# Energy-momentum tensor in scalar QED and the renormalization group

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(Received 23 October 1989)

We consider the renormalization-group equation satisfied by Green's functions of the energy-momentum tensor in scalar QED. We consider the renormalization-group covariance criterion suggested by Collins and show that this criterion (together with certain boundary conditions) when applied at a nontrivial fixed point $\lambda^*(\neq 0)$, $e^*$ of the theory *uniquely* determines an energy-momentum tensor. It is conjectured that this unique energy-momentum tensor may couple to an external gravity. We also consider the renormalization-group criterion in perturbation series in $\lambda$ and $e^2$ and show that no energy-momentum tensor exists satisfying this criterion. We consider also an additional form of boundary conditions which determine a unique solution to the renormalization-group criterion.

## I. INTRODUCTION

Energy-momentum tensors in quantum field theories are important objects and have been studied extensively.[1–11] In particular, so is the problem of finiteness and renormalization of energy-momentum tensors. The problem of finiteness of an energy-momentum tensor is important because the energy-momentum tensor acts as a source of gravity and external gravity couples to matter through an energy-momentum tensor. But an energy-momentum tensor in field theory is not unique. The energy-momentum tensor $\theta_{\mu\nu}$ that couples to an external gravitational field is however restricted by the requirement that its physical matrix elements between states of matter particles $|A\rangle$ and $|B\rangle$, viz., $\langle B|\theta_{\mu\nu}|A\rangle$ must be finite.[2] While this restricts $\theta^{\mu\nu}$, it does not uniquely fix it. In renormalizable theories that do not involve scalar fields, the situation is simple. Let $S[\phi]$ denote the flat-space action and let $S_{\min}[\phi,g]$ denote the minimal Einstein action obtained by the standard prescription of replacing ordinary derivatives by covariant derivatives etc. Then the energy-momentum tensor obtained via

$$\theta_{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\delta S_{\min}}{\delta g_{\mu\nu}}\Big|_{g_{\mu\nu}=\eta_{\mu\nu}} \tag{1.1}$$

leads to an energy-momentum tensor with finite Green's functions[3] and gauge-independent physical matrix elements and hence this $\theta_{\mu\nu}$ may be assumed to couple to a weak gravitational field.

In theories with scalar fields, the situation is more complicated. The energy-momentum tensor obtained via $S_{\min}$ does not lead to finite matrix elements. In $\lambda\phi^4$ theory, for example, one needs an improvement term of the form $(\partial_\mu\partial_\nu-\partial^2\eta_{\mu\nu})\phi^2$ (Ref. 2). The question here is whether the improvement term to be added can be obtained from an action that is a finite function of bare parameters of the flat-space theory. In the context of $\lambda\phi^4$ theory, this means that a finite energy-momentum tensor should be derivable from an action $S$ which is a finite function of the bare field $\phi$, bare mass $m_0$, and bare coupling $\lambda_0$ (Ref. 9). As is well known the term $(\partial_\mu\partial_\nu-\eta_{\mu\nu}\partial^2)\phi^2$ in $\theta_{\mu\nu}$ can be derived from a term in $S$ of the form $\frac{1}{2}\int R\phi^2 d^nx$. Thus one seeks an energy-momentum tensor

$$\theta_{\mu\nu}=\theta^C_{\mu\nu}-H_0\left[\lambda_0\mu^{-\epsilon},\frac{m_0^2}{\mu^2},\epsilon\right](\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^*\phi \tag{1.2}$$

derivable from an action

$$S[\phi;\lambda_0,m_0,\mu,g]=S_{\min}[\phi,\lambda_0,m_0,g]-\frac{1}{2}H_0\left[\lambda_0\mu^{-\epsilon},\frac{m_0^2}{\mu^2},\epsilon\right]\int R\phi^2 d^nx\sqrt{-g}\ . \tag{1.3}$$

If such a finite energy-momentum tensor exists then flat-space parameters are sufficient to fix the interaction with an external gravity. On the other hand, if one needs the improvement term to be added with divergent coefficients (involving negative powers of $\epsilon$), then this implies an infinite renormalization of $\theta_{\mu\nu}$ and signals a new parameter in the action (corresponding to the arbitrariness associated with an infinite renormalization), thus needing a new experimental input.

We shall call $\theta_{\mu\nu}$ of Eq. (1.2) with $H_0(\lambda_0\mu^{-\epsilon},m_0^2/\mu^2,\epsilon)$ a finite function of $\lambda_0,m_0$ at $\epsilon=0$ as obtained via a finite improvement program of type I. As shown by Collins,[7] this program does indeed work with a unique $H_0(\epsilon)$; a function of $\epsilon$ only. As shown in Ref. 9, Collins's energy-momentum tensor is a unique energy-momentum tensor of the form of Eq. (1.2).

An alternative finite improvement program has also been suggested in which $\theta_{\mu\nu}$ has the form[4,7]

$$\theta_{\mu\nu}=\theta^C_{\mu\nu}-H_0\left[\lambda,\frac{m}{\mu},\epsilon\right](\partial_\mu\partial_\nu-\partial^2\eta_{\mu\nu})\phi^2\ , \tag{1.4}$$

where $H_0(\lambda,m/\mu,\epsilon)$ is a finite number at $\epsilon=0$. This generalizes the idea of the Callan, Coleman, and Jackiw

 

(CCJ) improvement term in which a finite improvement coefficient $\frac{1}{6}$ appeared. As shown by Collins, his energy-momentum tensor is a unique finite energy-momentum tensor of this kind also. We shall call this a "finite improvement program of type II" and when it succeeds, it implies that interactions with external gravity are determined only by flat-space parameters.

The situation in theories with a scalar field and other fields (gauge fields, fermion fields or extra scalar fields) is however different.[10–12] Either type of improvement program has been applied to such theories and it has been shown that they fail. This implies that the interaction of such a theory with an external gravity will not be determined by flat-space parameters but one will need an independent renormalization fixed by the experimental data related to the "root-mean-square mass radius of the scalar particle."[4] This also means that in such theories there are an infinite number of finite energy-momentum tensors which will not be distinguished from each other by theoretical reasons but the correct one is selected only by experimental input.

The above would be true were it not for the fact that there is another independent theoretical criterion that $\theta_{\mu\nu}$ should satisfy, suggested by Collins.[7] This criterion is based on the renormalization-group transformation properties of the matrix elements of $\theta_{\mu\nu}$. Stated simply, it requires that the Green's functions of $\theta_{\mu\nu}$ satisfy a homogeneous renormalization-group equation (RGE) with a zero anomalous dimension for the operator. In other words an $n$-point Green's function of $\theta_{\mu\nu}$ satisfies the same RGE as the ordinary $n$-point Green's function; i.e., $\theta_{\mu\nu}$ is "RG covariant." This has the physical consequence that in the ultraviolet (infrared) limit, the Green's functions for the interaction with an external gravity scale as the ordinary Green's functions do. Collins has studied this criterion in the context of $\lambda\phi^4$ theory. Our aim is to study the same for scalar QED where it becomes especially important and relevant in view of the lack of finite improvement programs.

We state our result briefly. We show that the criterion of "RG covariance" together with the "boundary condition" that the Green's functions of $\theta_{\mu\nu}$ are analytic at the fixed point of the theory $\lambda^*$ ($\neq 0$) and $e^*$ does isolate a *unique* energy-momentum tensor. This, then, provides a theoretical criterion by which an external gravity may unambiguously couple to matter fields when scalar fields are involved and, if such is the case, the flat-space parameters would indeed fix interactions with external gravity.

We also supplement the discussion by considering certain other kinds of possible boundary conditions.

## II. PRELIMINARY

We shall work with a complex scalar field coupled to an Abelian gauge field described by the Lagrange density

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(D_\mu\phi)^*(D^\mu\phi)-\tfrac{1}{2}m_0^2\phi^*\phi-\frac{\lambda_0}{4!}(\phi^*\phi)^2-\tfrac{1}{2}\xi_0(\partial\cdot A)^2 , \tag{2.1}$$

where $D_\mu\phi$ is the covariant derivative defined by

$$D_\mu\phi=\partial_\mu\phi-ie_0A_\mu\phi .$$

We shall work with dimensionally regularized quantities and shall use the minimal subtraction (MS) scheme.[13]

$\mathcal{L}$ of Eq. (2.1) can be generalized to include a minimal interaction with an external gravity $g_{\mu\nu}(x)$ leading to $S_{\min}[\phi, A_\mu, g_{\mu\nu}]$. Via Eq. (1.1) this leads us to the energy-momentum tensor $\bar{\theta}_{\mu\nu}$, where

$$\begin{aligned}\bar{\theta}_{\mu\nu}=&-g_{\mu\nu}\mathcal{L}-F_{\mu\alpha}F^\alpha_\nu+\tfrac{1}{2}[(D_\mu\phi)^*(D_\nu\phi)+(D_\nu\phi)^*D_\mu\phi]\\&+\xi_0[\partial_\mu(\partial\cdot A)A_\nu+\partial_\nu(\partial\cdot A)A_\mu]\\&-g_{\mu\nu}\xi_0\partial^\rho(\partial\cdot A)A_\rho-g_{\mu\nu}\xi_0(\partial\cdot A)^2 .\end{aligned} \tag{2.2}$$

The improved energy-momentum tensor will have the most general form

$$\theta^{\text{imp}}_{\mu\nu}=\theta_{\mu\nu}-H_0(\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})\phi^*\phi , \tag{2.3}$$

with the most general possible $H_0$:

$$H_0\equiv H_0(\lambda,e^2,m^2/\mu^2,\xi,\epsilon) . \tag{2.4}$$

As shown in Ref. 3, $\theta_{\mu\nu}$ is finite to the zeroth and the first order in $q$ (the momentum entering via $\theta_{\mu\nu}$) and has divergences only in the second order in $q$. Thus one has

$$\begin{aligned}\{\langle\theta_{\mu\nu}\rangle\}^{\text{div}}&=G'(\lambda,e^2,\xi,\epsilon)(\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})\langle\phi^*\phi\rangle^R\\&=G'(\lambda,e^2,\xi,\epsilon)Z_m(\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})\langle\phi^*\phi\rangle\\&\equiv G(\lambda,e^2,\xi,\epsilon)(\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})\langle\phi^*\phi\rangle ,\end{aligned} \tag{2.5}$$

where we have used Ref. 7 that $\langle\phi^*\phi\rangle^R=Z_m\langle\phi^*\phi\rangle$ where $Z_m$ is defined via $m_0^2\equiv m^2Z_m$ and $m$ is the renormalized mass.

Here by construction $G'$ and hence $G(\lambda,e^2,\xi,\epsilon)$ has only poles in $\epsilon$. Thus the energy-momentum tensor

$$\theta'_{\mu\nu}=\theta_{\mu\nu}-G(\lambda,e^2,\xi,\epsilon)(\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})\phi^*\phi \tag{2.6}$$

is a finite energy-momentum tensor. But this is not the only one, for one could always add a term of the form

$$-k'\left[\lambda,e^2,\frac{m^2}{\mu^2},\xi,\epsilon\right](\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})\langle\phi^*\phi\rangle^R \tag{2.7}$$

(with $k'$ finite at $\epsilon=0$) and the resultant energy-momentum tensor will still be finite. Comparing Eqs. (2.6) and (2.7) with Eq. (2.4), one finds that

$$H_0=G(\lambda,e^2,\xi,\epsilon)+k'(\lambda,e^2,m^2/\mu^2,\xi,\epsilon)Z_m . \tag{2.8}$$

Now

$$k'(\lambda,e^2,m^2/\mu^2,\xi,\epsilon)=\bar{k}(\lambda,e^2,m^2/\mu^2,\xi)+O(\epsilon) \tag{2.9}$$

and the last term when substituted in (2.7), its contribution to the energy-momentum tensor vanishes at $n=4$.

Thus, if one is only concerned with $\langle\theta_{\mu\nu}\rangle$ at $n=4$; we can, without loss of generality, assume

$$H_0=G(\lambda,e^2,\xi,\epsilon)+\bar{k}\left[\lambda,e^2,\frac{m^2}{\mu^2},\xi\right]Z_m . \tag{2.10}$$

We define the renormalized coupling constant $h$ via Ref. 7:

$$\bar{k}\left[\lambda,e^2,\frac{m^2}{\mu^2},\xi\right]\equiv k\left[\lambda,e^2,\frac{m^2}{\mu^2},\xi\right]+h\ , \qquad (2.11)$$

where $h$ is independent of the parameters of the theory (except $\mu$). Thus, we have

$$H_0=G(\lambda,e^2,\xi,\epsilon)+\left[k\left[\lambda,e^2,\frac{m^2}{\mu^2},\xi\right]+h\right]Z_m\ . \qquad (2.12)$$

We now wish to make a contact with the notation of Ref. 10. There the improvement coefficient was expressed as

$$\theta_{\mu\nu}^{\rm imp}=\theta_{\mu\nu}+\left[\frac{n-2}{4(1-n)}+\frac{\tilde{g}}{1-n}\right](\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^*\phi\ . \qquad (2.13)$$

The improvement term $\tilde{g}$ is not necessary[10] up to $O(\lambda^3)$, $O(\lambda e^2)$, and $O(e^4)$. Thus up to these orders, one has that

$$\theta_{\mu\nu}+\frac{n-2}{4(1-n)}Z_m^{-1}(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)\langle\phi^*\phi\rangle^R \qquad (2.14)$$

is a finite operator. Comparison of Eqs. (2.3), (2.12), and (2.14) indicates that

$$P\left[GZ_m^{-1}+\frac{n-2}{4(1-n)}Z_m^{-1}\right]=0 \qquad (2.15)$$

up to $O(\lambda^3,\lambda e^2,e^4)$. Here $P$ stands for the pole part. Multiplying by $4(1-n)$ and using the finiteness of $\bar{k}$, one obtains

$$P[(G+\bar{k}Z_m)Z_m^{-1}4(1-n)+(n-2)Z_m^{-1}]$$
$$\equiv P[4XZ_m^{-1}(\epsilon-3)+(2-\epsilon)Z_m^{-1}]=0\ . \qquad (2.16)$$

This relation will be useful in Sec. IV.

We define renormalization-group quantities[10,12]

$$\mu\frac{\partial\lambda}{\partial\mu}\equiv\beta^\lambda(\lambda,e^2,\epsilon)=\bar{\beta}(\lambda,e^2)-\lambda\epsilon=-\lambda\epsilon+\tilde{\beta}_1\lambda^2+\beta_2^\lambda\lambda e^2+\beta_1^\lambda e^4+\cdots\ ,$$
$$\mu\frac{\partial}{\partial\mu}e=\beta^e(\lambda,e^2,\epsilon)=\bar{\beta}^e(\lambda,e^2)-\tfrac{1}{2}e\epsilon\equiv-\tfrac{1}{2}e\epsilon+\beta_3^e e^3+\cdots,\quad -\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_m\equiv\gamma_m(\lambda,e^2)\equiv\gamma_{m1}\lambda+\gamma_{m2}e^2+\cdots\ , \qquad (2.17)$$
$$\mu\frac{\partial}{\partial\mu}\xi=\gamma_\xi(\lambda,e^2)\xi,\quad Z_3^{-1}\mu\frac{\partial}{\partial\mu}Z_3=\gamma_3(\lambda,e^2),\quad Z^{-1}\mu\frac{\partial}{\partial\mu}Z=\gamma(\lambda,e^2,\xi)\ ,$$

where $Z$ and $Z_3$ are the wave-function renormalizations of the scalar and photon fields.

We quote the needed values

$$\beta_1^\lambda=\frac{9}{8\pi^2},\quad \tilde{\beta}_1^\lambda=\frac{1}{8\pi^2},\quad \beta_2^\lambda=-\frac{1}{\pi^2}\ ,$$
$$\gamma_{m1}=\frac{1}{48\pi^2}\equiv-\frac{1}{2}\alpha,\quad \gamma_{m2}=-\frac{3}{16\pi^2}\equiv-\tfrac{1}{2}k\ , \qquad (2.18)$$
$$\beta_3^e=\frac{1}{48\pi^2}\ .$$

Finally, we note that we should, on physical grounds, require that the physical matrix elements of $\theta_{\mu\nu}^{\rm imp}$ be independent of the gauge parameter $\xi$. As shown in the Appendix, this requires that $k$ is $\xi$ independent. Furthermore as shown in the Appendix, $G(\lambda,e^2,\xi,\epsilon)$ is also independent of $\xi$.

## III. RG EQUATION SATISFIED BY $\langle\theta_{\mu\nu}^{\rm imp}\rangle$ AND RG CONDITION ON $k$

In this section, we shall derive the RG equation satisfied by the proper vertices of $\theta_{\mu\nu}^{\rm imp}$. This RG equation will generally be an inhomogeneous equation, but it can be cast in an apparently homogeneous equation in which an additional term of the kind $\delta\partial/\partial h$ appears in the differential operator acting upon a proper vertex of $\theta_{\mu\nu}$ viz. $\Gamma_{\mu\nu}$. In order that $\Gamma_{\mu\nu}$ satisfies an ordinary RG-covariant equation, this term must disappear. This leads to a constraint in the form of a differential equation to be satisfied by $k(\lambda,e^2,m^2/\mu^2)$ [Collins[7] has suggested that, as there are two coupling constants, two mass scales $\mu_1$ and $\mu_2$ should be introduced, one for each coupling constant. This yields two differential equations for $\bar{k}$. But this also increases the number of independent variables, because $\bar{k}$ now depends on $x=\ln\mu_2/\mu_1$. Thus one still has one fewer equation than there are independent variables $\lambda$, $e^2$, and $x$ (omitting $m$). So this does not help.] The question of solutions to such an equation satisfied by $k$, the "boundary conditions" that are appropriate for the existence and uniqueness of its solution, are taken up in the subsequent sections.

The derivation of the RG equation for $\Gamma_{\nu\sigma}$ starts as usual, except now $\theta_{\nu\sigma}^{\rm imp}$ depends explicitly on $\mu$ when the improvement coefficient $H_0$ is expressed in terms of bare parameters. (We shall also allow $h$ to vary with $\mu$ though as yet $\mu(\partial h/\partial\mu)$ is unspecified, but the RG equation for $\Gamma_{\nu\sigma}$ turns out to be independent of such a term.) We express the unrenormalized proper vertices of $\theta_{\nu\sigma}$ as

$$\Gamma_{\nu\sigma}=\Gamma_{\nu\sigma}^{(1)}-H_0(\partial_\nu\partial_\sigma-g_{\nu\sigma}\partial^2)\Gamma^{(2)}\ , \qquad (3.1)$$

where, referring to Eq. (2.3), $\Gamma_{\nu\sigma}^{(1)}$ and $\Gamma^{(2)}$ are the unrenormalized proper vertices of $\theta_{\mu\nu}$ and $\phi^*\phi$, respectively. As $\theta_{\mu\nu}$ and $\phi^*\phi$ are functions independent of $\mu$ when bare quantities are held fixed,

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\nu\sigma}^{(1)}\Big|_{\rm bare}=0\ ,$$
$$\mu\frac{\partial}{\partial\mu}\Gamma^{(2)}\Big|_{\rm bare}=0\ . \qquad (3.2)$$

This leads us to

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\nu\sigma}\Big|_{\rm bare}=-\mu\frac{\partial}{\partial\mu}H_0\Big|_{\rm bare}(\partial_\nu\partial_\sigma-g_{\nu\sigma}\partial^2)\Gamma^{(2)}$$
$$=-Z_m^{-1}\mu\frac{\partial}{\partial\mu}H_0\Big|_{\rm bare}(\partial_\nu\partial_\sigma-\partial^2 g_{\nu\sigma})\times[Z_m\Gamma^{(2)}]\ . \qquad (3.3)$$

Noting that $\Gamma_{\nu\sigma}$ depends on $h$ as

$$\Gamma_{\nu\sigma}=\cdots h(\partial_\nu\partial_\sigma-g_{\nu\sigma}\partial^2)Z_m\Gamma^{(2)}\ . \qquad (3.4)$$

we can write

$$\mu\frac{\partial}{\partial\mu}\Gamma_{\nu\sigma}=Z_m^{-1}\mu\frac{\partial}{\partial\mu}H_0\Big|_{\rm bare}\frac{\partial}{\partial h}\Gamma_{\nu\sigma} \qquad (3.5)$$

thus turning the inhomogeneous term in Eq. (3.3) into an apparently homogeneous term.

Now consider, for concreteness, a proper vertex with $p$ photon lines and $2q$ scalar lines. Then

$$\Gamma_{\nu\sigma}^{(p,2q)R}=Z_3^{p/2}Z^q\Gamma_{\nu\sigma}^{(p,2q)}\ . \qquad (3.6)$$

Equation (3.5) then yields

$$\left[\mu\frac{\partial}{\partial\mu}+\beta^\lambda(\lambda,e^2,\epsilon)\frac{\partial}{\partial\lambda}+\beta^e(\lambda,e^2,\epsilon)\frac{\partial}{\partial e}+\xi\gamma_\xi(\lambda,e^2)\frac{\partial}{\partial\xi}+2\gamma_m m^2\frac{\partial}{\partial m^2}+\frac{p}{2}\gamma_3+q\gamma+\left[\mu\frac{\partial}{\partial\mu}h\right]\frac{\partial}{\partial h}\right]\Gamma_{\nu\sigma}^{(p,2q)}$$
$$=Z_m^{-1}\mu\frac{\partial}{\partial\mu}H_0\Big|_{\rm bare}\frac{\partial}{\partial h}\Gamma_{\nu\sigma}^{n}$$
$$=\left[Z_m^{-1}\left[\beta^\lambda(\lambda,e^2,\epsilon)\frac{\partial}{\partial\lambda}+\beta^e(\lambda,e^2,\epsilon)\frac{\partial}{\partial e}\right]G-2(h+k)\gamma_m+\mu\frac{\partial h}{\partial\mu}\right.$$
$$\left.+\left[\beta^\lambda(\lambda,e^2,\epsilon)\frac{\partial}{\partial\lambda}+\beta^e(\lambda,e^2,\epsilon)\frac{\partial}{\partial e}+2(\gamma_m-1)m^2\frac{\partial}{\partial m^2}\right]k\right]\frac{\partial}{\partial h}\Gamma_{\nu\sigma}^{(p,2q)}\ . \qquad (3.7)$$

Equation (3.7) simplifies to

$$\left[\mu\frac{\partial}{\partial\mu}+\beta^\lambda(\lambda,e^2\epsilon)\frac{\partial}{\partial\lambda}+\beta^e(\lambda,e^2,\epsilon)\frac{\partial}{\partial e}+\xi\gamma_\xi\frac{\partial}{\partial\xi}+2\gamma_m m^2\frac{\partial}{\partial m^2}+\left[\frac{p}{2}\gamma_3+q\gamma\right]+\delta\left[\lambda,e^2,\frac{m^2}{\mu^2},\epsilon\right]\frac{\partial}{\partial h}\right]\Gamma^{(p,2q)}=0\ . \qquad (3.8)$$

with

$$\delta\left[\lambda,e^2,\frac{m^2}{\mu^2},\epsilon\right]=-Z_m^{-1}\left[\beta^\lambda(\lambda,e^2,\epsilon)\frac{\partial}{\partial\lambda}+\beta^e(\lambda,e^2,\epsilon)\frac{\partial}{\partial e}\right]G+2(h+k)\gamma_m$$
$$-\left[\beta^\lambda(\lambda,e^2,\epsilon)\frac{\partial}{\partial\lambda}+\beta^e(\lambda,e^2,\epsilon)\frac{\partial}{\partial e}+(2\gamma_m-2)m^2\frac{\partial}{\partial m^2}\right]k\ . \qquad (3.9)$$

Now all the terms in the Eq. (3.8) except (possibly) the term $\delta(\partial/\partial h)\Gamma_{\nu\sigma}$ have a finite limit as $\epsilon\to 0$. Hence $\delta$ must also be finite at $\epsilon=0$. Thus putting $\epsilon=0$, Eq. (3.8) reads

$$\left[\mu\frac{\partial}{\partial\mu}+\bar\beta^\lambda\frac{\partial}{\partial\lambda}+\bar\beta^e\frac{\partial}{\partial e}+\xi\gamma_\xi\frac{\partial}{\partial\xi}+2\gamma_m m^2\frac{\partial}{\partial m^2}+\frac{p}{2}\gamma_3+q\gamma+\bar\delta\frac{\partial}{\partial h}\right]\Gamma_{\nu\sigma}^{(p,2q)}=0$$

with $\bar\delta=\delta(\lambda,e^2,m^2/\mu^2,\epsilon=0)$.

The above equation implies that the proper vertices of $\theta_{\nu\sigma}^{\rm imp}$ are RG covariant iff the coefficient of $(\partial/\partial h)\Gamma$ vanishes. This requires

$$-\bar\delta=0=\left[\bar\beta^\lambda\frac{\partial}{\partial\lambda}+\bar\beta^e\frac{\partial}{\partial e}+(2\gamma_m-2)m^2\frac{\partial}{\partial m^2}\right]\bar k-2\bar k\gamma_m$$
$$-\left[\lambda\frac{\partial}{\partial\lambda}+\frac{e}{2}\frac{\partial}{\partial e}\right]G^{(1)}(\lambda,e^2)=0\ , \qquad (3.10)$$

where $\bar k=k+h$; $G^{(1)}$ is the coefficient of the simple pole terms in $G(\lambda,e^2,\epsilon)$ and use has been made of the fact that $h$ is a constant independent of $\lambda$, $e^2$, $m$, $\epsilon$.

Equation (3.10) is the condition to be satisfied by $\bar k$ (or equivalently $k$) and we wish to seek solutions of Eq. (3.10) for $k$. In general, Eq. (3.10) is a *single* differential condition on a function of *two* independent variables and has an infinite number of solutions. If we impose a certain kind of "boundary conditions" there will be either no solution, a unique solution, or multiple (or infinite) solutions to Eq. (3.10). We wish to discuss physically meaningful "boundary conditions" that yield a unique solution for Eq. (3.10).

We shall briefly comment on the physical significance of the condition (3.10). In renormalizable theories without scalar fields, there exist energy-momentum tensors[3] which are (i) finite, (ii) finite functions of bare quantities, (iii) independent of $\mu$ when expressed in terms of bare quantities, and (iv) carry no parameters in addition to those of flat-space action. For such an energy-momentum tensor, the renormalization-group equation is homogeneous and with an operator anomalous dimension equal to zero. This leads to a certain high-energy behavior for the Green's functions of $\theta_{\mu\nu}$ that would enter a gravitational scattering. The condition (3.10) ensures that the energy-momentum tensor(s) in scalar QED so obtained leads to a similar RG equation and a similar high-energy behavior for Green's functions.

Alternatively, we could interpret the condition as follows. Unlike in the theories without scalar fields, in those with scalar fields the energy-momentum tensor depends on an extra parameter $h$. As shown by Collins, the RG condition of Eq. (3.10) fixes this arbitrary parameter at $h=0$, choosing a particular energy-momentum tensor from a set of an infinite number of them.

## IV. SOLUTION PERTURBATIVE IN $\lambda$ AND $e^2$

In this section, we shall attempt the most obvious boundary condition on $\bar{k}\,(\lambda,e^2,m^2/\mu^2)$ that (i) it should have a perturbative expansion in powers of $\lambda$ and $e^2$ and (ii) it should have a finite limit as $m\to 0$. Using the second part of the boundary condition, we shall first simplify Eq. (3.10). It can be rewritten as

$$2(\gamma_m-1)\frac{\partial}{\partial \ln m^2}\bar{k}=\left[\lambda\frac{\partial}{\partial\lambda}+\frac{1}{2}e\frac{\partial}{\partial e}\right]G^{(1)}+2\bar{k}\gamma_m-\left[\bar{\beta}^{\lambda}\frac{\partial}{\partial\lambda}+\bar{\beta}^{e}\frac{\partial}{\partial e}\right]\bar{k}\ .$$

Consider the above equation at $m=0$. By the second boundary condition the right-hand side has a finite limit at $m=0$. If it is not zero, then one has

$$2(\gamma_m-1)\frac{\partial}{\partial \ln m^2}\bar{k}\,\bigg|_{m=0}=A(\lambda,e^2)\neq 0\ .$$

Thus, $\bar{k}\sim[A(\lambda,e^2)/2(\gamma_m-1)]\ln m^2+\cdots$ indicating a singular behavior at $m=0$ contradicting the second part of the boundary condition. Thus one must have $A(\lambda,e^2)=0$; i.e.,

$$\left[\bar{\beta}^{\lambda}\frac{\partial}{\partial\lambda}+\bar{\beta}^{e}\frac{\partial}{\partial e}-2\gamma_m\right]\bar{k}(\lambda,e^2)=\left[\lambda\frac{\partial}{\partial\lambda}+e^2\frac{\partial}{\partial e^2}\right]G^{(1)}(\lambda,e^2)\ . \quad (4.1)$$

The pole parts in $G(\lambda,e^2,\epsilon)$ are related to those in $Z_m^{-1}$ via the relation (2.16) up to $O(\lambda^3,\lambda e^2,e^4)$. As this relation contains the combination $X=G+\bar{k}Z_m$ it is convenient to transform Eq. (4.1) to express it directly in terms of $X$ rather than $G$.

When this is done, one obtains

$$\left[(\bar{\beta}^{\lambda}-\lambda\epsilon)\frac{\partial}{\partial\lambda}+(\bar{\beta}^{e}-\tfrac{1}{2}e\epsilon)\frac{\partial}{\partial e}\right]X+\epsilon\lambda\frac{\partial}{\partial\lambda}\bar{k}Z_m+\frac{\epsilon e}{2}\frac{\partial}{\partial e}\bar{k}Z_m=0\ . \quad (4.2)$$

(In obtaining Eq. (4.2), we have made use of the fact, obtainable from the finiteness of $\delta$, that $Z_m^{-1}[\beta^{\lambda}(\partial/\partial\lambda)+\beta^{e}(\partial/\partial e)]G$ is finite and, as $G$ has only poles, it is $\epsilon$ independent. Hence it follows that

$$-Z_M^{-1}\left[\beta^{\lambda}\frac{\partial}{\partial\lambda}+\beta^{e}\frac{\partial}{\partial e}\right]G=\left[\lambda\frac{\partial}{\partial\lambda}+\frac{e}{2}\frac{\partial}{\partial e}\right]G^{(1)} \quad (4.3)$$

and this has been used.)

We expand

$$X=X^{(0)}+\tilde{X}^{(1)}\frac{\lambda}{\epsilon}+\hat{X}^{(1)}\frac{\lambda^2}{\epsilon}+\bar{X}^{2}\frac{\lambda^2}{\epsilon^2}+\tilde{A}^{(1)}\frac{e^2}{\epsilon}+\hat{A}^{(1)}\frac{e^4}{\epsilon}+\bar{A}^{(2)}\frac{e^4}{\epsilon^2}+\bar{B}^{(1)}\frac{\lambda e^2}{\epsilon}+\bar{B}^{(2)}\frac{\lambda e^2}{\epsilon^2}+\cdots\ , \quad (4.4)$$

$$Z_m^{-1}=1+\frac{\alpha\lambda}{\epsilon}+\frac{\tilde{\alpha}\lambda^2}{\epsilon^2}+\frac{d\lambda^2}{\epsilon}+\cdots+k\frac{e^2}{\epsilon}+\tilde{k}\frac{e^4}{\epsilon^2}+\frac{\tilde{d}e^4}{\epsilon}+\cdots+g\frac{\lambda e^2}{\epsilon}+\tilde{g}\frac{\lambda e^2}{\epsilon^2}+\cdots\ , \quad (4.5)$$

$$\bar{k}=\bar{k}_0+\bar{k}_1\lambda+\bar{k}_2e^2+\cdots\ . \quad (4.6)$$

Here, $\bar{k}$ has been expanded in powers of $\lambda$ and $e^2$, as this is the "boundary condition" we are trying. Direct calculation shows[2,7] that $\bar{k}_0=\frac{1}{6}=X^{(0)}$.

Comparing $O(\epsilon^0)$ terms in Eq. (4.2), we obtain

$$\left[-\lambda\frac{\partial}{\partial\lambda}-\frac{e}{2}\frac{\partial}{\partial e}\right]X^{(1)}+\lambda\frac{\partial\bar{k}}{\partial\lambda}Z_m^{(1)}+\frac{e}{2}\frac{\partial\bar{k}}{\partial e}Z_m^{(1)}=0\ , \quad (4.7)$$

where $X^{(1)}$ and $Z_m^{(1)}$ are the coefficient of simple pole terms in $X$ and $Z_m$, respectively.

We seek $\bar{k}$ of the form of Eq. (4.6), which satisfies Eq. (4.7). We assume, if possible, that such a $\bar{k}$ exists. Then Eq. (4.7) in $O(\lambda)$ leads to

$$\tilde{X}^{(1)}=0\ . \quad (4.8)$$

Further, Eq. (4.2) in $O(\lambda^2/\epsilon)$ implies

$$-2\bar{X}^{(2)}+\beta_1^{\lambda}\tilde{X}^{(1)}=0\ . \quad (4.9)$$

This, together with Eq. (4.8) implies

$$\bar{X}^{(2)}=0\ . \quad (4.10)$$

Now, consider Eq. (2.16) in $O(\lambda^2/\epsilon)$ leads to

$$4\tilde{\alpha}X^{(0)}-12\hat{X}^{(1)}-12\,dX^{(0)}+4\tilde{X}^{(1)}\alpha+4\bar{X}^{(2)}+2d-\alpha=0\ . \quad (4.11)$$

Using Eqs. (4.8), (4.10) and the value $X^{(0)}=\frac{1}{6}$ leads one to

$$\hat{X}^{(1)}=-\tfrac{1}{36}\tilde{\alpha}\ . \quad (4.12)$$

From the RGE satisfied by $Z_m^{-1}$, one obtains

$$\tilde{\alpha}=\tfrac{1}{2}\alpha(\bar{\beta}_1^{\lambda}+2\gamma_{m1}) \quad (4.13)$$

and thus one has

$$\hat{X}^{(1)}=-\frac{1}{72}\alpha(\bar{\beta}_1^{\lambda}+2\gamma_{m1})\ . \quad (4.14)$$

Equation (4.7), in $O(e^2)$, gives

$$\tilde{A}^{(1)}=0\ . \tag{4.15}$$

Equation (4.2) [or alternately Eq. (2.16) also] implies

$$\bar{A}^{(2)}=0\ . \tag{4.16}$$

Equation (2.16), in $O(e^4/\epsilon)$, yields

$$4X^{(0)}\tilde{k}-12\tilde{d}X^{(0)}+4\tilde{A}^{(1)}k-12\hat{A}^{(1)}+4\bar{A}^{(2)}+2\tilde{d}-\tilde{k}=0\ . \tag{4.17}$$

In view of Eqs. (4.15) and (4.16) and the value of $X^{(0)}$, this simplifies to

$$\hat{A}^{(1)}=-\tfrac{1}{36}\tilde{k}\ . \tag{4.18}$$

The RGE for $Z_m^{-1}$ yields

$$\tilde{k}=\tfrac{1}{2}[\beta_1^\lambda\alpha+k(2\beta_3^e+2\gamma_{m2})] \tag{4.19}$$

so that

$$\hat{A}^{(1)}=-\tfrac{1}{72}[\beta_1^\lambda\alpha+k(2\beta_3^e+2\gamma_{m2})]\ , \tag{4.20}$$

In a similar manner, Eq. (4.2) yields

$$\bar{B}^{(1)}=-\tfrac{1}{72}(\beta_2^\lambda\alpha+2\gamma_{m1}k+2\gamma_{m2}\alpha)\ . \tag{4.21}$$

Now, we obtain the relations that must be satisfied by $\bar{k}_1$ and $\bar{k}_2$. The crucial point is that there are only *two* variables and they must satisfy *three* equations obtained from Eq. (4.7) in orders $\lambda^2,\lambda e^2$, and $e^4$. This requires that they be consistent. The consistency requires a condition to be satisfied by calculable coefficients of the Eq. (4.25).

Equation (4.7) in $O(\lambda^2)$ yields

$$\bar{k}_1=-\frac{2\hat{X}^{(1)}}{\alpha}=\tfrac{1}{36}(\tilde{\beta}_1^\lambda+2\gamma_{m1})\ . \tag{4.22}$$

Equation (4.7) in $O(e^4)$ yields

$$\bar{k}_2=-\frac{2\hat{A}^{(1)}}{k}=\frac{1}{36}\left[\beta_1^\lambda\frac{\alpha}{k}+2\beta_3^e+2\gamma_{m2}\right]\ . \tag{4.23}$$

Equation (4.7) in $O(\lambda e^2)$ yields

$$\bar{k}_2\alpha+\bar{k}_1k=-2\bar{B}^{(1)}=\tfrac{1}{36}(\beta_2^\lambda\alpha+2\gamma_{m1}k+2\gamma_{m2}\alpha)\ . \tag{4.24}$$

As is easily verified, these equations are consistent iff

$$\beta_2^\lambda\alpha-\tilde{\beta}_1^\lambda k-\beta_1^\lambda\frac{\alpha^2}{k}-2\beta_3^e\alpha=0\ . \tag{4.25}$$

With the values given in Sec. II, it is easily verified that this condition is not met. Hence, no solution for $\bar{k}$ at $m=0$. Thus, the obvious perturbative boundary condition yields no solution for $\bar{k}$.

## V. SOLUTION PERTURBATIVE AROUND A NONTRIVIAL FIXED POINT $e^*,\lambda^*\neq0$

As Collins[7] has commented, it is more natural to impose boundary conditions at a fixed point. We assume the existence of a nontrivial fixed point $\lambda^*\neq0,e^*$ and require that the solution for $\bar{k}(\lambda,e^2,m^2/\mu^2)$ be perturbative in powers of $(\lambda-\lambda^*)$, and $(e^2-e^{2*})$ and that it is analytic in $m^2$.

We first consider the part of $\bar{k}$ independent of $m$. To show that a unique solution for $\bar{k}(\lambda,e^2)\equiv\bar{k}(\lambda,e^2,m=0)$ is obtained, we expand

$$\bar{k}(\lambda,e^2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}k_{mn}(\lambda-\lambda^*)^m(e^2-e^{*2})^n\ ,$$
$$\bar{\beta}^\lambda(\lambda,e^2)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\bar{\beta}^\lambda_{mn}(\lambda-\lambda^*)^m(e^2-e^{*2})^n\ ,$$
$$2e\bar{\beta}^e(\lambda,e^2)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\bar{\beta}^e_{mn}(\lambda-\lambda^*)^m(e^2-e^{*2})^n\ , \tag{5.1}$$
$$\gamma^m(\lambda,e^2)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\gamma^m_{pq}(\lambda-\lambda^*)^p(e^2-e^{*2})^q\ ,$$
$$\left[\lambda\frac{\partial}{\partial\lambda}+e^2\frac{\partial}{\partial e^2}\right]G^{(1)}(\lambda,e^2)\equiv\xi(\lambda,e^2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\xi_{mn}(\lambda-\lambda^*)^m\times(e^2-e^{*2})^n\ .$$

We wish to consider the equation

$$\left[\bar{\beta}^\lambda\frac{\partial}{\partial\lambda}+2e\bar{\beta}^e\frac{\partial}{\partial e^2}-2\gamma_m\right]\bar{k}(\lambda,e^2)=\xi(\lambda,e^2)\ . \tag{5.2}$$

The proof that a unique solution for $\bar{k}$ of the form of the first of Eqs. (5.1) exists is very straightforward.

Consider Eq. (5.2) in $O((\lambda-\lambda^*)^0(e^2-e^{*2})^0)$. One has

$$-2\gamma^m_{00}\bar{k}_{00}=\xi_{00}\ . \tag{5.3}$$

Assuming that $\gamma^m_{00}=\gamma^m(\lambda^*)\neq0$, one has

$$\bar{k}_{00}=-\frac{1}{2}\frac{\xi_{00}}{\gamma^m_{00}}\ . \tag{5.4}$$

Now we proceed by induction. Let us assume that $k_{pq}$ for $p+q\leq N$ have been uniquely fixed via Eq. (5.2). We wish to show that $k_{pq}$ with $p+q=N+1$ can be fixed via Eq. (5.2). To this end, we consider Eq. (5.2) in $O((\lambda-\lambda^*)^p(e^2-e^{*2})^q)$ with $0\leq p\leq N+1$, $p+q=N+1$. One has

$$(p\beta^\lambda_{10}+q\bar{\beta}^e_{01}-2\gamma^m_{00})k_{pq}+\text{terms known or already fixed uniquely}=\xi_{pq}\ . \tag{5.5}$$

This fixed $k_{pq}$ uniquely [unless $p\bar{\beta}^\lambda_{10}+q\bar{\beta}^e_{01}-2\gamma^m(\lambda^*)=0$ accidentally (these conditions cannot be verified as nothing is known about $\lambda^*$ and $\beta^\lambda_{10}$, etc.; but for such a condition to be satisfied is expected only as an accident)]. As the above proof is valid for any $p,q$ with $p+q=N+1$, $0\leq p$, $q\leq N+1$, $k_{pq}$ for *all* $p+q=N+1$ are determined. As the assumption made applies to $N=0$ via Eq. (5.4), the result is proved by induction, fixing $\bar{k}(\lambda,e^2)$ uniquely. Now, let

$$\bar{k}\left[\lambda,e^2,\frac{m^2}{\mu^2}\right]=\bar{k}(\lambda,e^2)+\sum_{n=1}^{\infty}\left[\frac{m^2}{\mu^2}\right]^n\bar{k}_n(\lambda,e^2)\ . \tag{5.6}$$

From Eqs. (3.10) and (4.1), it is easily seen that

$$\left[\bar{\beta}^{\lambda}\frac{\partial}{\partial\lambda}+\bar{\beta}^{e}\frac{\partial}{\partial e}+[(2n-2)\gamma_m-2n]\right]\bar{k}_n=0,\quad n\geq 1\ . \tag{5.7}$$

Following a similar procedure used in obtaining the unique solution for $\bar{k}(\lambda,e^2)$, one easily sees that the only perturbative [in $(\lambda-\lambda^*),(e^2-e^{*2})$] solution to Eqs. (5.7) are

$$\bar{k}_n(\lambda,e^2)=0,\quad n\geq 1\ . \tag{5.8}$$

Hence our boundary conditions fix $\bar{k}(\lambda,e^2,m^2/\mu^2)$ uniquely.

## VI. POSSIBLE ALTERNATE BOUNDARY CONDITIONS

In this section, we shall explore alternate choices of boundary conditions that also fix a unique solution to Eq. (3.10). Unlike the boundary condition used in the previous section the physical significance of these boundary conditions is, however, not clear.

*Perturbative expansion in powers of* $\lambda$. As in the previous section, we shall first consider $\bar{k}(\lambda,e^2)$, i.e., the $m$-independent part of $\bar{k}$ (as before, $\bar{k}$ is assumed to be analytic at $m=0$). $\bar{k}(\lambda,e^2)$ satisfies Eq. (5.2). We expand quantities in this equation in powers of $\lambda$:

$$\begin{aligned}\bar{k}(\lambda,e^2)&=\sum_{n=0}^{\infty}k_n(e^2)\lambda^n\ ,\\ \bar{\beta}^{\lambda}(\lambda,e^2)&=\sum_{n=0}^{\infty}\lambda^n\tilde{\beta}^{\lambda}_n(e^2)\ ,\\ 2e\beta^{e}(\lambda,e^2)&=\sum_{n=0}^{\infty}\lambda^n\tilde{\beta}^{e}_n(e^2)\ ,\\ \gamma_m(\lambda,e^2)&=\sum_{n=0}^{\infty}\lambda^n\tilde{\gamma}_{mn}(e^2)\ ,\\ \xi(\lambda,e^2)&=\sum_{n=0}^{\infty}\lambda^n\tilde{\xi}_n(e^2)\ .\end{aligned} \tag{6.1}$$

We compare successive powers of $\lambda$ in Eq. (5.2). They yield the set of equations

$$\tilde{\beta}^{\lambda}_0(e^2)k_1(e^2)+\tilde{\beta}^{e}_0(e^2)\frac{d}{de^2}k_0(e^2)-2\tilde{\gamma}_{m0}(e^2)k_0(e^2)=\xi_0(e^2)\ , \tag{6.2}$$

$$2\tilde{\beta}^{\lambda}_0(e^2)k_2(e^2)+\text{terms depending on } k_0 \text{ and } k_1=\xi_1(e^2)\ , \tag{6.3}$$

$$\vdots$$

$$(n+1)\tilde{\beta}^{\lambda}_0(e^2)k_{n+1}(e^2)+\text{terms depending on } k_0,k_1,\ldots,k_n=\xi_n(e^2),\ \text{etc.} \tag{6.4}$$

$$\vdots$$

Now we impose the boundary condition that $k_0(e^2)$ be so chosen that it corresponds to that in scalar electrodynamics with $\lambda=0$. This requires

$$\tilde{\beta}^{e}_0(e^2)\frac{d}{de^2}k_0(e^2)-2\tilde{\gamma}_{m0}(e^2)k_0(e^2)=\xi_0(e^2)\ . \tag{6.5}$$

The above equation determines $k_0(e^2)$ uniquely if we demand that $k_0(e^2)$ be a power series in $e^2$. This is seen easily by a series expansion of Eq. (6.5) [as mentioned below Eq. (5.5), determination of $k_0(e^2)$ is possible subject to conditions that $2n\beta^e_3-2\gamma_{m2}\neq 0$ for any $n$ which is readily verified from Eq. (2.18).] Then Eqs. (6.3), (6.4), etc. determine successively $k_1,k_2,\ldots,k_n,\ldots$ uniquely, as Eq. (6.4) is only an algebraic and not a differential constraint on $k_{n+1}$. It should be noted that $k_n$ $(n\geq 1)$ are not analytic at $e=0$ because $\tilde{\beta}^{\lambda}_0(e^2)$ starts out as $e^4$. This is as expected since it was shown in Sec. III, that a completely perturbative solution in powers of $\lambda$ and $e^2$ does not exist.

## APPENDIX

In this appendix, we shall show that the physical matrix elements of $\theta^{\text{imp}}_{\mu\nu}$ are gauge independent iff $\bar{k}$ is independent of $\xi$. To show this we note that

$$\theta^{\text{imp}}_{\mu\nu}=\theta^{(1)}_{\mu\nu}+\theta^{(2)}_{\mu\nu}+\theta^{(3)}_{\mu\nu}+\theta^{(4)}_{\mu\nu}\ , \tag{A1}$$

where

$$\begin{aligned}\theta^{(1)}_{\mu\nu}&=-g_{\mu\nu}[\mathcal{L}+\tfrac{1}{2}\xi_0(\partial\cdot A)^2]-F_{\nu\alpha}F^{\alpha}_{\mu}\\ &\quad+\tfrac{1}{2}[(D_\mu\phi)^*(D_\nu\phi)+(D_\nu\phi)^*(D_\mu\phi)]\ ,\\ \theta^{(2)}_{\mu\nu}&=\xi_0[\partial_\mu(\partial\cdot A)A_\nu+\partial_\nu(\partial\cdot A)A_\mu]\\ &\quad-g_{\mu\nu}\xi_0\partial^{\rho}(\partial\cdot A)A_\rho-g_{\mu\nu}\xi_0(\partial\cdot A)^2\ ,\\ \theta^{(3)}_{\mu\nu}&=-G(\lambda,e^2,\xi,\epsilon)(\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})\phi^*\phi\ ,\\ \theta^{(4)}_{\mu\nu}&=-\bar{k}\left[\lambda,e^2,\frac{m^2}{\mu^2},\xi\right](\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})(\phi^*\phi)^R\ .\end{aligned} \tag{A2}$$

Now $\theta^{(1)}_{\mu\nu}$ is a gauge-invariant operator. Hence its physical matrix elements (barring certain exceptional external momenta) are $\xi$ independent. We shall show that Ward-Takahashi (WT) identities imply that the physical matrix elements of $\theta^{(2)}_{\mu\nu}$ vanish $(q\neq 0)$. We shall also show that $\theta^{(2)}_{\mu\nu}$ does not mix with $(\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})(\phi^*\phi)$. Hence, from Eqs. (2.3) and (2.5),

$$\{\langle\theta_{\mu\nu}\rangle\}^{\mathrm{div}}=G'(\lambda,e^2,\xi,\epsilon)(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\langle\phi^*\phi\rangle^R$$
$$=\{\langle\theta^{(1)}_{\mu\nu}\rangle\}^{\mathrm{div}} . \tag{A3}$$

Thus $G'(\lambda,e^2,\xi,\epsilon)$ is the renormalization constant of a gauge-invariant operator $\theta^{(1)}_{\mu\nu}$ with another gauge-invariant operator $(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})(\phi^*\phi)$ and hence is $\xi$ independent. Also $Z_m^{-1}$, the multiplicative renormalization constant of a gauge-invariant operator $(\phi^*\phi)$, is also $\xi$ independent. Hence $G=G'Z_m$ is also $\xi$ independent. Thus $\theta^{(3)}_{\mu\nu}$ is a gauge-invariant operator with a $\xi$-independent coefficient and hence has $\xi$-independent physical matrix elements. Thus

$$\frac{\partial}{\partial\xi}\langle\theta^{\mathrm{imp}}_{\mu\nu}\rangle_{\mathrm{phy}}=\frac{\partial}{\partial\xi}\langle\theta^{(4)}_{\mu\nu}\rangle_{\mathrm{phy}}$$
$$=-\frac{\partial}{\partial\xi}\bar{k}\left[\lambda,e^2,\frac{m^2}{\mu^2},\xi\right]$$
$$\times(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\langle\phi^*\phi\rangle^R_{\mathrm{phy}}=0 \tag{A4}$$

iff $(\partial/\partial\xi)\bar{k}(\lambda,e^2,m^2/\mu^2,\xi)=0$ proving the result that $\bar{k}$ and hence $k$ is $\xi$ independent.

Finally, we prove the two points regarding $\theta^{(2)}_{\mu\nu}$ mentioned below Eq. (A2). $\theta^{(4)}_{\mu\nu}$ contributes only when $q\neq 0$. Hence, to determine the gauge independence of $\bar{k}$, it is sufficient to consider $\theta^{(2)}_{\mu\nu}$ at $q\neq 0$. Matrix elements of $\theta^{(2)}_{\mu\nu}$ can be determined from the WT identity derived in Eq. (B4) of Ref. 10. It reads

$$\Big\langle -\xi_0\int d^ny[\partial\cdot A(y)]\theta[A(y)]+\int J_\mu(x)\partial^\mu_x G(x,y)\theta[A(y)]d^nx\,d^ny$$
$$+\int J^*(x)ie_0\phi(x)G(x,y)\theta[A(y)]d^nx\,d^ny+\int J(x)(-ie_0)\phi^*(x)G(x,y)\theta[A(y)]d^nx\,d^ny\Big\rangle=0 . \tag{A5}$$

We exhibit the procedure for the first term in $\theta^{(2)}_{\mu\nu}$, viz., $\xi_0\partial_\mu(\partial\cdot A)A_\nu$. A similar procedure works for other terms. We let $\theta[A(y)]=\partial_\mu[A_\nu(y)\epsilon(y)]$. After integrating by parts and comparing the coefficient of $\epsilon(y)$, we obtain

$$\langle\xi_0\partial_\mu(\partial\cdot A)A_\nu(y)\rangle=\int J_\rho(x)\partial^y_\mu\partial^\rho_x G(x,y)A_\nu(y)d^nx+\int J^*(x)ie_0\phi(x)\partial^y_\mu G(x,y)A_\nu(y)d^nx$$
$$-\int J(x)ie_0\phi^x(x)\partial^y_\mu G(x,y)A_\nu(y)d^nx . \tag{A6}$$

It is easy to verify now that, when the on-shell truncated Green's functions on the right-hand side are considered for $q\neq 0$, they vanish either because $\epsilon\cdot k=0$ for a physical photon or because of the lack of a pole at the right position.

Further, following the discussion below Eq. (B6) of Ref. 10, it is easy to show that $\partial_\mu(\partial\cdot A)A_\nu$ cannot mix with $(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^*\phi$ when one notes the general form of the right-hand side of Eq. (B7) of Ref. 10.


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