Response function of accelerated monopole detector in $R\times T^3$ space-time

Osamu Abe
Physics Laboratory, Asahikawa College, Hokkaido University of Education, Asahikawa 070, Japan
(Received 28 August 1989)

The response function of a uniformly accelerated monopole detector interacting with a free scalar field is studied in $R\times T^3$ space-time. The response function depends not only on the periodicity lengths but also on the direction of the acceleration of the detector.

I. INTRODUCTION

Fang *et al.* pointed out the possibility that the Universe is multiply connected.[1,2] Further, the Universe may consist of the usual four-dimensional space-time and an extra dimensional compact space, such as $M^4\times S^1$, etc., as in Kaluza-Klein theory. If our Universe is multiply connected, it is interesting to investigate the properties of the vacuum or to study the periodicity lengths of the Universe.

It is generally believed that the global structure of space-times does not crucially affect local quantities. In fact, the mass spectra of quarkonia calculated on finite lattices coincide well with experimental results. But there are some local quantities which depend on the global structure of space-times. We are interested in whether the global structure is estimated by local quantities only.

In this paper, we consider the possibility that the size of space-time, or the periodicity length of space-time, can be observed by the use of a local quantity only. Throughout this paper, we have restricted ourselves to $R\times T^3$ space-time, which is one of the simplest models of flat, closed, and multiply connected space-time.

When the space-time under consideration has a nontrivial topology, the vacuum of space-time generally differs from the one of Minkowski space-time (M^4). It is well known that the quantized fields are modified in the presence of boundary conditions or periodicities in spatial directions and finite vacuum energies emerge by the existence of restrictions on field modes. These energies are well known as the Casimir energies.[3-16] So, the Casimir energy generally contains the topological quantity of space-time under consideration. Thus the Casimir energy is the first candidate to observe the size of space-time.

In the next section, we consider the Casimir energy in $R\times S^1$ and $R\times T^3$ space-times and discuss the observability of the size of space-time by the Casimir energy. In Sec. III we study the spectrum that the uniformly accelerated monopole detector measures in $R\times T^3$ space-time. The last section is devoted to a summary and discussion.

II. CASIMIR ENERGY IN MULTIPLY CONNECTED SPACE-TIMES

For the sake of simplicity, we consider the Casimir energy in the massless free scalar case in flat (1+1)-dimensional space-time with periodic boundary condition ($R\times S^1$) at first. By use of the point-splitting regularization method, the (0,0) component of the energy-momentum tensor can be written as

$$T_{00}=\tfrac{1}{2}\lim_{y\to x}(\partial_0^x\partial_0^y+\partial_1^x\partial_1^y)\phi(x)\phi(y)\ . \tag{2.1}$$

We introduce the projection operators $P_{1,L}$ and P_1 by

$$P_{1,L}=\sum_k a_k^\dagger|0,L\rangle\langle 0,L|a_k\ ,\qquad P_1=\int dk\, a_k^\dagger|0\rangle\langle 0|a_k\ , \tag{2.2}$$

which project any member of Fock space to one-particle states. Here $|0\rangle$ and $|0,L\rangle$ are vacua in M^2 and in $R\times S^1$, respectively, and L denotes the periodicity length. Since we have assumed free scalar theory, the quantized field ϕ excites the vacuum to the one-particle state only. Thus the renormalized vacuum expectation value of Eq. (2.1) in pseudo-Riemannian space $R\times S^1$ can be calculated as

$$\begin{aligned}\langle 0,L|T_{00}|0,L\rangle^{\text{ren}}&=\langle 0,L|T_{00}|0,L\rangle-\langle 0|T_{00}|0\rangle\\&=\tfrac{1}{2}\lim_{y\to x}(\partial_0^x\partial_0^y+\partial_1^x\partial_1^y)[\langle 0,L|\phi(x)P_{1,L}\phi(y)|0,L\rangle-\langle 0|\phi(x)P_1\phi(y)|0\rangle]\\&=-(2\pi)^{-1}\int_0^\infty d\omega\,\omega/[\exp(L\omega)-1]=-\pi/6L^2\ ,\end{aligned} \tag{2.3}$$

as stated in Refs. 9 and 15.

The explicit expressions of the Casimir energies in $R\times T^3$ are found in Refs. 8 and 12 and are given as

$$\langle 0,L|T_{00}|0,L\rangle^{\text{ren}}=-(2\pi^2)^{-1}\sum_{\mathbf{n}\neq\mathbf{0}}\frac{1}{\left[\sum_{i=1}^{3}(n_iL_i)^2\right]^2}\ , \tag{2.4}$$

where L_i's are the periodicity lengths of the three-dimensional torus T^3. We may expect that the size of space-time L or L_i's are observable by the use of Eqs. (2.3) or (2.4), but it is not so easy to observe them because of the following two reasons.

(1) For a given energy density, there are many sets of L_i's which reproduce the energy density, so we cannot know the size of the space-time. Further, we cannot distinguish whether our space-time is $R\times T^3$, $M^2\times T^2$, or $M^3\times S^1$.

(2) It is not so easy to observe the energy density itself, if the gravitational interaction is switched off.

Analogous to the usual Casimir effect in QED, we can only observe the difference between the energy density of the vacuum in the space-time under consideration and one in any other reference space-time, e.g., Minkowskian space-time; thus, the presence of the vacuum in the reference space-time or Minkowskian space-time is explicitly assumed in the above discussion. If we are truly in $R\times S^1$ or $R\times T^3$, we must set the renormalized vacuum energy to exactly zero, since the reference frame is the same space-time under consideration. So we cannot observe any vacuum energy, if we take no account of the gravitational interaction, even if we are in the multiply connected space-time. Even if we can observe the energy density itself, we cannot know the global structure of space-time. Thus, we must consider another physical quantity to observe the size of space-time.

In the next section, we consider the spectrum that the uniformly accelerated monopole detector measures in $R\times T^3$ space-time and the observability of the size of space-time by the use of the spectrum.

III. RESPONSE FUNCTION OF MONOPOLE DETECTOR

In this section, we consider the transition rate from the vacuum to excited states in $R\times T^3$ space-time in the case where the monopole detector is uniformly accelerated. The result obtained in the specific case where the detector is accelerated perpendicular to the S^1 axis in $M^3\times S^1$ space-time is found in Refs. 17 and 18. We start from the Lagrangian density described by

$$\mathcal{L}=-\tfrac{1}{2}\phi(x)\Box\phi(x)-gm(x)\phi(x)\;, \tag{3.1}$$

where $m(x)$ denotes the monopole detector[9,13] and g stands for the coupling constant and is assumed to be sufficiently small. We impose a periodic boundary condition on spatial directions, with a periodicity length L_1, L_2, and L_3. We calculate the transition rate from the lowest-energy state or a vacuum with energy $E_0, |0,E_0,\mathbf{L}\rangle$ to some excited state ψ with energy $E, |\psi,E,\mathbf{L}\rangle$. In the lowest order of perturbative approximation, the transition amplitude A is calculated to be

$$\begin{aligned}A&=ig\langle\psi,E,\mathbf{L}|\int_{-T/2}^{T/2}d\tau\, m(\tau)\phi(x(\tau))|0,E_0,\mathbf{L}\rangle\\&=ig\langle E|m(0)|E_0\rangle\int_{-T/2}^{T/2}d\tau\exp[i(E-E_0)\tau]\langle\psi,\mathbf{L}|\phi(x(\tau))|0,\mathbf{L}\rangle\;,\end{aligned} \tag{3.2}$$

where $x(\tau)$ is the trajectory of detector and τ is the proper time of the detector. The response function or transition rate R per unit time, from the state with energy E_0 to any state with energy E in the time interval from $-T/2$ to $T/2$ is calculated as

$$\begin{aligned}R&=\lim_{T\to\infty}T^{-1}g^2\sum_E|\langle E|m(0)|E_0\rangle|^2\int_{-T/2}^{T/2}d\tau\,d\tau'\exp[i(E-E_0)(\tau-\tau')]\int d\mu[\psi]\langle 0,\mathbf{L}|\phi(x(\tau'))|\psi\rangle\langle\psi|\phi(x(\tau))|0,\mathbf{L}\rangle\\&=-\lim_{T\to\infty}T^{-1}g^2(4\pi^2)^{-1}\sum_E|\langle E|m(0)|E_0\rangle|^2\\&\qquad\times\sum_{\mathbf{n}=-\infty}^{\infty}\int_{-T/2}^{T/2}d\tau\,d\tau'\frac{\exp[-i(E-E_0)(\tau-\tau')]}{[x_0(\tau')-x_0(\tau)]^2-[\mathbf{x}(\tau')-\mathbf{x}(\tau)+\mathbf{L_n}]^2-i\epsilon[x_0(\tau')-x_0(\tau)]}\;,\end{aligned} \tag{3.3}$$

where integration over all possible final states is performed and $\mathbf{L_n}$ denotes (n_1L_1,n_2L_2,n_3L_3). Now we assume that the detector is accelerated uniformly with acceleration α and we have the detector's trajectory as

$$x(\tau,\mathbf{n})=(\alpha\sinh(\tau/\alpha),\ \sin(\theta)\cos(\phi)[\alpha\cosh(\tau/\alpha)-\alpha],\ \sin(\theta)\sin(\phi)[\alpha\cosh(\tau/\alpha)-\alpha],\ \cos(\theta)[\alpha\cosh(\tau/\alpha)-\alpha])\,. \tag{3.4}$$

Then Eq. (3.3) becomes

$$\begin{aligned}R&=-\lim_{T\to\infty}T^{-1}g^2(4\pi^2)^{-1}\sum_E|\langle E|m(0)|E_0\rangle|^2\sum_{\mathbf{n}=-\infty}^{\infty}\int_{-T/2}^{T/2}d\tau\,d\tau'\exp[-i(E-E_0)\tau]\\&\qquad\times\left[4\alpha^2\sinh^2\left[\frac{\tau-\tau'}{2\alpha}\right]-4\alpha F(\mathbf{n},\mathbf{L},\theta,\phi)\sinh\left[\frac{\tau-\tau'}{2\alpha}\right]\sinh\left[\frac{\tau+\tau'}{2\alpha}\right]-G^2(\mathbf{n},\mathbf{L})-i\epsilon\xi\right]^{-1}\\&=-\lim_{T\to\infty}T^{-1}g^2(4\pi^2)^{-1}\sum_E|\langle E|m(0)|E_0\rangle|^2\\&\qquad\times\sum_{\mathbf{n}=-\infty}^{\infty}\int_{-T/2\alpha}^{T/2\alpha}d\xi\,d\zeta\frac{2\alpha^2\exp[-2i\alpha(E-E_0)\xi]}{4\alpha^2\sinh^2(\xi)-4\alpha F\sinh(\xi)\sinh(\zeta)-G^2-i\epsilon\xi}\;,\end{aligned} \tag{3.5}$$

where

$$\xi=(\tau-\tau')/2\alpha, \quad \zeta=(\tau+\tau')/2\alpha\ , \tag{3.6}$$

$$F(\mathbf{n},\mathbf{L},\theta,\phi)=n_1L_1\sin(\theta)\cos(\phi)+n_2L_2\sin(\theta)\sin(\phi)+n_3L_3\cos(\theta)\ ,$$
$$G(\mathbf{n},\mathbf{L})=(n_1^2L_1^2+n_2^2L_2^2+n_3^2L_3^2)^{1/2}=|\mathbf{L_n}|\ . \tag{3.7}$$

The integration with respect to the ξ can easily be performed. The integrand appearing in Eq. (3.5) has simple poles at

$$\xi=\xi_\pm-2ij\pi \text{ for } j=\text{integer and } \xi=-\xi_\pm-(2j-1)i\pi \text{ for } j=\text{integer}\ ,$$

where

$$\xi_\pm=\frac{F\sinh(\zeta)\pm[F^2\sinh^2(\zeta)+G^2]^{1/2}}{2\alpha} \tag{3.8}$$

with residues

$$\frac{\alpha\exp[-i(E-E_0)(\xi_\pm-2ij\pi)]}{2[2\alpha\sinh(\xi_\pm)-F\sinh(\zeta)]\cosh(\xi_\pm)} \text{ and } -\frac{\alpha\exp\{+i(E-E_0)[\xi_\pm+(2j-1)i\pi]\}}{2[2\alpha\sinh(\xi_\pm)-F\sinh(\zeta)]\cosh(\xi_\pm)}\ . \tag{3.9}$$

Using Eqs. (3.8) and (3.9) and symmetry of the integration, we can perform integration (3.5) and obtain, after some calculations,

$$\begin{aligned} R=&-\lim_{T\to\infty}T^{-1}g^2(4\pi^2)^{-1}\sum_E|\langle E|m(0)|E_0\rangle|^2\sum_{n=-\infty}^{\infty}\int_{-T/2\alpha}^{T/2\alpha}d\zeta \\ &\times\sum_{k=+,-}\sum_{j=1}^{\infty}(-2\pi i)\left[\frac{\alpha\exp[-i(E-E_0)(\xi_k-2ij\pi)]}{2[2\alpha\sinh(\xi_k)-F\sinh(\zeta)]\cosh(\xi_k)}-\frac{\alpha\exp\{+i(E-E_0)[\xi_k+(2j-1)i\pi]\}}{2[2\alpha\sinh(\xi_k)-F\sinh(\zeta)]\cosh(\xi_k)}\right] \\ =&\lim_{T\to\infty}T^{-1}g^2\sum_E|\langle E|m(0)|E_0\rangle|^2\frac{1}{4\pi\{\exp[2\alpha\pi(E-E_0)]-1\}}\sum_{n=-\infty}^{\infty}\int_{-T/2\alpha}^{T/2\alpha}\frac{2\alpha\sin[2\alpha(E-E_0)\xi_+]d\zeta}{2\alpha\sinh(\xi_+)-F\sinh(\zeta)\cosh(\xi_+)}\ . \end{aligned} \tag{3.10}$$

In order to evaluate the ζ integral in Eq. (3.10), we assume it has a Taylor series around $F=0$. From the symmetry of the integration, one can easily find that Eq. (3.10) can be expanded as a power series of F^2, and we have

$$\int_{-T/2\alpha}^{T/2\alpha}\frac{2\alpha\sin[2\alpha(E-E_0)\xi_+]d\zeta}{2\alpha\sinh(\xi_+)-F\sinh(\zeta)\cosh(\xi_+)}=I_0+I_2(F/2\alpha)^2+I_4(F/2\alpha)^4+O((F/2\alpha)^6)\ , \tag{3.11}$$

where

$$\begin{aligned} I_0=&\frac{\sin(\beta)}{2\alpha H(1+H^2)^{1/2}}\int_{-T/2\alpha}^{T/2\alpha}d\zeta\ , \\ I_2=&\frac{\cos(\beta)Hf(1-2H^2)(1+H^2)^{1/2}-\sin(\beta)(1+4H^2+H^2f^2+H^4f^2)}{2H(1+H^2)^{1/2}}\int_{-T/2\alpha}^{T/2\alpha}\sinh^2(\zeta)d\zeta\ , \\ I_4=&[\cos(\beta)Hf(4H^6f^2+16H^6-2H^4f^2+88H^4-6H^2f^2-42H^2-9)(1+H^2)^{1/2} \\ &+\sin(\beta)(H^8f^4+4H^8f^2+2H^6f^4+56H^6f^2+H^4f^4+55H^4f^2+144H^4+3H^2f^2+48H^2+9)] \\ &\times[24H^5(1+H^2)^{1/2}]^{-1}\int_{-T/2\alpha}^{T/2\alpha}\sinh^4(\zeta)d\zeta\ . \end{aligned} \tag{3.12}$$

Here, we set

$$\beta=2\alpha(E-E_0)\text{arcsinh}(H),\quad H=G/(2\alpha)\ ,$$

and

$$f=2\alpha(E-E_0)\ . \tag{3.13}$$

Note that Eq. (3.12) depends not only on the periodicity lengths L_i's but also on the direction of the acceleration. The counterpart of Eq. (3.11) in the case where space-time is $M^3\times S^1$ and the direction of the acceleration of the detector is perpendicular to the S^1 axis is obtainable as L_1 and L_2 become infinite and $\theta=\pi/2$, and we have

$$R = g^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 \times \sum_{n=-\infty}^{\infty} \frac{\sin\{2\alpha(E-E_0)\text{arcsinh}[nL/(2\alpha)]\}}{2\pi nL[1+n^2L^2/(4\alpha^2)]^{1/2}\{\exp[2\pi\alpha(E-E_0)]-1\}}, \tag{3.14}$$

which has the same expression as the one that appeared in Refs. 17 and 18. Note that Eq. (3.14) stands for $M^n \times S^1$ with $n \geq 2$ provided that the direction of the monopole detector is perpendicular to the S^1 axis.

IV. SUMMARY AND DISCUSSION

In the preceding sections we have seen that the transition rate Eq. (3.12) contains not only the size of space-time and the acceleration α of the detector but also the direction of the acceleration. Thus we can observe, in principle, the size of space-time, i.e., the periodicity length L_i's by the observation of transition rates by the monopole detectors with different accelerations and directions.

The effect on the response function due to the presence of the boundary condition is expected to be extremely small for enormous L_i's and/or enormous α, i.e., extremely small acceleration. Thus we cannot apply the above result directly to observe the size of the present-day Universe, because its periodicity length is estimated to be of the order of 600 Mpc (Refs. 1 and 2). However, there is a possibility that our Universe consists of the usual four-dimensional space-time attached by an extra compact space as in the case of Kaluza-Klein theory such as $M^4 \times S^1$, etc. If it is true, we expect that we can observe the trace of the presence of the external space, if we can accelerate the detector sufficiently, since the size of the external space is expected to be of the order of L_P, the Planck length.

It can be easily shown that the integrand appearing in Eq. (3.5) is invariant under time translation in the reference frame of the detector, i.e., $\tau \to \tau + \text{const}$, in the following cases: (1) detector's trajectory is in Minkowski space-time, i.e., L_i's$\to \infty$ limit; (2) detector's trajectory is in $M^3 \times S^1$ space-time with the direction of detector's acceleration being perpendicular to S^1 axis, i.e., L_1, $L_2 \to \infty$ and $\theta = \pi/2$. This means that the monopole detector is in equilibrium with the scalar field in the cases mentioned above. On the contrary, the integrand in Eq. (3.5) is not invariant under time translation in general. This causes the terms in Eq. (3.12) without a first term to depend on the time interval T and become infinite when the $T \to \infty$ limit is taken. It is a future problem to investigate why the detector is not in equilibrium with the scalar field in $M^3 \times S^1$ space-time in general.

[1]L.-Z. Fang, Y.-Q. Chu, Y. Liu, and Ch. Cao, Astron. Astrophys. **106**, 287 (1982).
[2]L.-Z. Fang and H. Sato, Commun. Theor. Phys. **2**, 1055 (1983); Acta Astron. Sin. **24**, 410 (1983); Gen. Relativ. Gravit. **17**, 1117 (1985).
[3]H. B. G. Casimir, Proc. Kon. Ned. Akad. Wet. **51**, 793 (1948).
[4]E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 94 (1956) [Sov. Phys. JETP **2**, 73 (1956)].
[5]R. Balian and B. Duplantier, Ann. Phys. (N.Y.) **104**, 300 (1977); **112**, 165 (1978).
[6]L. H. Ford, Phys. Rev. D **11**, 3370 (1975).
[7]B. S. Kay, Phys. Rev. D **20**, 3052 (1979).
[8]J. R. Ruggiero, A. Villani, and A. H. Zimerman, J. Phys. A **13**, 761 (1980).
[9]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[10]Y. Igarasi, Phys. Rev. D **30**, 1812 (1984).
[11]Y. Igarasi and T. Nonomiya, Phys. Lett. **161B**, 103 (1985); Prog. Theor. Phys. **77**, 427 (1987).
[12]O. Abe, Prog. Theor. Phys. **72**, 1225 (1984).
[13]P. C. W. Davies, in *Quantum Theory of Gravity,* edited by B. S. DeWitt (Adam Hilger, Bristol, 1984), p. 66.
[14]J. S. Dowker, in *Quantum Theory of Gravity* (Ref. 13), p. 103.
[15]L. H. Ford, Phys. Rev. D **38**, 528 (1988).
[16]P. C. Candelas and D. W. Sciama in *Quantum Theory of Gravity* (Ref. 13), p. 78.
[17]P. C. W. Davies, Z. X. Liu, and A. C. Ottewill, Class. Quantum Grav. **6**, 1041 (1989).
[18]O. Abe, Phys. Lett. B **229**, 202 (1989).