Pin groups in physics

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The double covers Pin(s,t) and Pin(t,s) of the orthonormal groups O(s,t) and O(t,s), respectively, are not necessarily isomorphic. They are locally isomorphic but, in general, not globally so. We compute the vacuum expectation values of several quantized fermionic currents on a nonorientable spacetime (Klein bottle$\times\mathbb{R}^2$) for both groups. They are strikingly different.

I. INTRODUCTION

A. Two pin groups

Introduction. The finite group generated by the gamma matrices that satisfy the equation

$$\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha\equiv\{\gamma_\alpha,\gamma_\beta\}=2\eta_{\alpha\beta}\mathbb{1}$$
$$\text{with } (\eta_{\alpha\beta})=\text{diag}(1,\ldots,1,-1) \quad (1.1)$$

and the finite group generated by the gamma matrices that satisfy the equation

$$\hat\gamma_\alpha\hat\gamma_\beta+\hat\gamma_\beta\hat\gamma_\alpha=2\hat\eta_{\alpha\beta}\mathbb{1}$$
$$\text{with } (\hat\eta_{\alpha\beta})=\text{diag}(1,-1,\ldots,-1) \quad (1.2)$$

are not, in general, isomorphic. For example, the group $(\pm\mathbb{1},\pm\gamma)$ with $\gamma^2=-\mathbb{1}$ is isomorphic to $\mathbb{Z}_4$, and the group $(\pm\mathbb{1},\pm\hat\gamma)$ with $\hat\gamma^2=\mathbb{1}$ is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$.

It follows that the *real* Clifford algebra $\mathcal{C}(s,t)$ generated by the γ_α's with

$$(\eta_{\alpha\beta})=\text{diag}(\overbrace{1,\ldots,1},\overbrace{-1,\ldots,-1})$$

is not, in general, isomorphic to the *real* Clifford algebra $\mathcal{C}(t,s)$ generated by the $\hat\gamma_\alpha$'s with

$$\hat\eta_{\alpha\beta}=\text{diag}(\overbrace{1,\ldots,1},\overbrace{-1,\ldots,-1}).$$

The Pin groups (defined in Sec. I B) Pin$(s,t)\subset\mathcal{C}(s,t)$ and Pin$(t,s)\subset\mathcal{C}(t,s)$, which double cover O$(s,t)$ and O(t,s), are not, in general, isomorphic. The groups being different, the Fermi fields ψ and $\hat\psi$ acted upon by Pin(s,t) or Pin(t,s), respectively, are different.

In this paper we make an explicit calculation of simple physical quantities, fermionic currents, and compare the results obtained when using Fermi fields ψ and $\hat\psi$. We have chosen to compute vacuum expectation values of fermionic currents on $\mathbb{R}_2\times$ Klein bottle for the following reasons: (i) The Klein bottle admits both kinds of Fermi fields;[1] (ii) the Klein bottle forces one to construct Fermi fields with nontrivial transformation laws on at least one of the overlaps of the coordinate patches; (iii) calculations by DeWitt, Hart, and Isham[2] of the vacuum expectation values of the energy-momentum stress tensor on a Klein bottle can be used in part, as well as several equations obtained previously by DeWitt.[3]

The setup. We call Spin$(n-1,1)$ and Spin$(1,n-1)$ the double covers of the orientation-preserving groups SO$(n-1,1)$ and SO$(1,n-1)$, respectively; Pin$(n-1,1)\equiv\Gamma$ and Pin$(1,n-1)\equiv\hat\Gamma$ are the double covers of O$(n-1,1)$ and O$(1,n-1)$, respectively. The interesting fact is that, although O$(n-1,1)$ and O$(1,n-1)$ are ismorphic and Spin$(n-1,1)$ and Spin$(1,n-1)$ are isomorphic, Pin$(n-1,1)$ and Pin$(1,n-1)$ are not necessarily isomorphic. They are locally isomorphic, as implied by the isomorphism of Spin$(n-1,1)$ with Spin$(1,n-1)$, but they are not necessarily globally isomorphic. To exhibit differences between the two Pin groups, it is sufficient to consider their subgroups Pin$^\uparrow$ which double cover orthochronous transformations. Time reversal does not bring any new physics in the example that we investigate.

Some properties of Γ *and* $\hat\Gamma$ (Refs. 4–7). First, we note that the sets of Hermitian conjugate matrices $\{\pm\gamma_\alpha^\dagger\}$, $\gamma_\alpha^\dagger=\gamma_\alpha^{\sim *}$, and complex-conjugate matrices $\{\pm\gamma_\alpha^*\}$, satisfy the same algebra (1.1) as the set $\{\pm\gamma_\alpha\}$. For n even, there is only one irreducible faithful representation of the gamma matrices, of dimension $2^{n/2}$; hence, there exist matrices $H_\pm$ and $C_\mp$ such that

$$H_\pm^{-1}\gamma_\alpha H_\pm=\pm\gamma_\alpha^\dagger\,, \quad (1.3)$$

$$C_\pm^{-1}\gamma_\alpha C_\pm=\pm\gamma_\alpha^*\,. \quad (1.4)$$

For n odd, there are two inequivalent irreducible faithful representations of the gamma matrices, of dimension $2^{(n-1)/2}$, and there may not exist matrices $H_\pm$ and $C_\pm$ satisfying those similarity transformations. Given the fact that, for n odd, the gamma matrices can be constructed from the gamma matrices for $n-1$ by adding the matrix

$$\gamma_{n-1}=\pm i^{n(n-3)/2}\gamma_0\gamma_1\cdots\gamma_{n-2}\,, \quad (1.5)$$

where the different signs correspond to different representations, it is sufficient to check whether the similarity transformations (1.3) and (1.4) have solutions for $\gamma_\alpha=\gamma_{n-1}$ given by (1.5). We obtain readily

$$H_\pm^{-1}\gamma_{n-1}H_\pm=-\gamma_{n-1}^\dagger\,. \quad (1.6)$$

Hence, for n odd, there exists only one solution, namely, H_-, to obtain the Hermitian conjugate gamma matrices. We shall henceforth use the letter η to label H_-, since, as can be anticipated, it will be used in Sec. I C to construct covariant pinors (copinors) from contravariant pinors and hence will play the role of a metric:

$$\eta \equiv H_- . \tag{1.7}$$

With γ_{n-1} given by (1.5), we obtain readily

$$C_\pm^{-1}\gamma_{n-1}C_\pm = (-1)^{n(n-3)/2} = \begin{cases} \gamma_{n-1}^* & \text{for } n = 3 \bmod 4 , \\ -\gamma_{n-1}^* & \text{for } n = 1 \bmod 4 . \end{cases} \tag{1.8}$$

Hence, the only solutions of (1.8) are

$$C_+ \quad \text{for } n = 3 \bmod 4 , \tag{1.9a}$$

$$C_- \quad \text{for } n = 1 \bmod 4 . \tag{1.9b}$$

A similar study can be made for the group $\hat{\Gamma}$. One obtains

$$\hat{H}_\pm^{-1}\hat{\gamma}_{n-1}\hat{H}_\pm = \hat{\gamma}_{n-1}^\dagger , \tag{1.10}$$

$$\hat{C}_\pm^{-1}\hat{\gamma}_{n-1}\hat{C}_\pm = -(-1)^{n(n-3)/2}\hat{\gamma}_{n-1}^* . \tag{1.11}$$

But it follows from the definitions of $H_\pm$, $\hat{H}_\pm$, $C_\pm$, $\hat{C}_\pm$ that

$$\hat{H}_\pm = H_\mp , \quad \hat{C}_\pm = C_\mp \tag{1.12}$$

and the choices (1.7) and (1.9) are valid for both Γ and $\hat{\Gamma}$.

The groups Γ and $\hat{\Gamma}$ are isomorphic for $n = 2 \bmod 8$.

Proof. It can be shown by explicit construction of the gamma matrices[4-7] that for $n = 2,3,4 \bmod 8$, the γ_α may be chosen real and the $\hat{\gamma}_\alpha$ imaginary, for $n = 8,9,10 \bmod 8$, the γ_α may be chosen imaginary and the $\hat{\gamma}_\alpha$ real. Hence, for $n = 2 \bmod 8$, the γ_α and the $\hat{\gamma}_\alpha$ may be chosen all real or all imaginary.

For $n = 5,6,7 \bmod 8$, neither set can be chosen all real or all imaginary. For $n = 2,3,4 \bmod 8$, $C_+C_+^* = 1$, for $n = 6,7,8 \bmod 8$, $C_+C_+^* = -1$. Recall that C_+ is not an acceptable choice for $n = 1 \bmod 4$. For $n = 0,1,2 \bmod 8$, $C_-C_-^* = 1$, for $n = 4,5,6 \bmod 8$, $C_-C_-^* = -1$. Recall that C_- is not an acceptable choice for $n = 3 \bmod 4$.

B. The Pin groups as double covers of $O(s,t)$ and $O(t,s)$, $s+t=n$

Pin groups are subsets of Clifford algebras. We shall consider Clifford algebras $\mathcal{C}(s,t)$ *over the reals,* whose symbols γ_α (neither real nor complex) satisfy

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}(s,t)\mathbf{1}, \qquad \eta_{\alpha\beta}(s,t) = \mathrm{diag}(\overbrace{1,\ldots,1}^{s\text{ copies}}, \overbrace{-1,\ldots,-1}^{t\text{ copies}}) . \tag{1.13}$$

We choose to work with real Clifford algebras because the n-dimensional subset $C^1(s,t)$ of $\mathcal{C}(s,t)$ generated by $\{\gamma_1,\ldots,\gamma_n\}$ is isomorphic to the Riemannian space $(\mathbb{R}^n,\eta)$—here spacetime is assumed to be real. *We shall work with real or complex representations of real Clifford algebras.*

There are two nonequivalent algebraic definitions of the Pin groups: the original one and the twisted one.[8] We recall briefly both definitions to justify our choosing the original one, in spite of the advantages of the twisted one and its growing popularity.

(i) The standard definition; $\Gamma(s,t)$.

The Pin(s,t) group $\Gamma(s,t)$ is the subset of invertible elements $\Lambda \in \mathcal{C}(s,t)$ such that

$$\Lambda\gamma_\alpha\Lambda^{-1} = \gamma_\beta O^\beta{}_\alpha(s,t) , \tag{1.14a}$$

$$\mathrm{Norm}(\Lambda) = \pm 1 , \tag{1.14b}$$

where $(O^\beta{}_\alpha)(s,t) \in O(s,t)$ and where the norm is defined by $N(\Lambda) = \Lambda^\tau\Lambda$, where τ: $\Lambda \mapsto \Lambda^\tau$ is an anti-involution. For instance, if $\Lambda = \gamma_{\alpha_1}\cdots\gamma_{\alpha_p}$, then[9] $\Lambda^\tau = \gamma_{\alpha_p}\cdots\gamma_{\alpha_1}$. Equation (1.14) defines a 2-to-1 homomorphism:

$$\mathcal{H}\colon\ \Gamma(s,t) \to O(s,t) \ \text{ by } \pm\Lambda \mapsto (O^\beta{}_\alpha) . \tag{1.15}$$

The drawback of this definition is that the homomorphism $\mathcal{H}$ is not surjective when $s+t=n$ is odd. Indeed, there is then no Λ corresponding to $(O^\beta{}_\alpha) = \mathrm{diag}(1,\ldots,1,-1,1,\ldots,1)$ with the minus entry at any position along the diagonal. When n is odd, the sign change of an odd number of gamma matrices cannot be effected by an equivalence transformation.

(ii) The twisted definition; $\tilde{\Gamma}(s,t)$.

The (twisted) Pin(s,t) group $\tilde{\Gamma}(s,t)$ is the subset of elements $\Lambda \in \mathcal{C}(s,t)$ such that

$$\alpha(\Lambda)\gamma_\alpha\Lambda^{-1} = \gamma_\beta O^\beta{}_\alpha(s,t) , \tag{1.16a}$$

$$\mathrm{Norm}(\Lambda) = \pm 1 , \tag{1.16b}$$

where α is the canonical automorphism[8] of $\mathcal{C}(s,t)$ such that

$$\alpha(\gamma_\alpha) = -\gamma_\alpha . \tag{1.17}$$

Equation (1.16) defines a 2-to-1 homomorphism

$$\tilde{\mathcal{H}}\colon\ \tilde{\Gamma}(s,t) \to O(s,t) \ \text{ by } \pm\Lambda \mapsto (O^\beta{}_\alpha) ,$$

which is surjective in all dimensions. Moreover, $\tilde{\mathcal{H}}$ maps $\pm\gamma_\alpha$ into the diagonal matrix, which reverses the direction of the α axis, a bookkeeping simplification. The drawback of this definition is the fact that the Lorentz invariance of the Dirac equation requires the gamma matrices to satisfy the similarity transformation (1.14a).

(iii) Comparison of $\Gamma(s,t)$ and $\tilde{\Gamma}(s,t)$ (Ref. 10).

The desire to work with the twisted Pin groups and to preserve the Lorentz invariance of the Dirac equation suggests the use of a twisted representation of the Pin group

$$\rho\colon\ \Gamma \to \Psi, \quad \psi(x) \in \Psi ,$$

such that

$$\rho(\Lambda)\gamma_\alpha[\rho(\Lambda)]^{-1} = \gamma_\beta O^\beta{}_\alpha(s,t) . \tag{1.18}$$

As already noted, this equation has no solution in odd dimensions. In even dimensions,

$$\rho(\Lambda) = \begin{cases} \Lambda & \text{if } \Lambda \text{ is even,} \\ \epsilon\Lambda & \text{if } \Lambda \text{ is odd,} \quad \epsilon = \gamma_1\cdots\gamma_n , \end{cases} \tag{1.19}$$

also known as the orientation matrix, solves (1.18). In addition,

$$\rho(\gamma_\alpha)\rho(\gamma_\alpha) = \gamma_\alpha^2 \quad \text{if } n = 0 \bmod 4 , \tag{1.20}$$

$$\rho(\gamma_\alpha)\rho(\gamma_\alpha) = -\gamma_\alpha^2 \quad \text{if } n = 2 \bmod 4 . \tag{1.21}$$

Hence, if $n=0 \bmod 4$. $\rho(\Lambda)$ defined an isomorphism between $\Gamma(s,t)$ and $\tilde{\Gamma}(s,t)$; if $n=2 \bmod 4$, $\rho(\Lambda)$ defines an isomorphism between $\hat{\Gamma}(t,s)$ and $\tilde{\Gamma}(s,t)$, where $\hat{\Gamma}(t,s)$ is the Pin group constructed with $\hat{\gamma}_\alpha$:

$$\{\hat{\gamma}_\alpha,\hat{\gamma}_\beta\}=2\hat{\eta}_{\alpha\beta}(t,s)\mathbb{1} ,$$
$$[\hat{\eta}_{\alpha\beta}(t,s)]=\mathrm{diag}(\overbrace{1,\ldots,1}^{t\text{ copies}},\overbrace{-1,\ldots,-1}^{s\text{ copies}}) .$$

In conclusion, for our purpose it is preferable to work with the usual definition (1.14). Also, we need not keep t aribtrary and we shall go back to the groups Γ and $\hat{\Gamma}$ introduced in the first section with $t=1$, $s=n-1$,

$$\begin{array}{ccc} \Gamma & & \hat{\Gamma} \\ \mathcal{H}\downarrow & & \downarrow\mathcal{H} \\ \mathrm{O}(n-1,1) & \simeq & \mathrm{O}(1,n-1) \end{array} . \tag{1.22}$$

Space Inversion. For n even, i.e., for odd space dimensions, space inversion in Γ can be defined by

$$\begin{aligned} &P\gamma_0P^{-1}=\gamma_0 ,\\ &P\gamma_aP^{-1}=-\gamma_a,\quad a=1,\ldots,n-1 \end{aligned} \tag{1.23}$$

and space inversion in $\hat{\Gamma}$ can be defined by

$$\begin{aligned} &\hat{P}\hat{\gamma}_0\hat{P}^{-1}=\hat{\gamma}_0 ,\\ &\hat{P}\hat{\gamma}_a\hat{P}^{-1}=-\hat{\gamma}_a,\quad a=1,\ldots,n-1 . \end{aligned} \tag{1.24}$$

It is easy to check that

$$P=\pm\gamma_0 \text{ and } \hat{P}=\pm\hat{\gamma}_0 \tag{1.25}$$

solve these equations. Note that

$$P^2=-\mathbb{1} \text{ and } \hat{P}^2=\mathbb{1} . \tag{1.26}$$

C. Pinor fields

Pinor fields are sections of a supervector bundle[11] associated with a principal Pin bundle by a representation of the Pin group defining the principal bundle. Since it is desirable, in several respects, to consider classical physics as the limit of quantum physics, we choose representations of the Pin groups on supervector spaces. A pinor field $\psi(x)$ is a section of a supervector bundle associated with a Γ bundle; a pinor field $\hat{\psi}(x)$ is a section of a supervector bundle associated with a $\hat{\Gamma}$ bundle. The components of (super)classical pinor fields are a numbers; one can choose the pinor fields either to be c type (i.e., the basis of the typical fiber consists of a-type vectors) or to be a type (i.e., the basis of the typical fiber consists of c-type vectors). We make the latter choice.

Tensor Fields Constructed from Pinor Fields. Let ψ be a contravariant pinor, i.e., Γ acts on ψ by

$$\psi\mapsto\Lambda\psi \tag{1.27}$$

and let η be defined (1.21) by

$$\eta^{-1}\gamma_\alpha\eta=-\gamma_\alpha^\dagger ; \tag{1.28}$$

then, under the action of the subgroup $\Gamma^\uparrow\subset\Gamma$, which double covers orthochronous transformations,

$$\psi^\dagger\eta \text{ is a covariant pinor field (copinor)} \tag{1.29}$$

and

$$\psi^\dagger\eta\gamma_\alpha\gamma_\beta\cdots\gamma_\delta\psi \text{ is a tensor field} .$$

If M is any matrix

$$(\psi^\dagger M\psi)^*=\psi^\dagger M^\dagger\psi \tag{1.30}$$

and (assuming all indices distinct)

$$\begin{aligned} &V\equiv\psi^\dagger\eta\psi \text{ is imaginary} ,\\ &V_\alpha\equiv\psi^\dagger\eta\gamma_\alpha\psi \text{ is real} ,\\ &V_{\alpha\beta}\equiv\psi^\dagger\eta\gamma_\alpha\gamma_\beta\psi \text{ is real} ,\\ &V_{\alpha\beta\gamma}\equiv\psi^\dagger\eta\gamma_\alpha\gamma_\beta\gamma_\gamma\psi \text{ is real} ,\\ &V_{\alpha\beta\gamma\delta}\equiv\psi^\dagger\eta\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\psi \text{ is imaginary, etc.} \end{aligned} \tag{1.31}$$

A similar discussion for the quantitites with a caret gives (recall that $\hat{\eta}=\hat{H}_+=H_-=\eta$)

$$\begin{aligned} &\hat{V}\equiv\hat{\psi}^\dagger\hat{\eta}\hat{\psi} \text{ is imaginary} ,\\ &\hat{V}_\alpha\equiv\hat{\psi}^\dagger\hat{\eta}\hat{\gamma}_\alpha\hat{\psi} \text{ is imaginary} ,\\ &\hat{V}_{\alpha\beta}\equiv\hat{\psi}^\dagger\hat{\eta}\hat{\gamma}_\alpha\hat{\gamma}_\beta\hat{\psi} \text{ is real} ,\\ &\hat{V}_{\alpha\beta\gamma}\equiv\hat{\psi}^\dagger\hat{\eta}\hat{\gamma}_\alpha\hat{\gamma}_\beta\hat{\gamma}_\gamma\hat{\psi} \text{ is real} ,\\ &\hat{V}_{\alpha\beta\gamma\delta}\equiv\hat{\psi}^\dagger\hat{\eta}\hat{\gamma}_\alpha\hat{\gamma}_\beta\hat{\gamma}_\gamma\hat{\gamma}_\delta\hat{\psi} \text{ is imaginary, etc.} \end{aligned} \tag{1.32}$$

Charge-Conjugate Currents. When $n\neq1 \bmod 4$, we can define the charge-conjugate pinor field of $\psi(x)$ by

$$\psi^{C_+}=C_+\psi^* , \tag{1.33}$$

or when $n\neq3 \bmod 4$ by

$$\psi^{C_-}=C_-\psi^* . \tag{1.34}$$

Remark. If ψ satisfies the equation

$$[\gamma_\alpha(\partial_\alpha+iqA_\alpha)-m]\psi=0 ,$$

then ψ^{C_+} and ψ^{C_-} satisfy

$$[\gamma_\alpha(\partial_\alpha-iqA_\alpha)-m]\psi^{C_+}=0 , \tag{1.35}$$

$$[\gamma_\alpha(\partial_\alpha-iqA_\alpha)+m]\psi^{C_-}=0 . \tag{1.36}$$

An antiparticle is usually defined as a solution of (1.35). See, for instance, Ref. 12 or 13. It is straightforward but somewhat tedious to check case by case that

$$C_+^\dagger\eta C_+=-\eta^\sim \tag{1.37}$$

and

$$C_-^\dagger\eta C_-=\eta^\sim . \tag{1.38}$$

Labeling $V^{C_\pm}, V_\alpha^{C_\pm},\ldots,$ the charge-conjugate currents of $V, V_\alpha,\ldots,$ it follows from (1.37) and (1.38) that

$$
\begin{aligned}
&V^{C_+} - V, \quad V^{C_-} = -V \,, \\
&V_\alpha^{C_+} = -V_\alpha, \quad V_\alpha^{C_-} = -V_\alpha \,, \\
&V_{\alpha\beta}^{C_+} = -V_{\alpha\beta}, \quad V_{\alpha\beta}^{C_-} = V_{\alpha\beta} \,, \\
&V_{\alpha\beta\gamma}^{C_+} = V_{\alpha\beta\gamma} \,, \quad V_{\alpha\beta\gamma}^{C_-} = V_{\alpha\beta\gamma} \,, \\
&V_{\alpha\beta\gamma\delta}^{C_+} = V_{\alpha\beta\gamma\delta}, \text{ etc.}, \quad V_{\alpha\beta\gamma\delta}^{C_-} = -V_{\alpha\beta\gamma\delta}, \text{ etc.}
\end{aligned}
\tag{1.39}
$$

Self-Charge-Conjugate Fields (Majorana Fields). Self-charge-conjugate fields satisfy, by definition, the equation

$$\psi^{C_\pm} = \psi \,, \tag{1.40}$$

which implies

$$C_\pm C_\pm^* = \mathbb{1} \,. \tag{1.41}$$

Hence, Majorana fields cannot be defined for $n=5,6,7 \bmod 8$. For $n=2,3,4 \bmod 8$, a Majorana field satisfies

$$\psi^{C_+} = \psi \tag{1.42}$$

and the following currents constructed with Majorana fields vanish:

$$V_\alpha = 0 \,, \quad V_{\alpha\beta} = 0 \,. \tag{1.43}$$

For $n=0,1,2 \bmod 8$, a Majorana field satisfies

$$\psi^{C_-} = \psi$$

and

$$V=0, \quad V_\alpha = 0 \,, \quad V_{\alpha\beta\gamma\delta} = 0 \,.$$

Remark 1. In quantum theory, the vanishing currents are replaced by zero-point constants.

Remark 2. For $n=2 \bmod 8$, there exist two distinct definitions for Majorana fields.

II. QUANTIZED FERMIONIC CURRENTS ON $\mathbb{R}^2\times$ KLEIN BOTTLE

A. The setup

We shall work with the real representation of $\Gamma(3,1)$ and the imaginary representation of $\hat{\Gamma}(1,3)$. We shall construct fermionic currents on $\mathbb{R}^2\times\mathbb{K}^2$, one of the simplest nonorientable flat spacetimes. To obtain this topology, we identify, in a Cartesian coordinate system, the points

$$(x^0,x^1,x^2,x^3) \text{ with } (x^0,x^1,x^2+ma,(-1)^m x^3+nb) \tag{2.1}$$

for all $m,n = \ldots, -2, -1, 0, 1, 2, \ldots$.

B. The Case $\{\gamma_\alpha,\gamma_\beta\} = 2\eta_{\alpha\beta}\mathbb{1}$, $(\eta_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1)$

A Majorana field ψ is an a-number-valued pinor field with the real Lagrangian

$$L = \tfrac{1}{2} i \psi^\sim \eta \left[\gamma^\alpha \frac{\partial}{\partial x^\alpha} + m \right] \psi \,. \tag{2.2}$$

It obeys the Dirac equation

$$i\eta \left[\gamma_\alpha \frac{\partial}{\partial x^\alpha} + m \right] \psi = 0 \,. \tag{2.3}$$

The vacuum expectation value of the chronological product is given by

$$\langle \psi(x)\psi^\sim(x) \rangle = -iG(x,x') \,, \tag{2.4}$$

where G is the Feynman Green's function satisfying

$$i\eta \left[\gamma^\alpha \frac{\partial}{\partial x^\alpha} + m \right] G(x,x') = -\mathbb{1}\delta(x,x') \,. \tag{2.5}$$

The Feynman Green's function can be represented in the form

$$G(x,x') = -i \left[\gamma^\alpha \frac{\partial}{\partial x^\alpha} - m \right] \mathcal{G}(x,x')\eta^{-1} \,, \tag{2.6}$$

where $\mathcal{G}(x,x')$ is the solution of the Klein-Gordon equation

$$(\Box - m^2)\mathcal{G}(x,x') = -\mathbb{1}\delta(x,x') \tag{2.7}$$

such that, for $m=0$ and Minkowskian spacetime,

$$\mathcal{G}(x,x') \equiv \mathcal{G}_0(x,x') = \frac{i}{(2\pi)^2} \frac{1}{(x-x')^2 + i\epsilon} \,. \tag{2.8}$$

Let M be an arbitrary matrix:

$$
\begin{aligned}
\langle \psi^\sim \eta M \psi \rangle &= -\mathrm{tr}[\eta M \langle \psi\psi^\sim \rangle] \\
&= i\, \mathrm{tr}[\eta M G(x,x)] \\
&= \mathrm{tr}\left[M \left[\gamma^\alpha \frac{\partial}{\partial x^\alpha} - m \right] \mathcal{G}(x,x') \right] \Bigg|_{x=x'} \,.
\end{aligned}
\tag{2.9}
$$

Henceforth we consider massless fields. With the representation (2.1) of $\mathbb{R}^2\times\mathbb{K}^2$, the direction of the local three-axis is reversed every time the coordinate x^2 is increased by an amount a. The elements of the Pin group that double cover the inversion of the 3-axis are

$$P = \pm\gamma_0\gamma_1\gamma_2, \quad P^2 = \mathbb{1} \,. \tag{2.10}$$

Both choices are possible; we consider first $P=\gamma_0\gamma_1\gamma_2$. The Fermi field on $\mathbb{R}^2\times\mathbb{K}^2$ then satisfies the periodicity condition

$$
\begin{aligned}
&\psi(x^0,x^1,x^2,x^3) \\
&\quad = (\gamma_0\gamma_1\gamma_2)^m \psi(x^0,x^1,x^2+2ma,(-)^m x^3+nb) \,.
\end{aligned}
\tag{2.11}
$$

It has been shown[2] that the renormalized vacuum expectation value of the chronological product (2.4) is

$$
\begin{aligned}
\langle \psi(x)\psi^\sim(x') \rangle_{\text{ren}} &= -iG_{\text{ren}}(x,x') \\
&= -\sum_{m,n}{}' G_0(x^0,x^1,x^2,x^3;x'^0,x'^1, \\
&\qquad \times x'^2+ma,(-1)^n x'^3+nb) \\
&\qquad \times (\gamma_2^\sim \gamma_1^\sim \gamma_0^\sim)^m \,,
\end{aligned}
\tag{2.12}
$$

where $\sum'_{m,n}$ denotes the sum over all values of m and n except $m=0$, $n=0$. It follows from (2.8) and (2.7) that

$$\langle \psi(x)\psi^\sim(x') \rangle_{\text{ren}} = \gamma^\alpha \frac{\partial}{\partial x^\alpha} \mathcal{G}_{\text{ren}}(x,x')\eta^{-1} \,, \tag{2.13}$$

where

$$\begin{aligned}\mathcal{G}_{\text{ren}}(x,x') &= \sum_{m,n}{}' \mathcal{G}_0(x^0,x^1,x^2,x^3;x'^0,x'^1,x'^2+ma,(-1)^m x'^3+nb)(\gamma_0\gamma_1\gamma_2)^m \\ &= \frac{1}{(2\pi)^2}\left[\sum_{m,n}{}' \frac{1}{-(x^0-x'^0)^2+(x^1-x'^1)^2+(x^2-x'^2+2ma)^2+(x^3-x'^3+nb)^2}\right. \\ &\qquad \left. + \sum_{m,n} \frac{\gamma_0\gamma_1\gamma_2}{-(x^0-x'^0)^2+(x^1-x'^1)^2+[x^2-x'^2+(2m+1)a]^2+(x^3+x'^3+nb)^2}\right] .\end{aligned}$$

Working out the derivatives of $\mathcal{G}_{\text{ren}}(x,x')$, one finds

$$\frac{\partial}{\partial x^a}\mathcal{G}_{\text{ren}}(x,x')=0 \quad \text{for } a=0,1,2 \ , \tag{2.14}$$

and

$$\begin{aligned}\left.\frac{\partial}{\partial x^3}\mathcal{G}_{\text{ren}}(x,x')\right|_{x=x'} &= -\frac{i}{2\pi^2}\left[\sum_{m,n}{}' \frac{nb}{(4m^2a^2+n^2b^2)^2}\mathbf{1} + \sum_{m,m} \frac{2x^3+nb}{[(2m+1)^2a^2+(2x^3+nb)^2]^2}\gamma_0\gamma_1\gamma_2\right] \\ &= i\gamma_0\gamma_1\gamma_2 F(x^3) \ ,\end{aligned} \tag{2.15}$$

where

$$F(x^3) = -\frac{1}{2\pi^2}\sum_{m,n}\frac{2x^3+nb}{[(2m+1)^2a^2+(2x^3+nb)^2]^2} \ .$$

The vanishing of the derivatives of $\mathcal{G}_{\text{ren}}$ with respect to x^0 and x^1 at the coincidence point $x=x'$ can be seen at a glance. The vanishing of the derivative with respect to x^2 comes from the cancellation of pairs of terms in the m series.

Remark. For a Möbius band of infinite width, i.e., for $b=\infty$, one can show that

$$F(x^3) = \frac{1}{32\pi a}\frac{\partial}{\partial x^3}\left[\frac{1}{x^3}\tanh\frac{\pi x^3}{a}\right] .$$

For finite b a plot of $F(x^3)$ appears in Fig. 1. It follows from (2.9) and (2.13)–(2.15) that

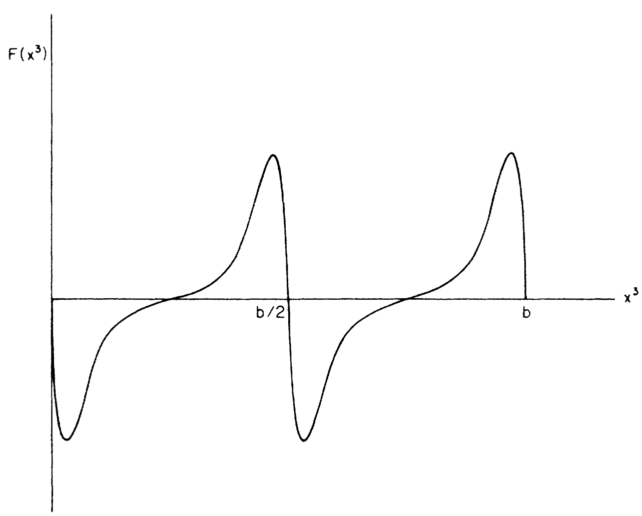

FIG. 1. The function $F(x^3)$.

$$\begin{aligned}&\langle i\psi^{\sim}\eta\psi\rangle_{\text{ren}}=0 \ , \\ &\langle \psi^{\sim}\eta\gamma^\alpha\psi\rangle_{\text{ren}}=0 \ , \\ &\langle \psi^{\sim}\eta G_{[\alpha\beta]}\psi\rangle_{\text{ren}}=0, \quad G_{[\alpha\beta]}\equiv\tfrac{1}{4}[\gamma_\alpha,\gamma_\beta] \ , \\ &\langle \psi^{\sim}\eta\gamma_5\psi\rangle_{\text{ren}}=-4F(x^3) \ , \\ &\langle \psi^{\sim}\eta\gamma_5\gamma^\alpha\rangle_{\text{ren}}=0 \ .\end{aligned} \tag{2.16}$$

If we had chosen $P=-\gamma_0\gamma_1\gamma_2$ instead of $P=\gamma_0\gamma_1\gamma_2$, the sign of the nonvanishing current would have been changed. These two choices correspond to two different Pin structures built with the group $\Gamma(3,1)$. Indeed, Pin structures are, by definition, a pair

(Principal Pin bundle, $\tilde{\mathcal{H}}$)

where $\tilde{\mathcal{H}}$ is a 2-to-1 bundle homomorphism.

$\tilde{\mathcal{H}}$: Principal Pin(s,t) bundle $\rightarrow$ Principal O(s,t) bundle .

The obstruction to constructing a Pin bundle that double covers a frame bundle is given by[10] a linear combination of second Stiefel-Whitney classes and cup products of first Stiefel-Whitney classes of the Pin bundle—the linear combination being a function of s and t, different in general for Pin(s,t), and Pin(t,s) bundles. The number of Pin structures (the number of inequivalent bundle homomorphisms $\tilde{\mathcal{H}}$) is equal to the number of elements of the first Stiefel-Whitney class. The assignments $P=-\gamma_0\gamma_1\gamma_2$ or $P=\gamma_0\gamma_1\gamma_2$ correspond to two different covers of O(3,1) by $\Gamma(3,1)$—hence to two different choices of $\tilde{\mathcal{H}}$. These are the only two possible choices for $\tilde{\mathcal{H}}$.

Remark. We could consider a complex field ψ built out of two real fields ψ_1,ψ_2:

$$\psi=\frac{1}{\sqrt{2}}(\psi_1+i\psi_2)$$

and a Lagrangian

$$L=i\psi^\dagger\eta\left[\gamma^\alpha\frac{\partial}{\partial x^\alpha}+m\right]\psi$$
$$=\frac{i}{2}\sum_{a=1,2}\psi_a^\sim\eta\left[\gamma^\alpha\frac{\partial}{\partial x^\alpha}+m\right]\psi_a\ .$$

Then

$$\langle\psi^\dagger\eta M\psi\rangle=\mathrm{tr}\left[M\left[\gamma^\alpha\frac{\partial}{\partial x^\alpha}-m\right]\mathcal{G}(x,x')\right]\Big|_{x=x'}\ .$$

$\mathcal{G}$ and $\mathcal{G}_{\rm ren}$ are the same as before. The previous equations remain valid if $\psi^\sim$ is replaced everywhere by $\psi^\dagger$.

C. The case $\{\hat\gamma_\alpha,\hat\gamma_\beta\}=2\hat\eta_{\alpha\beta}\mathbb{1}$, $(\hat\eta_{\alpha\beta})=\mathrm{diag}(1,-1,-1,-1)$

This case requires the use of a complex field ψ. Because of the above remark the computation proceeds along the same lines as before, but with the operator

$$P=\gamma_0\gamma_1\gamma_2$$

now replaced by

$$\hat P=\hat\gamma_0\hat\gamma_1\hat\gamma_2=-i\gamma_0\gamma_1\gamma_2\ . \tag{2.17}$$

The renormalized Green's function $\mathcal{G}_{\rm ren}$ is replaced by

$$\hat{\mathcal{G}}_{\rm ren}(x,x')=\frac{i}{(2\pi)^2}\left[\sum_{m,n}{}'\frac{(-1)^m}{-(x^0-x'^0)^2+(x^1-x'^1)^2+(x^2-x'^2+2ma)^2+(x^3-x'^3+nb)^2}\mathbb{1}\right.$$
$$\left.+\sum_{m,n}\frac{(-1)^m}{-(x^0-x'^0)^2+(x^1-x'^1)^2+[x^2-x'^2+(2m+1)a]^2+(x^3+x'^3+nb)^2}(-i\gamma_0\gamma_1\gamma_2)\right]\ . \tag{2.18}$$

The alternating sign in the m series is responsible for the *non*vanishing of the derivative of $\hat{\mathcal{G}}_{\rm ren}(x,x')$ with respect to x^2 at the coincidence point $x=x'$ and for the vanishing, in this case, of the derivative with respect to x^3,

$$\frac{\partial}{\partial x^a}\hat{\mathcal{G}}_{\rm ren}(x,x')\Big|_{x=x'}=0\quad\text{for } a=0,1,3\ ,$$
$$\frac{\partial}{\partial x^2}\hat{\mathcal{G}}_{\rm ren}(x,x')\Big|_{x=x'}=-\gamma_0\gamma_1\gamma_2\hat F(x^3)\ ,$$

where

$$\hat F(x^3)=\frac{1}{\pi^2}\sum_{m=0}^{\infty}\sum_{n=-\infty}^{+\infty}\frac{(-1)^m(2m+1)a}{[(2m+1)^2a^2+(2x^3+nb)^2]^2}\ .$$

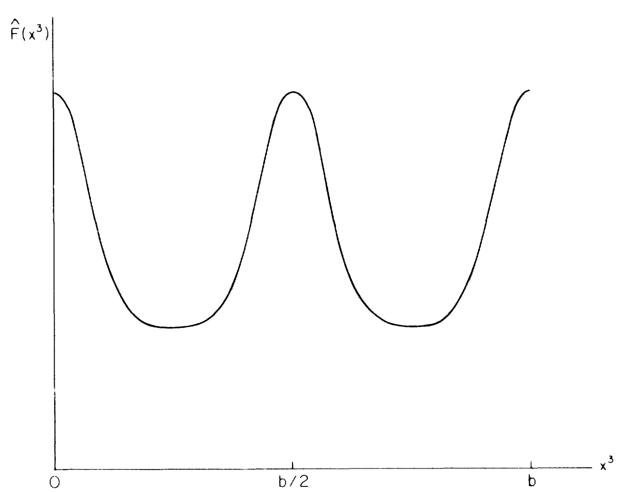

FIG. 2. The function $\hat F(x^3)$.

Remark. For a Möbius band of infinite width, i.e., $b=\infty$, one can show that

$$\hat F(x^3)=\frac{1}{4\pi a}\left[\cosh\frac{\pi x^3}{a}\right]^{-1}\ .$$

For finite b, a plot of $\hat F(x^3)$ appears in Fig. 2. Finally,

$$\langle i\hat\psi^\dagger\eta\hat\psi\rangle_{\rm ren}=0\ ,$$
$$\langle\hat\psi^\dagger\eta\hat\gamma^\alpha\hat\psi\rangle_{\rm ren}=0\ ,$$
$$\langle\hat\psi^\dagger\eta\hat G_{[01]}\hat\psi\rangle_{\rm ren}=2\hat F(x^3)\ ,$$
$$\text{the other components vanish}\ , \tag{2.19}$$
$$\langle\hat\psi^\dagger\eta\hat\gamma_5\hat\psi\rangle_{\rm ren}=0\ ,$$
$$\langle\hat\psi^\dagger\eta\hat\gamma_5\hat\gamma^\alpha\hat\psi\rangle_{\rm ren}=0\ .$$

If we had chosen $\hat P=-\hat\gamma_0\hat\gamma_1\hat\gamma_2$ instead of $\hat P=\hat\gamma_0\hat\gamma_1\hat\gamma_2$, the sign of the nonvanishing current would have been changed. Again, these two choices correspond to two different Pin structures built with the group $\hat\Gamma(1,3)$.

The only nonvanishing Γ current (2.16) is pseudoscalar; the only nonvanishing $\hat\Gamma$ current (2.19) is the (0,1) component of a tensor.

III. CONCLUSION

We chose the Pin groups Γ and $\hat\Gamma$ to be defined by (1.1) and (1.2). We could equally well have chosen them to be defined by

$$\{\gamma_\alpha,\gamma_\beta\}=-2\eta_{\alpha\beta}\mathbb{1},\quad \gamma_\alpha\in\mathrm{Pin}(s,t)\ , \tag{3.1}$$
$$\{\hat\gamma_\alpha,\hat\gamma_\beta\}=-2\hat\eta_{\alpha\beta}\mathbb{1},\quad \hat\gamma_\alpha\in\mathrm{Pin}(t,s)\ , \tag{3.2}$$

and the final results would have been interchanged; whatever has been said for $\mathrm{Pin}(s,t)$ would have applied to

$\mathrm{Pin}(t,s)$ and vice versa.

In conclusion, the choice of Pin groups depend jointly on the choice of the overall sign of the metric [(1.1) vs (1.2)] and the choice of Clifford algebra [(1.1), (1.2) vs (3.1), (3.2)]. For some systems (see, for instance, Ref. 1), the choice of the metric *alone* dictates the choice of the group, but this is not true in general. What *is* true in general is that the choice of Pin group, $\mathrm{Pin}(s,t)$, vs $\mathrm{Pin}(t,s)$ *is* physically relevant for systems in which space or time inversions play a role.

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APPENDIX: SOME REFERENCES MENTIONING EXPLICITLY OR IMPLICITLY THE EXISTENCE OF TWO PIN GROUPS AND ITS CONSEQUENCES

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