conjugate particle of arbitrary spin J (generalized Majorana particle), with the conclusion that such a particle can possess at most $2J$ single-photon form factors, which are the anapole and its higher moments. Moreover, in the event of P or T conservation, half of these form factors vanish. Thus, a generalized Majorana particle of spin 1 such as the Z boson, can have at most two form factors determining its electromagnetic structure and in the event where P or T are conserved, it has at most one form factor. On the other hand, it turns out[3] that a massless generalized Majorana particle does not have any single-photon form factors unless $J=\frac{1}{2}$, in which instance it possesses at most one form factor which is of the anapole type.

These considerations will be extended here to include the cases $S\geq 2$. In order to define the form factors of a spin-J particle, we will adopt a noncovariant procedure. The reason lies in the following: the usual covariant way for defining the form factors consists in writing down the most general expression allowed by Lorentz invariance for the matrix elements of the current operator taken between the initial and final states of the spin-J particle; this procedure becomes increasingly involved as the spins J and S take on higher values since one will have to write down all the allowed *independent* covariant tensor operators. Dependency among the tensor operators arises from algebraic identities and the mass-shell condition.

We find it more convenient for our purposes to adopt the noncovariant procedure consisting in a multipole expansion of the various components of $T_{\mu_1\cdots\mu_S}$ which transform among themselves under spatial rotations. This procedure allows us to define unambiguously the independent form factors as well as to classify and count their number. We define our framework in Sec. II; a generalization of the usual Wigner-Eckart theorem to relativistic particle states allows us to define the form factors of massive particles. For massless particles, the form factors are conveniently defined by transforming the current operator to a suitable basis in order to find out the various helicities it contains. Some known facts about the discrete symmetries have been included in order to unify the notation and for the sake of completeness.

The problem of finding a complete set of transverse and totally symmetric tensor spherical harmonics of arbitrary rank defined on S_2 is dealt with in Sec. III. We explicitly construct these harmonics and study their num-

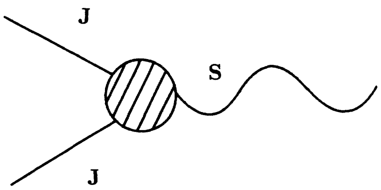

FIG. 1. Vertex of a spin-J particle coupling to a spin-S boson.

ber as well as their properties under parity transformation and complex conjugation. In Appendix B it is shown that these harmonics reproduce the usual vector harmonics when their rank is one; similarly, they reproduce the rank-2 spherical harmonics which exist in the literature dealing with classical gravitational radiation.

In Sec. IV we combine the results obtained in order to count and classify the form factors through which a spin-J particle couples to a spin-S boson through a conserved tensor current. It turns out that a generalized Majorana particle coupling to an odd-spin boson possesses only anapole moments; the theorems of Ref. 3 are recovered as a special case. For odd S, the electric- and magnetic-type form factors of a particle and its antiparticle are opposite in sign, while the anapole-type form factors are equal. The converse of the last statement is true if S is even. As a result, a particle (with $J\neq 0$) couples differently to the graviton than its antiparticle does. Only when both P and T (or alternatively C) are simultaneously conserved do particles and their antiparticles couple identically to the graviton, and for a spin-$\frac{1}{2}$ particle P conservation alone is sufficient to ensure the same coupling. However, the interaction leading to the asymmetric coupling of particles and their antiparticles is a contact interaction which does not affect the scattering on an exterior classical gravitational field such as the gravitational field of the Earth. We also investigate the coupling of massless particles to a spin-S boson. For example, we find that the graviton has no coupling to a single photon or to a single spin-3 boson.

We finally mention that no assumptions are made on the mass of the emitted spin-S boson. However, in the case where the emitted boson is massless, Weinberg has proven[4] that the current has to be conserved as a consequence of Lorentz invariance and that massless bosons with spin ≥ 3 do not couple at zero frequency, i.e., the lowest-order form factor of a particle coupling to such a boson vanishes at zero momentum transfer; another way of saying this is that massless particles with spin ≥ 3 cannot give rise to $1/r$ potentials.

II. FRAMEWORK

We consider the transition process

$$A(p_i)\rightarrow A(p_f)B(k)\ , \tag{2.1}$$

where $A(p)$ is a real particle with momentum p and spin J and $B(k)$ is a virtual boson of spin S and momentum k. It follows from energy-momentum conservation that $k=p_i-p_f$ and $(p_i+p_f)^2\geq 0$. The kinematics of the transition is greatly simplified in the Breit frame in which $p_i=(q_0,\mathbf{q})$, $p_f=(q_0,-\mathbf{q})$, and $k=(0,2\mathbf{q})$. For $k^2=0$, the Breit frame does not exist if $A(p)$ is a massless particle.

A. Massive particles

The state $|JM\rangle$ of a massive particle at rest, is specified by its spin J and the spin projection M along a fixed axis, say the $\hat{\mathbf{z}}$ axis. The state $|\mathbf{q};JM\rangle$ of a moving particle with momentum $(q_0,\mathbf{q})$ is constructed by applying a pure boost on the state $|JM\rangle$, i.e.,

$$|\mathbf{q};JM\rangle=e^{i\zeta\cdot\mathbf{K}}|JM\rangle\ , \tag{2.2}$$

where $\mathbf{K}$ is the boost generator and ζ is the rapidity. Under spatial rotations, $\mathbf{K}$ transforms like a vector operator,

$$\mathcal{R}K_i\mathcal{R}^\dagger=R_{ji}K_j \tag{2.3}$$

and $|JM\rangle$ transforms according to the Jth irreducible representation of the rotation group:

$$\mathcal{R}|JM\rangle=\sum_{M'}\mathcal{D}^J_{M'M}(R)|JM'\rangle$$

with $\mathcal{D}^J_{M'M}(R)$ being the Wigner D functions.[5] Therefore, the state (2.2) transforms as

$$\mathcal{R}|\mathbf{q};JM\rangle=\sum_{M'}\mathcal{D}^J_{M'M}(R)|\mathbf{q}';JM'\rangle\ , \tag{2.4}$$

where

$$q_i'=R_{ij}q_j\ .$$

The form factors describing the transition (2.1) will be defined through the multipole expansion of the matrix element

$$\langle -\mathbf{q};JM_f|T_{\mu_1\cdots\mu_S}(0)|\mathbf{q};JM_i\rangle\ , \tag{2.5}$$

where $T_{\mu_1\cdots\mu_S}$ is a covariant tensor operator assumed to be conserved and totally symmetric. The conservation equation $\partial_{\mu_1}T_{\mu_1\mu_2\cdots\mu_S}(x)=0$ implies the transversality of the matrix elements (in the Breit frame) with respect to the spatial indices:

$$q_{i_1}\langle -\mathbf{q};JM_f|T_{i_1\mu_2\cdots\mu_S}(0)|\mathbf{q};JM_i\rangle=0\ . \tag{2.6}$$

In order to define the form factors, we split the four-dimensional tensor $T_{\mu_1\cdots\mu_s}$ into a set of $S+1$ three-dimensional tensors: $T_{0\cdots 0},T_{i_10\cdots 0},\ldots,T_{i_1\cdots i_S}$. Clearly, this set exhausts all the components of the totally symmetric tensor $T_{\mu_1\cdots\mu_S}$. $T_{i_1\cdots i_k0\cdots 0}$ transforms under a rotation R according to

$$\mathcal{R}T_{i_1\cdots i_k0\cdots 0}\mathcal{R}^\dagger=\sum_{j_1\cdots j_k}R_{j_1i_1}\cdots R_{j_ki_k}T_{j_1\cdots j_k0\cdots 0}\ . \tag{2.7}$$

The above $S+1$ tensors will be called the "irreducible parts of $T_{\mu_1\cdots\mu_S}$." In what follows, we shall often omit the indices which are zero when dealing with the irreducible parts and simply write $T_{i_1\cdots i_k}$ for $T_{i_1\cdots i_k0\cdots 0}$ keeping in mind that $T_{i_1\cdots i_k}$ derives from a four-dimensional rank-S tensor. It follows from (2.6) that any irreducible part (except $T_{0\cdots 0}$ of course) is spatially transverse with respect to $\mathbf{q}$:

$$q_{i_1}\langle -\mathbf{q};JM_f|T_{i_1\cdots i_k}(0)|\mathbf{q};JM_i\rangle=0 \ .$$

We mention that the number of independent components of a symmetric and transverse tensor of rank k in three dimensions is equal to $(k+1)$. Thus the transverse and symmetrical tensor $T_{\mu_1\cdots\mu_S}$ will have $\sum_{k=0}^{S}(k+1)=\frac{1}{2}(S+1)(S+2)$ independent components. We now seek a multipole expansion of the irreducible parts. Let $\{\mathcal{Y}^{\lambda}_{LMi_1\cdots i_k}(\hat{\mathbf{q}})\}$ be a complete set of symmetric and transverse tensors defined on S_2 which satisfies

$$\int d^2\hat{\mathbf{q}} \sum_{i_1\cdots i_k} \mathcal{Y}^{\lambda *}_{LMi_1\cdots i_k}(\hat{\mathbf{q}})\mathcal{Y}^{\lambda'}_{L'M'i_1\cdots i_k}(\hat{\mathbf{q}})=\delta_{LL'}\delta_{MM'}\delta_{\lambda\lambda'} \tag{2.8}$$

and transforms under a rotation R according to

$$\mathcal{Y}^{\lambda}_{LMj_1\cdots j_k}(R\hat{\mathbf{q}}) = \sum_{M',l_1\cdots l_k} \mathcal{D}^{L}_{M'M}(R^{-1})R_{j_1l_1}\cdots R_{j_kl_k}\mathcal{Y}^{\lambda}_{LM'l_1\cdots l_k}(\hat{\mathbf{q}}) \ , \tag{2.9}$$

where λ denotes any other quantum numbers. One can write down the following multipole expansion for the matrix element:

$$\langle -\mathbf{q};JM_f|T_{i_1\cdots i_k}|\mathbf{q};JM_i\rangle = \sum_{LM\lambda} \tilde{Q}^{\lambda M_f M_i}_{JLM}(|\mathbf{q}|)\mathcal{Y}^{\lambda}_{LMi_1\cdots i_k}(\hat{\mathbf{q}}) \tag{2.10}$$

with

$$\tilde{Q}^{\lambda M_f M_i}_{JLM}(|\mathbf{q}|)=\int d^2\hat{\mathbf{q}} \sum_{i_1\cdots i_k} \mathcal{Y}^{\lambda *}_{LMi_1\cdots i_k}(\hat{\mathbf{q}}) \times\langle -\mathbf{q};JM_f|T_{i_1\cdots i_k}|\mathbf{q};JM_i\rangle$$

It follows from a generalized Wigner-Eckart theorem (proven in Appendix A) that $\tilde{Q}^{\lambda M_f M_i}_{JLM}(|\mathbf{q}|)$ can be factorized as

$$\tilde{Q}^{\lambda M_f M_i}_{JLM}(|\mathbf{q}|)=(-)^{J-M_i}\begin{bmatrix} J & L & J \\ -M_f & -M & M_i \end{bmatrix}\tilde{Q}^{\lambda}_{JL}(|\mathbf{q}|) \tag{2.11}$$

and therefore the multipole expansion (2.10) may be written as

$$\langle -\mathbf{q};JM_f|T_{i_1\cdots i_k}|\mathbf{q};JM_i\rangle = \sum_{L\lambda}(-)^{J-M_i}\begin{bmatrix} J & L & J \\ -M_f & M_f-M_i & M_i \end{bmatrix}\tilde{Q}^{\lambda}_{JL}(|\mathbf{q}|) \times\mathcal{Y}^{\lambda}_{LMi_1\cdots i_k}(\hat{\mathbf{q}}) \tag{2.12}$$

the sum being performed over a finite number of terms and $-M=M_f-M_i$, due to the occurrence of the Wigner 3-j symbol. As a consequence, no form factors with moments higher than $2J$ can exist. An example for $k=0$ is the expansion of the charge distribution according to the ordinary spherical harmonics $Y_{LM}(\hat{\mathbf{q}})$.

B. Massless particles

The state $|\mathbf{q};\lambda\rangle$ of a massless particle of spin J is labeled by the momentum $\mathbf{q}$ and the helicity λ which assumes only the two values $\pm J$. We consider the matrix elements

$$\langle -\mathbf{q};\lambda_f|T_{\mu_1\cdots\mu_S}(0)|\mathbf{q};\lambda_i\rangle \tag{2.13}$$

and split the tensor $T_{\mu_1\cdots\mu_S}$ into its irreducible parts, whose matrix elements are transverse to $\mathbf{q}$. For simplicity, we choose a coordinate system with $\mathbf{q}$ being parallel to the $\hat{\mathbf{z}}$ axis. The transversality condition then states that all the matrix elements in which one (or more) of the indices $i_1\cdots i_k$ takes on the value 3 vanish. The problem then reduces to studying matrix elements of the form

$$\langle -\mathbf{q};\lambda_f|T_{\underbrace{1\cdots 1}_{k-\nu\text{ times}}\underbrace{2\cdots 2}_{\nu\text{ times}}}|\mathbf{q};\lambda_i\rangle \tag{2.14}$$

with $0\le k\le S$ and $0\le \nu\le k$. For this purpose, it is convenient to decompose the representation of the group of rotations around the $\hat{\mathbf{z}}$ axis [acting on totally symmetric rank-k tensors in the (x,y) plane] into its irreducible components. This will lead to a new set of components $T_{\underbrace{+\cdots+}_{k-\nu\text{ times}}\underbrace{-\cdots-}_{\nu\text{ times}}}$ obtained from $T_{1\cdots 12\cdots 2}$ through a linear transformation and having the property of getting multiplied by

$$e^{i(k-2\nu)\psi}$$

under a rotation by ψ around the $\hat{\mathbf{z}}$ axis. It is instructive to consider the cases $k=1$ and 2. For $k=1$ we have

$$T^{\pm}=(T_1\pm iT_2) \ ,$$

while, for $k=2$,

$$T^{++}=(T_{11}-T_{22})+2iT_{12} \ ,$$

$$T^{+-}=(T_{11}+T_{22}) \ ,$$

$$T^{--}=(T_{11}-T_{22})-2iT_{12} \ .$$

For an arbitrary value of k, the components $T_{\underbrace{+\cdots+}_{k-\nu\text{ times}}\underbrace{-\cdots-}_{\nu\text{ times}}}$ can be readily found by noticing that they transform like the tensor product $T^{+}\otimes\cdots T^{+}\otimes T^{-}\cdots\otimes T^{-}$ where the $(+)$ sign occurs $k-\nu$ times and the $(-)$ sign ν times. This leaves us with investigating matrix elements of the form $\langle -\mathbf{q},\lambda_f|T_{+\cdots+\,-\cdots-}|\mathbf{q},\lambda_i\rangle$, which will be referred to as "canonical form" in what follows.

C. *C*, *P*, *T*, and *CPT*

In a theory where parity is conserved we have for a massive particle at rest:

$$P|JM\rangle=\eta_P(M)|JM\rangle \ . \quad (2.15)$$

$\eta_P(M)$ being a phase factor, since in the Poincaré group $[P,\mathbf{J}]=0$. Apply now a pure boost on both sides of the above equation:

$$e^{-i\zeta\cdot\mathbf{K}}P|JM\rangle=\eta_P(M)|-\mathbf{q};JM\rangle$$

and since $\{P,\mathbf{K}\}=0$, one gets

$$P|\mathbf{q};JM\rangle=\eta_P(M)|-\mathbf{q};JM\rangle \ . \quad (2.16)$$

This proves that the phases $\eta_P(M)$ for a massive particle are independent of $\mathbf{q}$. Moreover, the phases for various values of M are not independent. To see this, we apply a raising operator $\mathbf{J}_+$ to the angular momentum eigenstates $|JM\rangle$ in (2.15) (in the event where $M=J$, we apply of course $\mathbf{J}_-$). Using $[P,\mathbf{J}_+]=0$, one obtains

$$\eta_P(M+1)|JM+1\rangle=\eta_P(M)|JM+1\rangle$$

from which we infer that all the values of $\eta_P(M)$ are equal, leading to the relation

$$\eta_P^*(M)\eta_P(M')=1 \ . \quad (2.17)$$

In the same manner,

$$C|\mathbf{q};JM\rangle=\eta_C(M)\overline{|\mathbf{q};JM\rangle} \quad (2.18)$$

for a theory in which C is conserved, the bar denoting the antiparticle state. The same reasoning that led to (2.17) gives rise to

$$\eta_C^*(M)\eta_C(M')=1 \ . \quad (2.19)$$

For a theory in which T is a valid symmetry, we have similarly

$$T|\mathbf{q};JM\rangle=\eta_T(M)|-\mathbf{q};J-M\rangle \ . \quad (2.20)$$

T is represented by an antiunitary operator in the Hilbert space and satisfies $\mathbf{J}_+T=-T\mathbf{J}_-$. This leads to the phase relation

$$\eta_T^*(M)\eta_T(M')=(-)^{M-M'} \ . \quad (2.21)$$

For a CPT operation, which is also antiunitary and which we shall denote by Θ, we have

$$\Theta|\mathbf{q};JM\rangle=\eta_\Theta(M)\overline{|\mathbf{q};J-M\rangle} \quad (2.22)$$

and using the raising operator, one obtains

$$\eta_\Theta^*(M)\eta_\Theta(M')=(-)^{M-M'} \ . \quad (2.23)$$

We close this section by some general remarks on symmetry operations. Given an operator O in the Hilbert space of states $|\psi\rangle$ and a unitary operator U acting on the states $|\psi_U\rangle\equiv U|\psi\rangle$, one has

$$\langle\psi|O|\psi'\rangle=\langle\psi_U|UOU^\dagger|\psi'_U\rangle \ ,$$

whereas for an antiunitary operator such as Θ we have

$$\langle\psi|O|\psi'\rangle=\langle\psi'_\Theta|\Theta O^\dagger\Theta^{-1}|\psi_\Theta\rangle \ . \quad (2.24)$$

A Hermitian operator $T_{\mu_1\cdots\mu_S}$ transforms under Θ according to

$$\Theta T_{\mu_1\cdots\mu_S}(0)\Theta^{-1}=(-)^S T_{\mu_1\cdots\mu_S}(0) \quad (2.25)$$

and in the event where P or T are conserved we have, for a Hermitian tensor (and not for a pseudotensor),

$$PT_{\mu_1\cdots\mu_S}(0)P^{-1}=(-)^k T_{\mu_1\cdots\mu_S}(0) \ ,$$
$$TT_{\mu_1\cdots\mu_S}(0)T^{-1}=(-)^k T_{\mu_1\cdots\mu_S}(0) \ , \quad (2.26)$$

k being the number of spacelike indices among $\mu_1\cdots\mu_S$. Finally, Θ acting in the space of a spin-J particle, obeys

$$\Theta^2=(-)^{2J} \ . \quad (2.27)$$

III. TRANSVERSE TENSOR HARMONICS ON S_2 OF ARBITRARY RANK

In this section, we construct a complete set of transverse and symmetrical tensor spherical harmonics of arbitrary rank in terms of which any symmetrical and transverse tensor field defined on S_2 can be expanded. We will further analyze the behavior of these harmonics under rotation, space reflection, and complex conjugation. We first discuss vector harmonics and rank-2 tensor harmonics and then go over to discuss the case of arbitrary rank.

The transverse vector spherical harmonics may be generated by applying a combination of the operators $\mathbf{q}$ and ∇ to the ordinary spherical harmonics Y_{LM}:

$$\mathbf{Y}_{LM}^0\equiv\frac{1}{\sqrt{L(L+1)}}\mathbf{q}\times\nabla Y_{LM}, \quad L\geq 1 \ ,$$

$$\mathbf{Y}_{LM}^1\equiv\hat{\mathbf{q}}\times\mathbf{Y}_{LM}^0, \quad L\geq 1 \ .$$

They transform according to the Lth irreducible representation of the rotation group, i.e.,

$$[\mathcal{R}\mathbf{Y}_{LM}^\lambda(\hat{\mathbf{q}})]_i\equiv R_{ij}\mathbf{Y}_{LM,j}^\lambda(R^{-1}\hat{\mathbf{q}})$$
$$=\sum_{M'}\mathcal{D}_{M'M}^L(R)\mathbf{Y}_{LM',i}^\lambda(\hat{\mathbf{q}}) \ ,$$

and carry parities

$$\mathbf{Y}_{LM}^\lambda(-\hat{\mathbf{q}})=(-)^{L+\lambda}\mathbf{Y}_{LM}^\lambda(\hat{\mathbf{q}}) \ .$$

They are mutually orthogonal

$$\int d^2\hat{\mathbf{q}}\,\mathbf{Y}_{LM}^{\lambda *}(\hat{\mathbf{q}})\mathbf{Y}_{L'M'}^{\lambda'}(\hat{\mathbf{q}})=\delta_{LL'}\delta_{MM'}\delta_{\lambda\lambda'}$$

and form a complete set in the space of transverse vector fields on S_2. These vector harmonics play an important role in the theory of electromagnetic radiation.

In the case of $k=2$, one can still try to generate the desired tensor spherical harmonics by applying the operators $\hat{\mathbf{q}}$ and ∇ to the ordinary spherical harmonics. After some quite lengthy manipulations, one ends up with the following set of tensor harmonics:

$$Y_{LM;ij}^0(\hat{\mathbf{q}})\equiv\frac{1}{\sqrt{2}}(\delta_{ij}-\hat{q}_i\hat{q}_j)Y_{LM}(\hat{\mathbf{q}}), \quad L\geq 0 \ ,$$

$$Y_{LM;ij}^1(\hat{\mathbf{q}})\equiv\frac{1}{\sqrt{2(L-1)L(L+1)(L+2)}}$$
$$\times(\epsilon_{imn}\hat{q}_m\hat{\nabla}_j\hat{\nabla}_n+\epsilon_{jmn}\hat{q}_m\hat{\nabla}_i\hat{\nabla}_n)Y_{LM}(\hat{\mathbf{q}}) \ ,$$
$$L\geq 2 \ ,$$

$$Y^2_{LM;ij}(\hat{\mathbf{q}}) \equiv \frac{1}{\sqrt{2(L-1)L(L+1)(L+2)}} \times [2(\hat{q}_i\hat{\nabla}_j + \hat{\nabla}_j\hat{\nabla}_i)Y_{LM}(\hat{\mathbf{q}}) + \sqrt{2}L(L+1)Y^0_{LM;ij}(\hat{\mathbf{q}})], \quad L \geq 2 ,$$

where $\hat{\nabla} \equiv |\mathbf{q}|\nabla - \mathbf{q}(\partial/\partial|\mathbf{q}|)$. These tensor fields are symmetric and transverse. The set is orthonormal,

$$\int d^2\hat{\mathbf{q}}\,\mathrm{Tr}[Y^{\lambda *}_{LM}(\hat{\mathbf{q}})Y^{\lambda'}_{L'M'}(\hat{\mathbf{q}})] = \delta_{LL'}\delta_{MM'}\delta_{\lambda\lambda'} ,$$

and complete (for the proof, see Appendix B and the lemma in Sec. III) in the space of rank-2 symmetric and transverse tensors on S_2. These harmonics realize the Lth representation of the rotation group. Their parities are given by

$$Y^{\lambda}_{LM;ij}(-\hat{\mathbf{q}}) = (-)^{L+\lambda}Y^{\lambda}_{LM;ij}(\hat{\mathbf{q}}) .$$

There are also other versions of the tensor harmonics of rank 2 in the literature[6] for the purpose of studying classical gravitational radiation.

As the rank S takes on higher values, the procedure for constructing tensor harmonics used above gets more and more involved. This procedure also fails to give a clear understanding of the number of existing harmonics for $L<S$. (For $S=1$, we do not have any transverse vector harmonics for $L=0$ whereas for $S=2$ we do have one such harmonic for $L=0$ and one for $L=1$.) To construct the tensor harmonics with higher rank, we follow a different route which is more systematic. First of all, we require each tensor harmonic to be an eigenfunction of the square of the total angular momentum operator and of the $\hat{\mathbf{z}}$ projection of the total angular momentum: i.e.,

$$\mathbf{J}^2\mathcal{T}_{LMi_1\cdots i_k}(\hat{\mathbf{q}}) = L(L+1)\mathcal{T}_{LMi_1\cdots i_k}(\hat{\mathbf{q}}) ,$$

$$\mathbf{J}_z\mathcal{T}_{LMi_1\cdots i_k}(\hat{\mathbf{q}}) = M\mathcal{T}_{LMi_1\cdots i_k}(\hat{\mathbf{q}}) .$$

Upon a rotation R we have

$$\mathcal{R}\mathcal{T}_{LMi_1\cdots i_k}(\hat{\mathbf{q}}) \equiv \sum_{j_1\cdots j_k} R_{i_1j_1}\cdots R_{i_kj_k}\mathcal{T}_{LMj_1\cdots j_k}(R^{-1}\hat{\mathbf{q}}) = \sum_{M'}\mathcal{D}^L_{M'M}(R)\mathcal{T}_{LM'i_1\cdots i_k}(\hat{\mathbf{q}}) .$$

Choose R to be a rotation bringing the $\hat{\mathbf{z}}$ axis to $\hat{\mathbf{q}}$, i.e., $\hat{\mathbf{q}} = R\hat{\mathbf{z}}$. The above equation then reads

$$\sum_{M'}\mathcal{D}^L_{M'M}(R)\mathcal{T}_{LM'i_1\cdots i_k}(\hat{\mathbf{q}}) = \sum_{j_1\cdots j_k} R_{i_1j_1}\cdots R_{i_kj_k}\mathcal{T}_{LMj_1\cdots j_k}(\hat{\mathbf{z}}) . \tag{3.1}$$

We now construct an orthogonal basis consisting of $k+1$ elements in the space of rank-k symmetric tensors which are transverse to the $\hat{\mathbf{z}}$ axis. This will allow us to expand any tensor $\mathcal{T}_{LMj_1\cdots j_k}(\hat{\mathbf{z}})$ in terms of the $k+1$ basis elements defined as follows: introduce the canonical vectors

$$\hat{\mathbf{e}}_{\pm} \equiv \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})$$

and define a set of rank-k tensors by totally symmetrizing the tensor product formed out of the vectors $\hat{\mathbf{e}}_{\pm}$:

$$T^{\nu} \equiv \left[\frac{\nu!(k-\nu)!}{k!}\right]^{1/2} \sum_P P(\hat{\mathbf{e}}_+ \otimes \cdots \hat{\mathbf{e}}_+ \otimes \hat{\mathbf{e}}_- \cdots \otimes \hat{\mathbf{e}}_-) .$$

The product which has been symmetrized above consists of ν vectors $\hat{\mathbf{e}}_+$ and $k-\nu$ vectors $\hat{\mathbf{e}}_-$, $0 \leq \nu \leq k$; the sum itself contains

$$\begin{bmatrix} k \\ \nu \end{bmatrix}$$

terms. The tensor T^{ν} acquires a phase $e^{i\lambda\psi}$ with $\lambda = (2\nu - k)$ under a rotation by an angle ψ around the $\hat{\mathbf{z}}$ axis ($\hat{\mathbf{z}} \| \hat{\mathbf{q}}$ in this frame) and is therefore an eigenfunction of the helicity operator $\hat{\mathbf{q}}\cdot\mathbf{S}$. The allowed helicities for a transverse rank-k tensor are therefore $k, k-2, \ldots, -k+2, -k$. Taking $\mathcal{T}_{LMj_1\cdots j_k}(\hat{\mathbf{z}})$ to be one of the T^{ν} and inverting Eq. (3.1), one obtains

$$\mathcal{T}^{\lambda}_{LMi_1\ldots i_k}(\hat{\mathbf{q}}) = \mathcal{N}_L\mathcal{D}^{L*}_{M\lambda}(R) \sum_{j_1\cdots j_k} R_{i_1j_1}\cdots R_{i_kj_k}T^{\nu}_{j_1\cdots j_k} , \tag{3.2}$$

where $\mathcal{N}_L = \sqrt{(2L+1)/4\pi}$ is the normalization constant. $\mathcal{T}^{\lambda}_{LMi_1\cdots i_k}(\hat{\mathbf{q}})$ is a simultaneous eigenfunction of the square of the total angular momentum, the $\hat{\mathbf{z}}$ projection of the total angular momentum and the helicity. The foregoing construction is quite analogous to the construction of the wave function of a symmetric top in quantum mechanics.

Let ψ, θ, ϕ be the three Euler angles of the rotation R with $\hat{\mathbf{q}} = R\hat{\mathbf{z}}$. Clearly, θ and ϕ are the polar angles of $\hat{\mathbf{q}}$ and ψ is unspecified. Introduce the unit vectors $\hat{\mathbf{e}}_{\theta}$ and $\hat{\mathbf{e}}_{\phi}$ of the polar coordinates system:

$$\hat{\mathbf{e}}_{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) ,$$

$$\hat{\mathbf{e}}_{\phi} = (-\sin\phi, \cos\phi, 0)$$

and define

$$\hat{\epsilon}_{\pm} \equiv \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_{\theta} \pm i\hat{\mathbf{e}}_{\phi}) .$$

With this notation, it can be shown that the tensor $T^{\lambda}(\hat{\mathbf{q}})$ defined through

$$T^{\lambda}_{i_1\cdots i_k}(\hat{\mathbf{q}}) \equiv \sum_{j_1\cdots j_k} R_{i_1j_1}\cdots R_{i_kj_k}T^{\nu}_{j_1\cdots j_k} \tag{3.3}$$

can be expressed as

$$T^{\lambda}(\hat{\mathbf{q}}) = e^{-i\lambda\psi}\left[\frac{\nu!(k-\nu)!}{k!}\right]^{1/2} \times \sum_P P(\hat{\epsilon}_+ \otimes \cdots \hat{\epsilon}_+ \otimes \hat{\epsilon}_- \cdots \otimes \hat{\epsilon}_-) . \tag{3.4}$$

On the other hand, the Wigner $\mathcal{D}$ function can be written as

$$\mathcal{D}^{L*}_{M\lambda}(R) = e^{iM\phi}d^L_{M\lambda}(\theta)e^{i\lambda\psi} .$$

Putting the last two equations together in (3.2), one ends up with an explicit expression for the tensor spherical harmonics of rank k:

$$\mathcal{T}^{\lambda}_{LM}(\hat{\mathbf{q}})=\left[\frac{2L+1}{4\pi}\right]^{1/2}\mathcal{D}^{L*}_{M\lambda}(R)T^{\lambda}(\hat{\mathbf{q}}) \tag{3.5}$$

or equivalently

$$\mathcal{T}^{\lambda}_{LM}(\hat{\mathbf{q}})=\left[\frac{2L+1}{4\pi}\right]^{1/2}\left[\frac{\nu!(k-\nu)!}{k!}\right]^{1/2}e^{iM\phi}d^{L}_{M\lambda}(\theta)\times\sum_{P}P(\hat{\epsilon}_{+}\otimes\cdots\hat{\epsilon}_{+}\otimes\hat{\epsilon}_{-}\cdots\otimes\hat{\epsilon}_{-})\ , \tag{3.6}$$

which depends only on the polar angles of $\hat{\mathbf{q}}$ and thereby removes the ambiguity of specifying ψ. A number of properties of $\mathcal{T}^{\lambda}_{LMi_1\cdots i_k}(\hat{\mathbf{q}})$ can be deduced from the properties of $\mathcal{D}^{L}_{M\lambda}(R)$ and are summarized in the following lemma.

Lemma. (i) The set $\mathcal{T}^{\lambda}_{LM}$ is orthonormal according to the scalar product defined in (2.8).

(ii) The set $\mathcal{T}^{\lambda}_{LM}$ is complete in the space of totally symmetric and transverse tensors of rank k defined on S_2.

(iii) For given L and k one has $-L\le M\le L$ and $-\min(L,k)\le\lambda\le\min(L,k)$. Furthermore, the consecutive values λ assumes obey the rule $\Delta\lambda=2$ and λ is even (odd) if k is even (odd).

(iv) $\mathcal{Y}^{\lambda\pm}_{LM}\equiv 1/\sqrt{2}(\mathcal{T}^{\lambda}_{LM}\pm\mathcal{T}^{-\lambda}_{LM})$ are parity eigenstates with

$$\mathcal{Y}^{\lambda}_{LM}(-\hat{\mathbf{q}})=(-)^{\Pi(\lambda)}\mathcal{Y}^{\lambda}_{LM}(\hat{\mathbf{q}})\ ,$$

where $\Pi(\lambda_{+})=\Pi(0)=L$ and $\Pi(\lambda_{-})=L+1$. For $\lambda=0$ there is only one such combination $\mathcal{Y}^{0}_{LM}=\mathcal{T}^{0}_{LM}$. For $L\ge k$ and k even, there are $\frac{1}{2}k+1$ tensor harmonics with parity $(-)^{L}$ and $\frac{1}{2}k$ with parity $(-)^{L}$; for k odd, there are $\frac{1}{2}(k+1)$ tensor harmonics with parity $(-)^{L}$ and $\frac{1}{2}(k+1)$ with parity $(-)^{L+1}$.

(v) Under complex conjugation,

$$\mathcal{Y}^{\lambda*}_{LM}=(-)^{M+\Lambda(\lambda)}\mathcal{Y}^{\lambda}_{L-M}$$

with $\Lambda(\lambda_{+})=k$, $\Lambda(0)=0$, and $\Lambda(\lambda_{-})=k+1$.

Proof. (i) We have

$$(T^{\lambda},T^{\lambda'})\equiv\sum_{i_1\cdots i_k}T^{\lambda*}_{i_1\cdots i_k}(\hat{\mathbf{q}})T^{\lambda'}_{i_1\cdots i_k}(\hat{\mathbf{q}})=\delta_{\lambda\lambda'}\ .$$

Consequently,

$$\begin{aligned}\int d^2\hat{\mathbf{q}}(\mathcal{T}^{\lambda}_{LM},\mathcal{T}^{\lambda'}_{L'M'})&=\frac{2L+1}{4\pi}\int d^2\hat{\mathbf{q}}(T^{\lambda},T^{\lambda'})\mathcal{D}^{L*}_{M\lambda}(R)\mathcal{D}^{L'}_{M'\lambda'}(R)\\&=\delta_{\lambda\lambda'}\frac{2L+1}{4\pi}\int d^2\hat{\mathbf{q}}\mathcal{D}^{L*}_{M\lambda}(R)\mathcal{D}^{L'}_{M'\lambda}(R)\\&=\delta_{\lambda\lambda'}\delta_{LL'}\delta_{MM'}\ .\end{aligned}$$

(ii) Consider a transverse and symmetric rank-k tensor field $\mathcal{A}(\hat{\mathbf{q}})$ defined on S_2. For fixed $\hat{\mathbf{q}}$, $T^{\lambda}(\hat{\mathbf{q}})$ is an orthonormal basis in the space of transverse and symmetrical tensors of rank k which are defined in the tangent space at $\hat{\mathbf{q}}$. With (3.4) we can thus write

$$\mathcal{A}(\hat{\mathbf{q}})=\sum_{\lambda=-k}^{k}e^{i\lambda\psi}c_{\lambda}(\theta,\phi)T^{\lambda}(\hat{\mathbf{q}})\ .$$

By the Peter-Weyl theorem,[7] the set $\mathcal{D}^{L*}_{MM'}(\theta,\phi,\psi)$ is complete on SO(3) and one can write

$$e^{i\lambda\psi}c_{\lambda}(\theta,\phi)=\sum_{LMM'}c^{L\lambda}_{MM'}\mathcal{D}^{L*}_{MM'}(\theta,\phi,\psi)$$

with $c^{L\lambda}_{MM'}$ being constant coefficients. However, since the functions $e^{iM'\psi}$ are linearly independent, the above expansion actually reads

$$e^{i\lambda\psi}c_{\lambda}(\theta,\phi)=\sum_{LM}c_{LM\lambda}\mathcal{D}^{L*}_{M\lambda}(\theta,\phi,\psi)$$

with $c_{LM\lambda}\equiv c^{L\lambda}_{M\lambda}$. For $\mathcal{A}(\hat{\mathbf{q}})$ this implies that

$$\mathcal{A}(\hat{\mathbf{q}})=\sum_{LM\lambda}c_{LM\lambda}\mathcal{D}^{L*}_{M\lambda}(R)T^{\lambda}(\hat{\mathbf{q}})$$

and this was to be proven.

(iii) This follows from the fact that $\mathcal{D}^{L*}_{M\lambda}$ vanishes unless $-L\le M,\ \lambda\le L$ and that $-k\le\lambda\le k$. The second part of the statement is true because $\lambda=(2\nu-k)$, ν being an integer.

(iv) A parity transformation means $\phi\to\pi+\phi$ and $\theta\to\pi-\theta$. But $d^{L}_{M\lambda}(\pi-\theta)=(-)^{L-M}d^{L}_{M-\lambda}(\theta)$ and $e^{i\lambda\psi}T^{\lambda}(-\hat{\mathbf{q}})=e^{i\lambda\psi}T^{-\lambda}(\hat{\mathbf{q}})$ since under parity $\hat{\epsilon}_{\pm}$ transforms into $\hat{\epsilon}_{\mp}$. Putting this together into the definition of $\mathcal{Y}^{\lambda}_{LM}$ leads to the desired result. Moreover, the counting rule for the allowed harmonics given under (iii) gives the number of parity eigenstates harmonics existing for any given L and k.

(v) Under conjugation, $\mathcal{D}^{L*}_{M\lambda}(R)=(-)^{-M-\lambda}\mathcal{D}^{L}_{-M-\lambda}(R)$ and $T^{\lambda*}=T^{-\lambda}$ since $\hat{\epsilon}^{*}_{\pm}=\hat{\epsilon}_{\mp}$.

Some remarks are in order.

(1) Since parity and helicity do not commute, the parity eigenstates $\mathcal{Y}^{\lambda}_{LM}$ are in general different from the helicity eigenstates $\mathcal{T}^{\lambda}_{LM}$.

(2) The importance of the lemma above lies in the fact that it determines the number and parities of the existing transverse and symmetric tensor harmonics for $L<k$ and for $L\ge k$. For instance, for $k=0$ we have from (i) that $\lambda=0$ and $-L\le M\le L$. The scalar spherical harmonic $\mathcal{Y}^{0}_{LM}$ has [from (ii)] parity $(-)^{L}$. Note that $\mathcal{Y}^{\lambda}_{LM}$ reduce for $\lambda=0$ to $\sqrt{(2L+1)/4\pi}\mathcal{D}^{L*}_{M0}(R)=Y_{LM}(\theta,\phi)$ which are the usual spherical harmonics.

For $k=1$, we find from the lemma that there are no transverse harmonics for $L=0$, while for $L\ge 1$ we find that there exist exactly two transverse vector harmonics, one with parity $(-)^{L}$ and the other with parity $(-)^{L+1}$. In Appendix B it is shown that $\mathcal{Y}^{\lambda\pm}_{LM}$ (for $k=1$) are precisely the transverse vector harmonics given at the beginning of this section.

For $k=2$, we find for $L=0$ one transverse tensor harmonic, having parity $(-)^{L}$. For $L=1$, we find again only one tensor harmonic with parity $(-)^{L}$. For a given $L\ge 2$, there exist three transverse and symmetrical harmonics of rank 2, two of them having parity $(-)^{L}$ and one having parity $(-)^{L+1}$. In Appendix B it is shown that the rank-2 spherical harmonics given at the beginning of this section are identical to $\mathcal{Y}^{\lambda\pm,0}_{LM}$ (for $k=2$).

(3) It is evident that the condition of transversality may be relaxed in (3.5). This formula then explicitly represents all the tensor harmonics (for example, the

symmetric and traceless ones) on S_2, by taking the tensor T^{ν} to be constructed accordingly (for example symmetric and traceless). The construction can also be generalized to the case of the spinor spherical harmonics by taking T^{ν} to be a higher-rank spinor. Moreover, by reversing the procedure used in Appendix B, one obtains a systematic way to derive the arbitrary rank tensor harmonics as being expressed in terms of the operators $\hat{\mathbf{q}}$ and $\hat{\nabla}$ acting on the ordinary spherical harmonics $Y_{LM}(\hat{\mathbf{q}})$.

IV. NUMBER AND CLASSIFICATION OF THE FORM FACTORS

In this section we combine the results obtained in the last two sections in order to count and to classify the form factors which characterize the radiation of a single spin-S boson through the conserved tensor current of a spin-J particle. We first study massive particles.

A. Massive particles

Consider the expansion (2.12) and rephase the form factors $\tilde{Q}^{\lambda}_{JL}(|\mathbf{q}|)$ according to

$$\tilde{Q}^{\lambda}_{JL}(|\mathbf{q}|)=(i)^{\Sigma(\lambda)}Q^{\lambda}_{JL}(|\mathbf{q}|)$$

with $\Sigma(\lambda_+)=\Sigma(\lambda_-)=L+k$, $\Sigma(0)=L$. Then, we have the following statement: For a Hermitian operator $T_{i_1\cdots i_k}$ all the form factors occurring in the expansion

$$\langle -\mathbf{q};JM_f|T|\mathbf{q};JM_i\rangle=\sum_{L\lambda}(-)^{J-M_i}\begin{bmatrix} J & L & J\\ -M_f & -M & M_i\end{bmatrix}\times(i)^{\Sigma(\lambda)}Q^{\lambda}_{JL}(|\mathbf{q}|)\mathcal{Y}^{\lambda}_{LM}(\hat{\mathbf{q}}) \quad (4.1)$$

are real. Here λ and L run throughout the values allowed by the lemma of Sec. III and $L\le 2J$.

Proof. By Hermiticity, $\langle -\mathbf{q};JM_f|T|\mathbf{q};JM_i\rangle^*=\langle \mathbf{q};JM_i|T|-\mathbf{q};JM_f\rangle$. Expanding both sides according to (4.1) leads to

$$\sum_{L\lambda}(-)^{J-M_i+\Sigma(\lambda)}\begin{bmatrix} J & L & J\\ -M_f & M_f-M_i & M_i\end{bmatrix}\times(i)^{\Sigma(\lambda)}Q^{\lambda *}_{JL}(|\mathbf{q}|)\mathcal{Y}^{\lambda *}_{LM}(\hat{\mathbf{q}}) ,$$

$$=\sum_{L\lambda}(-)^{J-M_f}\begin{bmatrix} J & L & J\\ -M_i & M_i-M_f & M_f\end{bmatrix}\times(i)^{\Sigma(\lambda)}Q^{\lambda}_{JL}(|\mathbf{q}|)\mathcal{Y}^{\lambda}_{L-M}(-\hat{\mathbf{q}}) .$$

We now compare the coefficients of $\mathcal{Y}^{\lambda}_{L-M}(\hat{\mathbf{q}})$ by using the transformation law under parity and conjugation given in the lemma of Sec. III and taking into account the identity

$$\begin{bmatrix} J & L & J\\ -M_f & M_f-M_i & M_i\end{bmatrix}=\begin{bmatrix} J & L & J\\ -M_i & M_i-M_f & M_f\end{bmatrix} .$$

One finds

$$(-)^{\Sigma(\lambda)+\Lambda(\lambda)}Q^{\lambda *}_{JL}(|\mathbf{q}|)=(-)^{\Pi(\lambda)}Q^{\lambda}_{JL}(|\mathbf{q}|) ,$$

i.e.,

$$Q^{\lambda *}_{JL}(|\mathbf{q}|)=Q^{\lambda}_{JL}(|\mathbf{q}|) ,$$

which was to be proven.

We are now in a position to give the allowed number of form factors in general for the vertex of Fig. 1.

(i) For $2J\ge S$, the number of form factors involved in the process $A(p_i)\rightarrow A(p_f)B(k)$ of a spin-J particle coupling through a conserved and symmetric tensor current to a spin S boson, is at most equal to

$$J(S+1)(S+2)-\tfrac{1}{12}(S^2-4)(2S+3) \quad \text{for } S \text{ even} ,$$

$$J(S+1)(S+2)-\tfrac{1}{12}(S+1)(2S^2+S-9) \quad \text{for } S \text{ odd} .$$

(ii) For $2J<S$, the number of form factors is at most equal to $\frac{1}{2}(2J+1)^2(S-2J)+P_J$ where

$$P_J=\tfrac{1}{3}(J+1)(8J^2+7J+3) \quad \text{for } J \text{ integer, } S \text{ even} ,$$

$$P_J=\tfrac{1}{3}(J+1)(8J^2+7J+3)-\tfrac{1}{2} \quad \text{for } J \text{ integer, } S \text{ odd} ,$$

$$P_J=\tfrac{1}{12}(2J+1)(16J^2+22J+9) \quad \text{for } J\ \tfrac{1}{2}\text{-odd integer} .$$

Proof. Let us denote for $2J\ge k$ by E_k the number of form factors of a rank-k symmetrical and transverse tensor operator with k even and by O_k with k odd. Using the lemma of Sec. III and taking into account that no form factors exist for $L\ge 2J$, one finds

$$E_k=-\tfrac{1}{2}k^2+2Jk+(2J+1) ,$$

$$O_k=-\tfrac{1}{2}k^2+2Jk+(2J+\tfrac{1}{2}) .$$

By decomposing the rank-S tensor into its irreducible parts, one obtains for the total number of form factors of a four-dimensional symmetric and transverse tensor of even rank S:

$$\sum_{\substack{k=0\\ k\text{ even}}}^{S}E_k+\sum_{\substack{k=1\\ k\text{ odd}}}^{S-1}O_k=[J(S+1)(S+2)-\tfrac{1}{12}(S^2-4)(2S+3)] .$$

For S odd, we have to replace S in the first sum above by $(S-1)$ and replace $(S-1)$ in the second sum by S.

(ii) For a rank-k tensor, we have for $2J<k$ the following number of form factors:

$$2J^2+2J+1 \quad \text{for } J \text{ integer, } k \text{ even} ,$$

$$2J^2+2J \quad \text{for } J \text{ integer, } k \text{ odd} , \qquad (4.2)$$

$$2J^2+2J+\tfrac{1}{2} \quad \text{for } J\ \tfrac{1}{2}\text{-odd integer} .$$

We now divide the irreducible parts of the rank-S tensor into two sets: a first set consisting of tensor with rank less or equal to $2J$ and a second set with tensors whose rank is greater than $2J$:

$$T_{0\cdots 0},\ldots,T_{i_1\cdots i_{2J}0\cdots 0} \quad (\text{first set}) ,$$

$$T_{i_1\cdots i_{2J+1}0\cdots 0},\ldots,T_{i_1\cdots i_S} \quad (\text{second set}) .$$

The number of form factors by which the first set contributes, is now given by part (i) by replacing the S there by

$2J$. To count the number of form factors contributed by the second set, we have to distinguish between the following cases.

(A) J is integer and S is even. In that case, there will be $\frac{1}{2}(S-2J)$ tensors of even rank and $\frac{1}{2}(S-2J)$ tensors of odd rank among the tensors of the second set. Taking (4.2) into account, the total number of form factors contributed by the second set will be $\frac{1}{2}(2J+1)^2(S-2J)$.

(B) J is integer and S is odd. There will be $\frac{1}{2}(S-2J+1)$ tensors of odd rank and $\frac{1}{2}(S-2J-1)$ tensors of even rank in the second set. With (4.2), this gives $\frac{1}{2}(2J+1)^2(S-2J)-\frac{1}{2}$ form factors.

(C) J is $\frac{1}{2}$-odd integer and S is even. There will be $\frac{1}{2}(S-2J+1)$ tensors of even rank and $\frac{1}{2}(S-2J-1)$ tensors of odd rank in the second set. With (4.2), this leads to $\frac{1}{2}(2J+1)^2(S-2J)$ form factors.

(D) J is $\frac{1}{2}$-odd integer and S is odd. In this case, the number of tensors of even and odd rank is the same as under (A), and one obtains $\frac{1}{2}(2J+1)^2(S-2J)$ form factors. Adding the contribution of the first and the second set completes the proof of (ii). We illustrate the above result by considering some special cases. For $S=1$, we find one form factor for a spin-0 particle. The covariant matrix element of the current operator is in that case proportional to (p_i+p_f), the proportionality constant (properly normalized) being the charge form factor; on the other hand, if $2J\geq 1$, we find $(6J+1)$ form factors. This leads to four form factors for a Dirac particle. For $S=2$ and $J=0$, we find two form factors; the first one stems from the expansion of T_{00} (for $L=0$) and may be called the mass form factor, while the second one stems from the expansion of T_{ij} (for $L=0$) and is thus the coefficient of the rank-2 transverse spherical harmonic (see Sec. II) $(\delta_{ij}-\hat{q}_i\hat{q}_j)$. The covariant matrix element will thus contain two form factors being the coefficients of $(p_i+p_f)_\mu(p_i+p_f)_\nu$ and $(g_{\mu\nu}-q_\mu q_\nu/q^2)$. The last term gives rise to a contact interaction potential in real space. However, such an interaction affects only quantum-mechanical systems. This may be seen by noticing that in classical physics, the addition of a delta function potential to the Newton potential does not affect the Kepler orbits, since these orbits are sharply defined. In contrast, the quantum-mechanical S states have a nonvanishing probability density at the origin and will thus be affected by a contact interaction. In that sense, the above correction may be called a *quantum* correction to Newtonian gravity.[8]

Concerning generalized Majorana particles, we have the following statement.

(iii) If S is odd and the underlying theory is CPT invariant and the initial and final particle states are CPT self-conjugate then $Q_{JL}^{\lambda+}=Q_{JL}^{0}=0$ and only one set of multipole moments $Q_{JL}^{\lambda-}$ of parity type $(-)^{L+1}$ exists. For $2J\geq S$, the number of these form factors can be at most

$$\tfrac{1}{2}J(S+1)^2-\tfrac{1}{24}(S^2-1)(2S+3)\ ,$$

while for $2J<S$ it can be at most

$$\tfrac{1}{2}J[\tfrac{1}{3}(J+1)(8J+1)+(2J+1)(S-2J)+1] \quad \text{for } J \text{ integer}\ ,$$

$$\tfrac{1}{2}(2J+1)[\tfrac{1}{12}(16J^2+10J+3)+J(S-2J)] \quad \text{for } J\ \tfrac{1}{2}\text{-odd integer}\ .$$

(iv) If S is even and under the same assumptions on the theory and particle states as in (iii), $Q_{JL}^{\lambda-}=0$ and only the multipole moments $Q_{JL}^{\lambda+}$ and Q_{JL}^{0} of parity type $(-)^L$ can exist. The number of these form factors for $2J\geq S$ is

$$\tfrac{1}{2}J(S+2)^2-\tfrac{1}{24}(S+2)(2S^2-S-12)$$

and, for $2J<S$,

$$\tfrac{1}{2}(J+1)[\tfrac{1}{3}(8J^2+13J+6)+(2J+1)(S-2J)] \quad \text{for } J \text{ integer}\ ,$$

$$\tfrac{1}{2}(2J+1)[\tfrac{1}{12}(2J+3)(8J+5)+(J+1)(S-2J)+\tfrac{1}{2}] \quad \text{for } J\ \tfrac{1}{2}\text{-odd integer}\ .$$

Proof. Using (2.22)–(2.24) we obtain

$$\langle -\mathbf{q};JM_f|T|\mathbf{q};JM_i\rangle =(-)^{M_f-M_i+1}\langle \mathbf{q};J-M_i|T|-\mathbf{q};J-M_f\rangle\ .$$

We now expand both sides of this equation according to (4.1) and use the symmetry property

$$\begin{bmatrix} J & L & J \\ M_i & M & -M_f \end{bmatrix}=(-)^{2J+L}\begin{bmatrix} J & L & J \\ -M_f & M & M_i \end{bmatrix}$$

and the transformation law

$$\mathcal{Y}_{LM}^{\lambda}(-\hat{\mathbf{q}})=(-)^{\Pi(\lambda)}\mathcal{Y}_{LM}^{\lambda}(\hat{\mathbf{q}})$$

in order to compare the coefficients of $\mathcal{Y}_{LM}^{\lambda}(\hat{\mathbf{q}})$. This leads to the equalities

$$Q_{JL}^{\lambda}=(-)^{\Pi(\lambda)+L+1}Q_{JL}^{\lambda}$$

and thus only the set $Q_{JL}^{\lambda-}$ which belongs to $\mathcal{Y}_{LM}^{\lambda-}$ exists. With this selection rule, the counting of the nonvanishing form factors proceeds in the same manner as was done in the proof of (i).

(iv) The proof here is the same as in (iii) except that the tensor current is now even under CPT. This leads to

$$Q_{JL}^{\lambda}=(-)^{\Pi(\lambda)+L}Q_{JL}^{\lambda}$$

so that only the two sets $Q_{JL}^{\lambda+}$ and Q_{JL}^{0} of parity type $(-)^L$ can exist.

A case of physical interest is $S=1$. Here, we find from the above statement that a generalized Majorana particle of spin J can have at most $2J$ form factors. Thus, the neutral pion has no single-photon form factors, the Majorana neutrino can have at most one form factor and the Z boson can have at most two form factors. These form factors are of the type $Q_{JL}^{\lambda-}$ and are known under the name "anapole moments" or "charge radius moments." Their counterpart in the classical theory are the so-called[9] "toroidal moments." The origin of this set of form factors lies in the following: in the classical theory (as well as in quantum mechanics) the multipole expansion of a *conserved* current J_μ is determined by *three* sets of multipoles: The electric charge multipoles, the magnetic charge multipoles, and the toroidal multipoles. While the electric charge multipoles are determined by the charge distribution ρ, the magnetic and toroidal multipole moments are determined by the current distribution $\mathbf{J}$. Thus, a system with no electric charge, no electric or magnetic dipole moments, can still possess a toroidal dipole moment. The simplest example of such a system is a coil shaped in the form of torus, for which the magnetic moments of loops opposite to each other cancel, respectively, to first order. Thus, a pure toroidal dipole moment [i.e., letting the dimensions of the torus go to zero while keeping the product (current)×(dimensions of the torus) fixed] possesses no electric or magnetic moments and will thus not interact with a given external field but will only scatter when it "hits" an external source, since the fields are confined within the coil. In this sense the interaction of a toroidal moment may be called a contact interaction. This behavior also holds quantum mechanically: for a Dirac particle, the interaction with an external current J_μ (one-photon exchange) is $\langle p_f|j_\mu|p_i\rangle(1/q^2)J_\mu$ and contains four terms: the first three are the Coulomb interaction and the interaction of the electric and magnetic dipole moments. The fourth term is the anapole dipole moment and its interaction at low frequencies is $(q\cdot\gamma q_\mu-\gamma_\mu q^2)\gamma_5(1/q^2)J_\mu=\gamma_5\gamma_\mu J_\mu$ since the external current is also conserved. The Fourier transform of this interaction is a delta function in the coordinate space leading to a contact interaction with the external source. As a measurable physical consequence, all the S states in an atom in which the bound particle possesses an anapole moment will be shifted due to the point interaction at the origin. In the classical theory the radiation of the anapole moment is negligible in the wave zone when compared with the radiation of the electric or magnetic dipoles. It has to be taken into account, however, when the emitted wavelength is short (compared to the characteristic length of the emitting source), such as in nuclei.[10]

For $S=2$ and $J=0$, we find for a Majorana particle two form factors, which is the same number as for any other particle with the same spin. We thus see that to lowest order in the multipole expansion, the coupling of the particle and its antiparticle to the even spin field is the same. That this property is not true in general is the content of the next statement. Note that in contrast, any odd spin S field, always gives 0 for the charge (i.e., the lowest-order form factor) of a spin-J Majorana particle coupling to it.

(v) If the underlying theory is CPT is invariant, the form factors Q^λ_{JL} of a particle and $\bar{Q}^\lambda_{JL}$ of its antiparticle satisfy

$$Q^{\lambda_+}_{JL}=-\bar{Q}^{\lambda_+}_{JL},\quad Q^0_{JL}=-\bar{Q}^0_{JL},\quad Q^{\lambda_-}_{JL}=\bar{Q}^{\lambda_-}_{JL}\quad \text{for } S \text{ odd}\ ,$$

$$Q^{\lambda_+}_{JL}=\bar{Q}^{\lambda_+}_{JL},\quad Q^0_{JL}=\bar{Q}^0_{JL},\quad Q^{\lambda_-}_{JL}=-\bar{Q}^{\lambda_-}_{JL}\quad \text{for } S \text{ even}\ .$$

Proof. Using (2.22)–(2.25) we obtain

$$\langle -\mathbf{q};JM_f|T|\mathbf{q};JM_i\rangle =(-)^{M_f-M_i+S}\langle\overline{\mathbf{q};J-M_i}|T|\overline{-\mathbf{q};J-M_f}\rangle\ .$$

Expanding both sides according to (4.1), we then compare the coefficients of the tensor harmonics (see the previous proof). This leads to the equations

$$Q^{\lambda_+}_{JL}=(-)^S\bar{Q}^{\lambda_+}_{JL},\quad Q^0_{JL}=(-)^S\bar{Q}^0_{JL},\quad Q^{\lambda_-}_{JL}=(-)^{S+1}\bar{Q}^{\lambda_-}_{JL}\ ,$$

which are the content of the above statement.

Some remarks are in order.

(A) For $S=0$, we know from the general counting that a spin-J particle possesses at most $(2J+1)$ form factors; however, they all are of the type Q^0_{JL} and the previous statement asserts that a spin-0 field couples in the same way to particles and antiparticles. Thus, an interaction mediated by a spin-0 boson is independent of the matter-antimatter nature of the particles. This fact is only true for a spin-0 field.

(B) For S odd, unless the particle is a generalized Majorana particle, the field mediating the interaction always distinguishes between particles and antiparticles.

(C) The equalities $Q^0_{JL}=\bar{Q}^0_{JL}$, which are valid for S even, show that all the mass multipoles (e.g., the mass, the mass dipole, the mass quadrupole, . . .) are equal for a particle and its antiparticle. Furthermore, for S even, we find the following: if $J=0$ or the spin-J particle is a generalized Majorana particle, the coupling is independent of the particle-antiparticle nature; if $J\geq\frac{1}{2}$ and the particle is distinct from its antiparticle, form factors of the type $Q^{\lambda_-}_{JL}$ may be present. This has the consequence that in a quantum theory of gravity the scattering of a particle mediated by graviton exchange will be different in general from the scattering of the antiparticle in the same experiment. To gain more insight, we consider the case $J=\frac{1}{2}$ and find from (ii) that there are at most six form factors for the coupling to the graviton. With $p=(p_i+p_f)$ and $q=(p_i-p_f)$, the expression

$$\overline{u_f}\left[f(q^2)p_\mu p_\nu+F(q^2)(p_\mu\sigma_{\nu\lambda}q_\lambda)_s\gamma_5+g(q^2)(p_\mu\sigma_{\nu\lambda}q_\lambda)_s +G(q^2)[p_\mu(q\cdot\gamma q_\nu-q^2\gamma_\nu)]_s\gamma_5+[h(q^2)+H(q^2)\gamma_5]q^2\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\right]u_i$$

is thus the most general one which can be written down for the matrix elements of the conserved and symmetrical current, since the six tensors occurring in it are independent. One can also establish this statement explicitly, with no reference to (ii): by using Gordon identities and reduction formulas for the product of γ matrices, one can check that all other symmetric and transverse tensors such as $p_\mu p_\nu\gamma_5, \ldots$ lead to nothing new in the expression above. $f(q^2)$ and $F(q^2)$ are the form factors of the mass and the mass dipole, respectively. $g(q^2)$ is the dipole of the momentum flow distribution and its electromagnetic analog is the magnetic dipole. The form factor $G(q^2)$ may be identified with $Q_{JL}^{\lambda-}$ and its electromagnetic counterpart is the anapole moment. In the low-frequency limit, this terms gives rise to a contact interaction in real space with the external source. This will affect the S-wave scattering of particles mediated by graviton exchange and the S states of bound systems. However, $G(q^2)$ does not contribute (in the low-frequency limit) to scattering processes where there is no contact with the external source, such as in the scattering of particles on an exterior classical gravitational field. This implies that under CPT, protons and antiprotons or neutrons and antineutrons behave identically —say—when falling in the gravitational field of the Earth. The next statements show that the form factors $Q_{JL}^{\lambda-}$ do occur in parity- (or time-reversal-) violating theories.

(vi) If the underlying theory is C conserving, the form factors of a particle and its antiparticle are equal in magnitude and sign if S is even and are equal in magnitude but opposite in sign if S is odd. If S is odd, and the initial and final particle states are C self-conjugate, all the form factors vanish.

(vii) If the theory is P conserving, the form factors of the irreducible part $T_{i_1\cdots i_k}$ satisfy $Q_{JL}^{\lambda+,0}=(-)^L Q_{JL}^{\lambda+,0}$, $Q_{JL}^{\lambda-}=(-)^{L+1}Q_{JL}^{\lambda-}$ if k is even and $Q_{JL}^{\lambda+}=(-)^{L+1}Q_{JL}^{\lambda+}$, $Q_{JL}^{\lambda-}=(-)^L Q_{JL}^{\lambda-}$ if k is odd. For $2J\geq S$, a spin-J particle can have at most $\frac{1}{2}J(S+1)(S+2)-\frac{1}{24}X_{J,S}$ form factors where

$$X_{J,S}=(S+2)(2S^2-S-12) \quad \text{for } S \text{ even, } J \text{ integer},$$

$$X_{J,S}=(S+2)(S-2)(2S+3) \quad \text{for } S \text{ even, } J \ \tfrac{1}{2}\text{-odd integer},$$

$$X_{J,S}=(S+1)(S+3)(2S-5) \quad \text{for } S \text{ odd, } J \text{ integer},$$

$$X_{J,S}=(S+1)(2S^2+S-9) \quad \text{for } S \text{ odd, } J \ \tfrac{1}{2}\text{-odd integer}.$$

If $2J<S$, the number of form factors allowed for $2J$ even is $\frac{1}{2}(2J^2+2J+1)(S-2J)+Y_{J,S}$ where

$$Y_{J,S}=\tfrac{1}{6}(J+1)(8J^2+7J+6) \quad \text{for } S \text{ even},$$

$$Y_{J,S}=\tfrac{1}{6}(J+1)(8J^2+7J+6)-\tfrac{1}{2} \quad \text{for } S \text{ odd},$$

while for $2J$ odd the number is

$$\tfrac{1}{4}(2J+1)^2(S-2J)+\tfrac{1}{24}(2J+1)(16J^2+22J+9).$$

(viii) If the underlying theory is T conserving, the form factors of the irreducible part $T_{i_1\cdots i_k}$ satisfy $Q_{JL}^{\lambda}=(-)^L Q_{JL}^{\lambda}$ if k is even and $Q_{JL}^{\lambda}=(-)^{L+1}Q_{JL}^{\lambda}$ if k is odd. For $2J\geq S$, a spin J particle can have at most $\frac{1}{2}J(S+1)(S+2)-\frac{1}{12}V_{J,S}$ form factors where

$$V_{J,S}=(S+2)(S^2-2S-6) \quad \text{for } S \text{ even, } J \text{ integer},$$

$$V_{J,S}=(S+1)(S^2-S-9) \quad \text{for } S \text{ odd, } J \ \tfrac{1}{2}\text{-odd integer},$$

$$V_{J,S}=(S+1)(S+2)(S-3) \quad \text{for } S \text{ even (odd), } 2J \text{ odd (even)}.$$

For $2J<S$, the particle has at most $\frac{1}{2}(J+1)(2J+1)(S-2J)+W_{J,S}$ form factors where

$$W_{J,S}=\tfrac{1}{3}(J+1)(4J^2+5J+3) \quad \text{for } S \text{ even, } J \text{ integer},$$

$$W_{J,S}=\tfrac{1}{12}(2J+1)(8J^2+14J+9) \quad \text{for } S \text{ odd, } J \ \tfrac{1}{2}\text{-odd integer},$$

$$W_{J,S}=\tfrac{1}{6}(2J+1)(J+1)(4J+3) \quad \text{for } S \text{ even (odd), } 2J \text{ odd (even)}.$$

Proof. (vi) Using (2.18) and (2.19) we obtain, for a rank-k tensor,

$$\langle p_f;JM_f|T|p_i;JM_i\rangle=(-)^S\langle\overline{p_f;JM_f}|T|\overline{p_i;JM_i}\rangle$$

from which the statements immediately follow.

(vii) Using (2.16), (2.17), and (2.26), we obtain for the irreducible part of T:

$$\langle -\mathbf{q};JM_f|T_{i_1\cdots i_k}|\mathbf{q};JM_i\rangle =(-)^k\langle \mathbf{q};JM_f|T_{i_1\cdots i_k}|-\mathbf{q};JM_i\rangle .$$

Expanding both sides according to (4.1), we obtain the desired relation

$$Q_{JL}^{\lambda}=(-)^{k+\Pi(\lambda)}Q_{JL}^{\lambda} .$$

With this selection rule, the counting of the form factors proceeds in the same manner as was done in the proof of (i).

(viii) Using (2.20), (2.21), and (2.26) we have

$$\langle -\mathbf{q};JM_f|T_{i_1\cdots i_k}|\mathbf{q};JM_i\rangle =(-)^{M_f-M_i+k}\langle -\mathbf{q};J-M_i|T_{i_1\cdots i_k}|\mathbf{q};J-M_f\rangle .$$

Upon expanding both sides and comparing the coefficients, we find

$$Q_{JL}^{\lambda}=(-)^{k+L}Q_{JL}^{\lambda} .$$

Some remarks are in order.

(A) From the selection rules above, we also see that if P and T or alternatively C are conserved, all the $Q_{JL}^{\lambda-}$ have to vanish. Half of the form factors $Q_{JL}^{\lambda-}$ are P-violating quantities while the other half are T violating. Moreover, we see that for a spin-$\frac{1}{2}$ particle, P conservation alone is sufficient to ensure the vanishing of the $Q_{JL}^{\lambda-}$ form factors.

(B) The selection rules given above allow one to directly infer which form factors violate which symmetry. For example, they show that if P is conserved, no mass dipole, no electric dipole moment, no magnetic quadrupole moment, no anapole dipole moment, are allowed, etc. On the other hand, they show that if T is conserved, no mass dipole, no electric dipole, no magnetic quadrupole, etc., can exist. However, they show that the anapole dipole moment is a T-conserving but a P-violating quantity. They also show that an electric or a mass quadrupole are

P- and T-conserving quantities, etc.

We finally give the number M_J of single-photon form factors and the number N_J of single graviton form factors which a Majorana particle is allowed to possess in the event where P or T are conserved.

(a) If the underlying theory is P conserving,

$$M_J=J \quad N_J=4J+2 \quad J \text{ integer },$$

$$M_J=J-\tfrac{1}{2} \quad N_J=4J+1 \quad J \ \tfrac{1}{2}\text{-odd integer }.$$

(b) If the underlying theory is T conserving,

$$M_J=J \quad N_J=4J+2 \quad J \text{ integer },$$

$$M_J=J+\tfrac{1}{2} \quad N_J=4J+1 \quad J \ \tfrac{1}{2}\text{-odd integer }.$$

B. Massless particles

As explained in Sec. II, the form factors of the vertex of a massless on-shell spin-J particle, coupling through a tensor current to a spin-S boson, may be defined through the matrix elements

$$Q^J_{\lambda_f\lambda_i}(\mathbf{q},k,\nu)\equiv\langle -\mathbf{q};\lambda_f|T(k,\nu)|\mathbf{q};\lambda_i\rangle \tag{4.3}$$

of the canonical components $T(k,\nu)\equiv T_{\underbrace{+\cdots+}_{k-\nu \text{ times}}\ \underbrace{-\cdots-}_{\nu \text{ times}}}$ with $0\le\nu\le k$ and $0\le k\le S$. Because of Hermiticity, not all the matrix elements in (4.3) are independent. To see this, we relate the state $|-\mathbf{q};\lambda\rangle$ to the state $|\mathbf{q};\lambda\rangle$ through a rotation by π around the $\hat{\mathbf{y}}$ axis:

$$|-\mathbf{q};\lambda\rangle=c(\mathbf{q},\lambda)e^{-i\pi \mathbf{J}_y}|\mathbf{q};\lambda\rangle \tag{4.4}$$

with $c(\mathbf{q},\lambda)$ being a phase factor and

$$e^{-2i\pi\mathbf{J}_y}=(-)^{2J}. \tag{4.5}$$

We then have

$$Q^J_{\lambda_f\lambda_i}(\mathbf{q},k,\nu)=(-)^k c^*(\mathbf{q},\lambda_i)c(-\mathbf{q},\lambda_f)Q^{J*}_{\lambda_i\lambda_f}(\mathbf{q},k,\nu).$$

Proof. The operator $T(k,\nu)$ transforms under a rotation by π around the $\hat{\mathbf{y}}$ axis according to

$$e^{-i\pi J_y}T(k,\nu)e^{i\pi J_y}=(-)^k T(k,k-\nu). \tag{4.6}$$

Inserting a rotation in (4.3) and making use of (4.4) and (4.6), we obtain the relation

$$Q^J_{\lambda_f\lambda_i}(\mathbf{q},k,\nu) =(-)^k c^*(\mathbf{q},\lambda_i)c(-\mathbf{q},\lambda_f)Q^J_{\lambda_f\lambda_i}(-\mathbf{q},k,k-\nu).$$

But the Hermiticity of $T_{\mu_1\cdots\mu_S}$ implies

$$Q^J_{\lambda_f\lambda_i}(\mathbf{q},k,\nu)=Q^{J*}_{\lambda_i\lambda_f}(-\mathbf{q},k,k-\nu).$$

Combining the last two equations yields the result. We next give the general number of nonvanishing matrix elements $Q^J_{\lambda_f\lambda_i}$ for a massless spin-J particle. The matrix elements of a symmetric and conserved tensor current, through which a massless particle of spin J couples to a spin-S boson, possess the following properties:

(i) The matrix elements of the tensor current between equal helicity states vanish if $2J>S$.

(ii) If $J=0$ or $2J>S$, there are at most $\frac{1}{2}(S+2)$ form factors for S even and $\frac{1}{2}(S+1)$ form factors for S odd.

(iii) If $0<2J\le S$, the number of form factors allowed is at most $\frac{3}{2}S-2J+\Gamma_{J,S}$ where

$$\Gamma_{J,S}=3 \quad \text{for } S \text{ even and } J \text{ integer },$$

$$\Gamma_{J,S}=2 \quad \text{for } S \text{ even and } J \ \tfrac{1}{2}\text{-odd integer },$$

$$\Gamma_{J,S}=\tfrac{3}{2} \quad \text{for } S \text{ odd and } J \text{ integer },$$

$$\Gamma_{J,S}=\tfrac{5}{2} \quad \text{for } S \text{ odd and } J \ \tfrac{1}{2}\text{-odd integer }.$$

Proof. Under a rotation by ϕ around the $\hat{\mathbf{z}}$ axis, the states and the tensor transform according to

$$\mathcal{R}(\phi)|\mathbf{q},\lambda_i\rangle=e^{i\lambda_i\phi}|\mathbf{q},\lambda_i\rangle,$$

$$\mathcal{R}(\phi)|-\mathbf{q},\lambda_f\rangle=e^{-i\lambda_f\phi}|\mathbf{q},\lambda_f\rangle,$$

$$\mathcal{R}T(k,\nu)\mathcal{R}^{-1}=e^{i(k-2\nu)\phi}T(k,\nu).$$

Inserting such a rotation in the matrix element (4.3), we obtain

$$Q^J_{\lambda_f\lambda_i}(\mathbf{q},k,\nu)=e^{i(\lambda_i+\lambda_f+k-2\nu)\phi}Q^J_{\lambda_f\lambda_i}(\mathbf{q},k,\nu).$$

Therefore, if $Q^J_{\lambda_f\lambda_i}$ is not to vanish, the initial and final helicities have to obey

$$\lambda_i+\lambda_f=2\nu-k.$$

Since $(\lambda_i+\lambda_f)$ can only assume the values 0 and $\pm 2J$, the only nonvanishing matrix elements are

$$Q^J_{J-J}(\mathbf{q},k,\tfrac{1}{2}k)$$

for k even $(0\le k\le S)$, and

$$Q^J_{JJ}(\mathbf{q},k,\nu) \quad \text{and} \quad Q^J_{-J-J}(\mathbf{q},k,k-\nu)$$

for k such that $2J\le k\le S$ and such that $\nu=(J+\frac{1}{2}k)$ is an integer. Equipped with these selection rules, it is straightforward to deduce the statements (i)–(iii).

We note that the concept "diagonal elements" in (i) is frame independent, since the helicity is a Lorentz invariant for massless particles. The theorems of Weinberg and Witten[11] are contained in (i), namely, for the case where T_μ and $T_{\mu\nu}$ are, respectively, taken to be the covariant charge current and the covariant energy-momentum tensor. These theorems state that in a theory allowing the construction of a conserved and Lorentz-*covariant* charge current, no massless charged particles with spin greater than $\frac{1}{2}$ exist and that in a theory allowing the construction of a conserved and Lorentz-*covariant* energy-momentum tensor, no massless particles with spin greater than 1 exist. We next give the number of nonvanishing form factors in the event where the particle is a massless *CPT* eigenstate, such as the photon, the graviton, and the massless gravitino.

If the underlying theory is *CPT* invariant, the nonvanishing form factors of a massless generalized Majorana particle coupling through a symmetric and conserved tensor current to a spin-S boson satisfy

$$Q^J_{J-J}(\mathbf{q},k,\tfrac{1}{2}k)=(-)^S Q^J_{J-J}(\mathbf{q},k,\tfrac{1}{2}k)\ ,$$

$$Q^J_{JJ}(\mathbf{q},k,\nu)=(-)^{2J+k+S}c^*(\mathbf{q},J)c(\mathbf{q},-J)\times Q^J_{-J-J}(\mathbf{q},k,k-\nu)\ .$$

(iv) If S is even and $J=0$ or $2J>S$, the number of form factors is at most $\frac{1}{2}(S+2)$, while for $0<2J\le S$ it is at most

$(S-J+2)$ for S even, J integer ,

$(S-J+\frac{3}{2})$ for S even, J $\frac{1}{2}$-odd integer .

(v) If S is odd and $J=0$ or $2J>S$, the particle has no coupling at all to the single boson. If $0<2J\le S$, the number of form factors could be at most

$(\frac{1}{2}S-J+\frac{1}{2})$ for S odd, J integer ,

$(\frac{1}{2}S-J+1)$ for S odd, J $\frac{1}{2}$-odd integer .

Proof. Using (2.22), (2.24), (4.3)–(4.6), we have, for the nonvanishing form factors,

$$Q^J_{J-J}(\mathbf{q},k,\tfrac{1}{2}k)=c^*(\mathbf{q},J)\langle \mathbf{q};J|e^{i\pi \mathbf{J}_y}T(k,\tfrac{1}{2}k)|\mathbf{q};-J\rangle=(-)^S c^*(\mathbf{q},J)\eta^*_\Theta(\mathbf{q},-J)\eta_\Theta(\mathbf{q},J)\langle \mathbf{q};J|T(k,\tfrac{1}{2}k)e^{-i\pi \mathbf{J}_y}|\mathbf{q};-J\rangle\ .$$

Because of Eq. (2.27), $\eta^*_\Theta(\mathbf{q},-J)\eta_\Theta(\mathbf{q},J)=(-)^{2J}$, leading to

$$\begin{aligned}Q^J_{J-J}(\mathbf{q},k,\tfrac{1}{2}k)&=(-)^S c^*(\mathbf{q},J)(-)^{2J}\langle \mathbf{q};J|e^{-i\pi \mathbf{J}_y}e^{i\pi \mathbf{J}_y}T(k,\tfrac{1}{2}k)e^{-i\pi \mathbf{J}_y}|\mathbf{q};-J\rangle\\&=(-)^{S+k}c^*(\mathbf{q},J)(-)^{2J}\langle \mathbf{q};J|e^{-i\pi \mathbf{J}_y}T(k,\tfrac{1}{2}k)|\mathbf{q};-J\rangle\\&=(-)^{S+k}c^*(\mathbf{q},J)\langle \mathbf{q};J|e^{i\pi \mathbf{J}_y}T(k,\tfrac{1}{2}k)|\mathbf{q};-J\rangle\\&=(-)^{S+k}\langle -\mathbf{q};J|T(k,\tfrac{1}{2}k)|\mathbf{q};-J\rangle\ ,\end{aligned}$$

which proves the first part of the equalities. On the other hand,

$$\begin{aligned}Q^J_{JJ}(\mathbf{q},k,\nu)=c^*(\mathbf{q},J)\langle \mathbf{q};J|e^{i\pi \mathbf{J}_y}T(k,\nu)|\mathbf{q};J\rangle&=(-)^S c^*(\mathbf{q},J)\langle \mathbf{q};-J|T(k,\nu)e^{-i\pi \mathbf{J}_y}|\mathbf{q};-J\rangle\\&=(-)^{S+k}c^*(\mathbf{q},J)\langle \mathbf{q};-J|e^{-i\pi \mathbf{J}_y}T(k,k-\nu)|\mathbf{q};-J\rangle\\&=(-)^{S+k+2J}c^*(\mathbf{q},J)c(\mathbf{q},-J)\langle -\mathbf{q};-J|T(k,k-\nu)|\mathbf{q};-J\rangle\ .\end{aligned}$$

With these selection rules, the statements (iv) and (v) are straightforward to derive. We list some examples. For $S=1$, we recover the theorem of Ref. 3; the massless gravitino and the graviton, for which $2J>1$, have consequently no single-photon form factors. For $S=3$, the massless gravitino can have at most one form factor, while the graviton has no coupling at all to a single spin-3 particle.

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APPENDIX A

In order to prove (2.12), we make use of the transformation property of the various objects involved. First, the particle states transform under a rotation R according to (2.4). The Wigner matrices $\mathcal{D}^J$ are unitary and possess the symmetry

$$\mathcal{D}^{J*}_{M'M}(R)=(-)^{M'-M}\mathcal{D}^J_{-M'-M}(R)$$

and the product of three $\mathcal{D}$'s has the property

$$\int dR\,\mathcal{D}^{J_1}_{M_1'M_1}(R)\mathcal{D}^{J_2}_{M_2'M_2}(R)\mathcal{D}^{J_3}_{M_3'M_3}(R)=\begin{pmatrix}J_1 & J_2 & J_3\\ M_1' & M_2' & M_3'\end{pmatrix}\begin{pmatrix}J_1 & J_2 & J_3\\ M_1 & M_2 & M_3\end{pmatrix} \qquad \text{(A1)}$$

the integral being taken over the whole rotation group with a normalized measure. Next, the tensor operator transforms according to (2.7). We now use (2.4) and (2.7) in order to relate the matrix elements of $T_{i_1\cdots i_k}$ in two different frames differing from each other by a rotation R:

$$\langle -\mathbf{q};JM_f|T_{i_1\cdots i_k}|\mathbf{q};JM_i\rangle=\sum_{\substack{M'M''\\ j_1\cdots j_k}}\mathcal{D}^{J*}_{M''M_f}(R)\mathcal{D}^J_{M'M_i}(R)R_{j_1i_1}\cdots R_{j_ki_k}\times\langle -R\mathbf{q};JM''|T_{j_1\cdots j_k}|R\mathbf{q};JM'\rangle\ . \qquad \text{(A2)}$$

We now expand both sides of the above equation according to (2.10) in tensor harmonics and obtain

$$\sum_{LM\lambda} \tilde{Q}^{\lambda M_f M_i}_{JLM}(|q|)\mathcal{Y}^{\lambda}_{LMi_1\cdots i_k}(\hat{\mathbf{q}})$$
$$= \sum_{\substack{M'M''LM\lambda\\ j_1\cdots j_k}} \mathcal{D}^{J*}_{M''M_f}(R)\mathcal{D}^{J}_{M'M_i}(R)R_{j_1i_1}\cdots R_{j_ki_k}$$
$$\times \tilde{Q}^{\lambda M''M'}_{JLM}(|\mathbf{q}|)\mathcal{Y}^{\lambda}_{LMj_1\cdots j_k}(R\hat{\mathbf{q}}) . \tag{A3}$$

On the right-hand side, one can now replace $\mathcal{Y}^{\lambda}_{LMj_1\cdots j_k}(R\hat{\mathbf{q}})$ using Eq. (2.9) by the quantities $\mathcal{Y}^{\lambda}_{LMj_1\cdots j_k}(\hat{\mathbf{q}})$. This enables us to compare the coefficients of $\mathcal{Y}^{\lambda}_{LM}$ on both sides of (A3). We thus find that the form factors possess the property

$$\tilde{Q}^{\lambda M_f M_i}_{JLM}(|\mathbf{q}|) = \sum_{M'M''m} \mathcal{D}^{J*}_{M''M_f}(R)\mathcal{D}^{L}_{Mm}(R^{-1})\mathcal{D}^{J}_{M'M_i}(R) \times \tilde{Q}^{\lambda M''M'}_{JLm}(|\mathbf{q}|) ,$$

which can be rewritten using the symmetry properties of the $\mathcal{D}$'s as

$$\tilde{Q}^{\lambda M_f M_i}_{JLM}(|\mathbf{q}|) = \sum_{M'M''m} (-)^{M''+m-M_f-M}\mathcal{D}^{J}_{-M''-M_f}(R) \times \mathcal{D}^{L}_{-m-M}(R) \times \mathcal{D}^{J}_{M'M_i}(R)\tilde{Q}^{\lambda M''M'}_{JLm}(|\mathbf{q}|) .$$

As a final step, we integrate both sides of this equation over the whole rotation group and make use of (A1). This leads to the factorization property (2.11) with the reduced matrix element $\tilde{Q}^{\lambda}_{JL}(|\mathbf{q}|)$ being defined as

$$\tilde{Q}^{\lambda}_{JL}(|\mathbf{q}|) \equiv \sum_{M'M''m} (-)^{m+M''-J} \begin{pmatrix} J & L & J \\ -M'' & -m & M' \end{pmatrix} \times \tilde{Q}^{\lambda M''M'}_{JLm}(|\mathbf{q}|)$$

a quantity only depending on J, L, and λ. This proves the expansion (2.12).

APPENDIX B

Here we show that the tensor spherical harmonics of rank k constructed in Sec. III, reduce for $k=1$ and 2 to the ones constructed with the help of the operators $\hat{\mathbf{q}}$ and $\hat{\nabla}$. First, we derive two useful recurrence relations between the Wigner functions $d^{L}_{MM'}(\theta) \equiv \langle LM|e^{-i\mathbf{J}_y\theta}|LM'\rangle$. Define the operator

$$\mathbf{j}_{\pm} \equiv \left[\frac{d}{d\theta} \pm \frac{M}{\sin\theta} \mp M'\cot\theta \right] .$$

Then the following equation holds:

$$\mathbf{j}_{\pm} d^{L}_{MM'}(\theta) = \mp\sqrt{(L\pm M'+1)(L\mp M')}\, d^{L}_{MM'\pm 1}(\theta) . \tag{B1}$$

To see this, we apply the operator $(d/d\theta \pm M/\sin\theta)$ to $d^{L}_{MM'}$ and obtain, as a result,

$$\langle LM|e^{-i\mathbf{J}_y\theta}(-i\mathbf{J}_y)|LM'\rangle \pm \frac{1}{\sin\theta}\langle LM|\mathbf{J}_z e^{-i\mathbf{J}_y\theta}|LM'\rangle$$
$$= \langle LM|e^{-i\mathbf{J}_y\theta}(-i\mathbf{J}_y)|LM'\rangle \pm \frac{1}{\sin\theta}\langle LM|e^{-i\mathbf{J}_y\theta}(\cos\theta\mathbf{J}_z - \sin\theta\mathbf{J}_x)|LM'\rangle$$
$$= \pm M'\cot\theta d^{L}_{MM'}(\theta) \mp \langle LM|e^{-i\mathbf{J}_y\theta}\mathbf{J}_{\pm}|LM'\rangle .$$

Acting with $\mathbf{J}_{\pm}$ on $|LM'\rangle$ yields the result. Another identity we need is

$$\hat{\nabla}_{\pm} \equiv \frac{1}{\sqrt{2}}(\hat{\nabla} \pm i\hat{\mathbf{q}}\times\hat{\nabla}) = \hat{\epsilon}_{\pm}\left[\partial_\theta \mp \frac{i}{\sin\theta}\partial_\phi\right] . \tag{B2}$$

With the vector harmonics given at the beginning of Sec. III, we obtain, using (B1) and (B2),

$$\mathbf{Y}^{\pm}_{LM}(\hat{\mathbf{q}}) \equiv \frac{1}{\sqrt{2}}[-\mathbf{Y}^{1}_{LM}(\hat{\mathbf{q}}) \pm i\mathbf{Y}^{0}_{LM}(\hat{\mathbf{q}})]$$
$$= \frac{1}{\sqrt{L(L+1)}}\frac{1}{\sqrt{2}}(\hat{\nabla} \pm i\hat{\mathbf{q}}\times\hat{\nabla})Y_{LM}(\theta,\phi)$$
$$= \frac{1}{\sqrt{L(L+1)}}\hat{\epsilon}_{\pm}\left[\partial_\theta \mp \frac{i}{\sin\theta}\partial_\phi\right]\left[\frac{2L+1}{4\pi}\right]^{1/2} d^{L}_{M0}(\theta)e^{iM\phi}$$
$$= \frac{1}{\sqrt{L(L+1)}}\left[\frac{2L+1}{4\pi}\right]^{1/2}[\mathbf{j}_{\pm}d^{L}_{M0}(\theta)]e^{iM\phi}\hat{\epsilon}_{\pm} = \mp\left[\frac{2L+1}{4\pi}\right]^{1/2} d^{L}_{M\pm 1}(\theta)e^{iM\phi}\hat{\epsilon}_{\pm} , \tag{B3}$$

which is, up to a sign, $\mathcal{T}^{\pm}_{LM}(\hat{\mathbf{q}})$ for $k=1$. We now go over to the rank-2 harmonics. We first note the identity

$$\hat{\nabla}_i\hat{q}_j = (\delta_{ij} - \hat{q}_i\hat{q}_j) \tag{B4}$$

so that the harmonic $Y^{0}_{LM;ij}$ may be written in a more compact form as

$$Y^{0}_{LM}(\hat{\mathbf{q}}) = Y_{LM}(\hat{\mathbf{q}})\left[\frac{1!(2-1)!}{2!}\right]^{1/2}\hat{\nabla}\otimes\hat{\mathbf{q}} . \tag{B5}$$

To compute the tensor product $\hat{\nabla}\otimes\hat{\mathbf{q}}$ we decompose $\hat{\nabla}$ according to

$$\hat{\nabla}=\frac{1}{\sqrt{2}}(\hat{\nabla}_{+}+\hat{\nabla}_{-}) \tag{B6}$$

and use the identities

$$\frac{1}{\sqrt{2}}\hat{\nabla}_{+}\otimes\hat{\mathbf{q}}=\hat{\epsilon}_{+}\otimes\hat{\epsilon}_{-}\ , \tag{B7}$$

$$\frac{1}{\sqrt{2}}\hat{\nabla}_{-}\otimes\hat{\mathbf{q}}=\hat{\epsilon}_{-}\otimes\hat{\epsilon}_{+}\ . \tag{B8}$$

Putting (B6)–(B8) into (B5) shows that the set Y^0_{LM} is identical with the set $\mathcal{Y}^0_{LM}=\mathcal{T}^0_{LM}$ for $k=2$. To proceed further, we need the identities

$$\hat{\nabla}_{\pm}\otimes\hat{\epsilon}_{\mp}=\cot\theta\hat{\epsilon}_{\pm}\otimes\hat{\epsilon}_{\mp}\ , \tag{B9}$$

$$\hat{\nabla}_{\pm}\otimes\hat{\epsilon}_{\pm}=-\sqrt{2}\hat{\epsilon}_{\pm}\otimes\hat{\mathbf{q}}-\cot\theta\hat{\epsilon}_{\pm}\otimes\hat{\epsilon}_{\pm}\ , \tag{B10}$$

which are straightforward to derive. Using (B1), (B3), (B9), and (B10), we find the expressions

$$\begin{aligned}\hat{\nabla}_{\pm}\otimes\mathbf{Y}^{\pm}_{LM}&=\left[\frac{2L+1}{4\pi}\right]^{1/2}\sqrt{(L+2)(L-1)}d^L_{M\pm 2}\\&\quad\times e^{iM\phi}\hat{\epsilon}_{\pm}\otimes\hat{\epsilon}_{\pm}-\sqrt{2}\mathbf{Y}^{\pm}_{LM}\otimes\hat{\mathbf{q}}\ ,\\\hat{\nabla}_{\pm}\otimes\mathbf{Y}^{\mp}_{LM}&=-\sqrt{L(L+1)}Y_{LM}(\hat{\mathbf{q}})\hat{\epsilon}_{\pm}\otimes\hat{\epsilon}_{\mp}\ .\end{aligned} \tag{B11}$$

We now rewrite, using (B4), the rank-2 harmonic Y^1_{LM} in a more compact form:

$$Y^1_{LM}=\frac{1}{\sqrt{2(L-1)(L+2)}}[(\hat{\nabla}+\hat{\mathbf{q}})\otimes\mathbf{Y}^0_{LM}]_S\ , \tag{B12}$$

where the subsript s stands for symmetrization. We decompose $\hat{\nabla}$ according to (B6) and express $\mathbf{Y}^0_{LM}$ in terms of $\mathbf{Y}^{\pm}_{LM}$. This leads to

$$\begin{aligned}(\hat{\nabla}\otimes\mathbf{Y}^0_{LM})_S&=\frac{i}{2}(\hat{\nabla}_{+}\otimes\mathbf{Y}^{-}_{LM}-\hat{\nabla}_{-}\otimes\mathbf{Y}^{+}_{LM})_S\\&\quad-\frac{i}{2}(\hat{\nabla}_{+}\otimes\mathbf{Y}^{+}_{LM}-\hat{\nabla}_{-}\otimes\mathbf{Y}^{-}_{LM})_S\ ,\end{aligned} \tag{B13}$$

which can be simplified by using the expressions (B11):

$$[\hat{\nabla}\otimes\mathbf{Y}^0_{LM}]_S=-i\sqrt{(L+2)(L-1)}\sqrt{2}\mathcal{Y}^{\lambda_-}_{LM}-[\mathbf{Y}^0_{LM}\otimes\mathbf{q}]_S$$

and this gives, in view of (B12),

$$iY^1_{LM}=\mathcal{Y}^{\lambda_-}_{LM}\ .$$

To rewrite the tensor $Y^2_{LM;ij}$ in a compact form, we use the commutation rule which is valid at $\hat{\mathbf{q}}$,

$$[\hat{\nabla}_i,\hat{\nabla}_j]=\hat{q}_i\hat{\nabla}_j-\hat{q}_j\hat{\nabla}_i\ ,$$

and obtain

$$\begin{aligned}Y^2_{LM}(\hat{\mathbf{q}})=&-\frac{1}{\sqrt{2(L-1)(L+2)}}\\&\times\{[(\hat{\nabla}+\hat{\mathbf{q}})\otimes\mathbf{Y}^1_{LM}]_s\\&\quad-\sqrt{L(L+1)}Y_{LM}(\hat{\mathbf{q}})\hat{\nabla}\otimes\hat{\mathbf{q}}\}\ .\end{aligned} \tag{B14}$$

Next, we write

$$\begin{aligned}(\hat{\nabla}\otimes\mathbf{Y}^1_{LM})_s=&-\tfrac{1}{2}(\hat{\nabla}_{+}\otimes\mathbf{Y}^{+}_{LM}+\hat{\nabla}_{-}\otimes\mathbf{Y}^{-}_{LM}+\hat{\nabla}_{+}\otimes\mathbf{Y}^{-}_{LM}\\&+\hat{\nabla}_{-}\otimes\mathbf{Y}^{+}_{LM})_S\end{aligned}$$

and simplify using the expressions (B11). One finds

$$\begin{aligned}(\hat{\nabla}\otimes\mathbf{Y}^1_{LM})_S=&-\sqrt{(L-1)(L+2)}\sqrt{2}\mathcal{Y}^{\lambda_+}_{LM}-(\mathbf{Y}^1_{LM}\otimes\hat{\mathbf{q}})_S\\&+\sqrt{L(L+1)}Y_{LM}\hat{\nabla}\otimes\hat{\mathbf{q}}\ .\end{aligned}$$

By substituting this expression in (B14), we obtain

$$Y^2_{LM}=\mathcal{Y}^{\lambda_+}_{LM}\ .$$

We finally point out that the repeated application of (B1) gives a simple derivation for the Wigner matrices being expressed as usual by the Jacobi polynomials.

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