# Bonnor solution in five-dimensional gravity

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From Bonnor's solution of Einstein-Maxwell theory, a new solution to five-dimensional Kaluza-Klein equations which refers to a massive source carrying a magnetic and an electric dipole is constructed.

## I. INTRODUCTION

As is well known, in the 1920s, Kaluza and Klein[1,2] proposed a unified theory of gravitation and electromagnetism by assuming that the world has four spatial dimensions, one which is compactified to a circle.

Recently many-dimensional extensions of the Kaluza-Klein theory have been given in order to achieve a geometrical unification of all the fundamental interactions.[3] Also, as a matter of fact, searching for exact solutions to the five-dimensional Kaluza-Klein equations has received new impetus.

In this paper we shall use a theorem given by Matos[4] to obtain the corresponding Bonnor dipole solution for a Kaluza-Klein field. Matos's proof is highly nontrivial; we shall prove his theorem in a very simple form in Sec. II.

## II. POTENTIAL FORMALISM

We shall work with a variant of the five-dimensional theory in which it is assumed to admit one non-null Killing vector field $X^\mu$ and the five-dimensional metric $\gamma_{\mu\nu}$ reads

$$\gamma_{\mu\nu}=\begin{bmatrix} I^{-1}g_{ab}+I^2A_aA_b & I^2A_a \\ I^2A_b & I^2 \end{bmatrix}, \quad \mu,\nu=1,\ldots,5, \quad a,b=1,\ldots,4, \tag{1}$$

where $A_a$ is the electromagnetic four-potential, $g_{ab}$ is the space-time metric, and $I$ is the scalar field. Then vacuum field equations are characterized by

$$R^{(5)}_{\mu\nu}=0 \tag{2}$$

where $R^{(5)}_{\mu\nu}$ is the five-dimensional Ricci tensor.[5]

Stationary fields admit a second Killing vector field $Y^\mu$ with $Y^\mu Y_\mu<0$. In this case, one can define in a covariant manner five real potentials in the form[6]

$$\begin{aligned} &\mathcal{H}^{4/3}=I^2=X^\mu X_\mu, \quad f=-IY^\mu Y_\mu+I^{-1}(X^\mu Y_\mu)^2, \\ &\psi=-I^{-2}X_\mu Y^\mu, \quad \chi_{,\mu}=\epsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta X^{\gamma;\delta}, \\ &\epsilon_{,\mu}=\epsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta Y^{\gamma;\delta}, \end{aligned} \tag{3}$$

where $f,\psi,\chi,\epsilon$ are the gravitational, electrostatic, magnetostatic, and rotational potentials, respectively, and $\epsilon_{\alpha\beta\gamma\delta\mu}$ is the five-dimensional Levi-Civita pseudotensor.

The five-dimensional field equations (2), in function of the potentials (3), can be derived from the Lagrangian[6]

$$\begin{aligned} \mathcal{L}^{(5)}=&\frac{\rho}{2f^2}[f_{,a}f^{,a}+(\epsilon_{,a}+\psi\chi_{,a})(\epsilon^{,a}+\psi\chi^{,a})] \\ &+\frac{\rho}{2f}(\mathcal{H}\psi_{,a}\psi^{,a}+\mathcal{H}^{-2}\chi_{,a}\chi^{,a})+\frac{2}{3}\frac{\rho}{\mathcal{H}^2}\mathcal{H}_{,a}\mathcal{H}^{,a}, \end{aligned} \tag{4}$$

where

$$\det\gamma=-\rho^2=\det\begin{vmatrix}\gamma_{33}\cdots \\ \cdots\gamma_{55}\end{vmatrix}.$$

The potentials (3) are analogous to the Ernst potentials $(\xi,\Phi)$ and generate a five-dimensional Riemannian potential space (see Ref. 6).

The Ernst equations for stationary Einstein-Maxwell fields outside the sources can be deduced from a variational principle with the Lagrangian

$$\mathcal{L}=\frac{\rho}{2}F^{-2}(\xi_{,a}+2\Phi_{,a}\bar{\Phi})(\bar{\xi}^{,a}+2\Phi\bar{\Phi}^{,a})+2\rho F^{-1}\Phi_{,a}\bar{\Phi}^{,a} \tag{5}$$

(see Ref. 7). For electrostatic fields $\xi=\bar{\xi}$ and $\Phi=\bar{\Phi}$ (an overbar denotes complex conjugation) then the Lagrangian is reduced to

$$\mathcal{L}=\frac{\rho}{2}F^{-2}F_{,a}F^{,a}+2\rho F^{-1}\Phi_{,a}\Phi^{,a}. \tag{6}$$

Now we shall prove the following.[4]

*Theorem 1.* For each electrostatic (magnetostatic) field of the Einstein-Maxwell theory $(\xi,\Phi)$ (with $\xi=\bar{\xi}$ and $\Phi=\bar{\Phi}$) there is one corresponding field of the five-dimensional theory by the substitution

$$\begin{aligned} &\mathcal{H}=\mathcal{H}_0=\text{const}, \quad \psi=a\Phi, \quad \chi=b\Phi, \\ &\epsilon=-c\Phi^2, \quad f=\xi+\Phi^2, \end{aligned} \tag{7a}$$

where $a$, $b$, and $c$ are constants on the condition that

$$ab=2c \quad\text{and}\quad \mathcal{H}^2a^2+\mathcal{H}^{-2}b^2=4. \tag{7b}$$

*Proof.* It can easily be checked that the substitution (7a) reduces the Lagrangian (4) to the Lagrangian (6) if the condition (7b) holds.

In the magnetic case $\Phi$ is imaginary, but theorem 1 can be applied in virtue of the invariance of (5) by the transformation $\Phi \to i\Phi$ (Ref. 7).

### III. THE BONNOR SOLUTION

As a solution of the Einstein-Maxwell equations we take the Bonnor solution;[8] it reads

$$dS^2 = \frac{Y^2P^2}{\Delta^3}\left[\frac{dr^2}{Z}+d\theta^2\right] + Z\left[\frac{Y}{P}\right]^2 \sin^2\theta\, d\phi^2 - \left[\frac{P}{Y}\right]^2 dt^2 , \tag{8a}$$

where

$$\begin{aligned} &P = r^2 - 2mr + e^2\cos^2\theta ,\\ &\Delta = (r-m)^2 + \sigma^2\cos^2\theta ,\\ &Y = r^2 + e^2\cos^2\theta, \quad \sigma^2 = e^2 - m^2 ,\\ &Z = r^2 - 2mr + e^2 \end{aligned} \tag{8b}$$

in Boyer-Lindquist coordinates ($X^1 = r, X^2 = \theta, X^3 = \phi, X^4 = t$). The Ernst and the electromagnetic potentials for this solution are

$$\xi = \frac{P^2 - (2me\cos\theta)^2}{Y^2}, \quad \Phi = i\frac{2me\cos\theta}{Y^2} . \tag{9}$$

Using theorem 1, we can easily find the corresponding five-dimensional potentials, one arrives at

$$\begin{aligned} &f = \frac{P^2}{Y^2}, \quad \psi = -\frac{2ame\cos\theta}{Y}, \quad \chi = -\frac{2bme\cos\theta}{Y}\\ &\epsilon = -\frac{4cm^2e^2\cos^2\theta}{Y^2}, \quad I = \mathcal{H}_0^{2/3} . \end{aligned} \tag{10}$$

Elements of the matrix $\gamma$ are obtained using (3) and one finds

$$\begin{aligned} &g_{44} = -\frac{P^2}{Y^2}, \quad g_{34} = 0, \quad g_{33} = \frac{Y^2 Z\sin^2\theta}{P^2} ,\\ &A_3 = \frac{2bmer\sin^2\theta}{I^3P}, \quad A_4 = \frac{2ame\cos\theta}{Y} . \end{aligned} \tag{11}$$

The five-dimensional metric can be written as

$$\begin{aligned} dS^{(5)2} = e^{2k}dz\,d\bar{z} + \frac{Y^2Z\sin^2\theta}{IP^2}d\phi^2 - \frac{P^2}{IY^2}dt^2\\ + I^2(A_4dt + A_3d\phi + dy)^2 , \end{aligned} \tag{12}$$

where $z = \rho + i\zeta$, $\rho = \sqrt{r^2 - 2mr + e^2}\sin\theta$, $\zeta = (r-m)\cos\theta$, and $y$ is the fifth coordinate. In order to find the function $k$, one has to integrate the differential equation[9]

$$2k_{,z} = \frac{1}{(\ln\rho)_{,z}}[(\ln\rho)_{,zz} + \tfrac{1}{4}\mathrm{Tr}\, A^2] , \tag{13}$$

where the matrix $A = \gamma_{,z}\gamma^{-1}$. After the integration and the choice $a = b = \sqrt{2}$, $c = \mathcal{H}_0 = 1$, the five-dimensional metric reads

$$\begin{aligned} dS^{(5)2} = \frac{Y^2P^2}{\Delta^3}\left[\frac{dr^2}{Z}+d\theta^2\right]\\ + Z\left[\frac{Y}{P}\right]^2\sin^2\theta\, d\phi^2 - \left[\frac{P}{Y}\right]^2 dt^2\\ + \left[\frac{2\sqrt{2}mer\sin^2\theta}{P}d\phi\right.\\ \left. + \frac{2\sqrt{2}me\cos\theta}{Y}dt + dy\right]^2 . \end{aligned} \tag{14}$$

This solution has the appearance of the extreme Bonnor form in four-dimensional Einstein-Maxwell theory. Bonnor's solution refers to a massive source carrying a magnetic dipole or (after using a known theorem[10]) a mass distribution carrying an electric dipole. The five-dimensional solution (14) has both an electric and a magnetic dipole and their moments are all the same $2\sqrt{2}me$. The parameters $a$ and $b$ are arbitrary but are parameters of electrical charge and magnetic charge, respectively; unfortunately for the time being we do not have a clear meaning of $c$. The parameter $m$ is interpreted as the mass of the source. It may be easily seen that the metric (14) is asymptotically flat and for $m = 0$ and $e = 0$ the metric becomes flat.

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