Corrections to the thin-wall approximation in general relativity

David Garfinkle*
Department of Physics, University of Florida, Gainesville, Florida 32611

Ruth Gregory
NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Box 500, Batavia, Illinois 60510
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We consider the question of whether the thin-wall formalism of Israel applies to the gravitating domain walls of a $\lambda\phi^4$ theory. The coupled Einstein-scalar equations that describe the thick gravitating wall are expanded in powers of the thickness of the wall. The solutions of the zeroth-order equations reproduce the results of the usual Israel thin-wall approximation for domain walls. The solutions of the first-order equations provide corrections to the expressions for the stress energy of the wall and to the Israel thin-wall equations. The modified thin-wall equations are then used to treat the motion of spherical and planar domain walls.

I. INTRODUCTION

There has recently been renewed interest in the properties of domain walls in general relativity. Most of the work thus far has used the thin-wall approximation of Israel,[1] treating the walls as idealized zero-thickness objects. However, recent work of Raychaudhuri and Mukherjee[2] has cast some doubt on the applicability of the thin-wall approximation to domain walls. If the thin-wall approximation applies to domain walls, then thick domain walls should have a zero-thickness limit that reduces to the Israel thin-wall formalism. For instance, the consistency of the Israel formalism requires that the stress-energy tensor of a zero-thickness wall have no components that are normal to the wall; however, the work of Raychaudhuri and Mukherjee indicates that thick domain walls do have components that are orthogonal to the wall. Unless these components vanish in the zero-thickness limit, the thin-wall approximation is not applicable to domain walls. It has been shown by Geroch and Traschen[3] that a regular zero-thickness limit requires the vanishing of these orthogonal components; but it is not clear that the equations describing thick domain walls are compatible with a regular zero-thickness limit.

This potential conflict was resolved by Widrow in the case of domain walls with planar symmetry.[4] Widrow treats the Einstein-scalar equations for a gravitating domain wall assuming that the solution has planar symmetry, but places no restrictions on the thickness of the wall. He then takes the zero-thickness limit of his solution and shows that the orthogonal components of the stress-energy tensor become negligible in that limit and that the solution reduces to the Vilenkin[5]-Ipser-Sikivie[6] zero-thickness domain-wall solution.

In this paper we treat the gravitating domain walls of a $\lambda\phi^4$ theory, making no assumption of symmetry for the solutions. We treat only walls whose thickness is small compared to their radius of curvature. In such a way, we hope to address the problem of the zero-thickness limit for arbitrary domain walls. Section II introduces the notation and the equations for these walls. In Sec. III we expand the equations in powers of the thickness of the wall. We show that the zeroth-order equations reproduce all the results of the Israel thin-wall formalism for domain walls including the usual expression for the domain-wall stress-energy tensor. We then find the first-order corrections to the stress energy of the wall and to the wall's equation of motion. Section IV contains a treatment of spherical and planar walls using the corrected equations of motion. Our conclusions are presented in Sec. V.

II. EINSTEIN-SCALAR EQUATIONS

In an attempt to model the behavior of a thick domain wall we will consider the $\lambda\phi^4$ kink, i.e., walls made of a real scalar field ϕ with the Lagrangian

$$L=-\tfrac{1}{2}\nabla_a\phi\nabla^a\phi-\lambda(\phi^2-\eta^2)^2 , \tag{2.1}$$

where λ and η are constants (we use units where $G=c=1$). From the Lagrangian we have the equation of motion

$$\nabla_a\nabla^a\phi-4\lambda\phi(\phi^2-\eta^2)=0 \tag{2.2}$$

and the stress-energy tensor

$$T_{ab}=\nabla_a\phi\nabla_b\phi-\tfrac{1}{2}g_{ab}[\nabla_c\phi\nabla^c\phi+2\lambda(\phi^2-\eta^2)^2] . \tag{2.3}$$

A wall is characterized by the presence of two or more asymptotic regions in which ϕ attains separate vacuum expectation values, the boundary between such regions being the domain wall. For example, in flat space we could have $\phi\to\pm\eta$ as $z\to\pm\infty$ where z is the usual Cartesian coordinate. Defining the quantities ϵ and σ for future convenience by

$$\epsilon\equiv\frac{1}{\sqrt{2\lambda}\,\eta} , \tag{2.4}$$

$$\sigma\equiv\tfrac{4}{3}\sqrt{2\lambda}\,\eta^3 , \tag{2.5}$$

we may write one solution of Eq. (2.2) as[7]

$$\phi = \eta \tanh(z/\epsilon) \ . \tag{2.6}$$

This solution represents an infinite planar domain wall centered at $z=0$. From Eq. (2.3) it follows that the stress-energy tensor of this solution is

$$T_{ab} = -\frac{3}{4}\frac{\sigma}{\epsilon\cosh^4(z/\epsilon)}(g_{ab} - \nabla_a z \nabla_b z) \ . \tag{2.7}$$

This shows that σ represents the energy per unit area of the wall and that the stress-energy tensor has no components orthogonal to the $z=$const surfaces. We also see that the energy density is strongly peaked around $z=0$, and has an effective cutoff at $z=\epsilon$. We can thus think of ϵ as the effective thickness of the wall.

This is the solution for a flat domain wall in flat spacetime; we however are interested in curved gravitating walls. In order to investigate these, we will assume that the radii of curvature of the wall are much larger than its thickness. We may therefore expect that locally the wall will resemble the planar nongravitating wall. Let Σ be the surface on which $\phi=0$. Then we suspect that ϕ and the metric components are quickly varying in directions orthogonal to Σ and slowly varying in directions tangential to Σ. Thus we would like to split the field equations into their components orthogonal and tangential to Σ. We use the Gauss-Codazzi formalism to achieve this aim.

Let n^a be a unit geodesic vector field orthogonal to Σ. We wish to define an analog of the Cartesian coordinate z for the curved gravitating wall and do so as follows: let z be the length parameter along the integral curves of n^a. Each $z=$const surface then has unit normal n_a, intrinsic metric h_{ab} and extrinsic curvature K_{ab} given by

$$\nabla_a z = n_a \ , \tag{2.8}$$

$$h_{ab} = g_{ab} - n_a n_b \ , \tag{2.9}$$

$$K_{ab} = h_a{}^c \nabla_c n_b = \nabla_a n_b \ . \tag{2.10}$$

We may now write the coupled Einstein-scalar equations as differential equations for h_{ab}, K_{ab}, and ϕ. (This is essentially a spacelike analog of the usual "3+1" formalism for Einstein's equation.)

Define D_a and ${}^{(3)}R_{ab}$ to be, respectively, the derivative operator and Ricci tensor associated with h_{ab}. From Eqs. (2.9) and (2.10) it follows that

$$\mathcal{L}_n h_{ab} = 2K_{ab} \ , \tag{2.11}$$

where $\mathcal{L}_n$ is the Lie derivative with respect to the vector field n^a.

Now, the Gauss-Codazzi equations for the wall imply

$${}^{(3)}R_{ac} = h^b{}_a h^d{}_c R_{bd} - R_{abcd} n^b n^d + KK_{ac} - K_{ab}K^b{}_c \ .$$

But

$$\begin{aligned} R_{abcd} n^b n^d &= n^d(\nabla_c \nabla_d - \nabla_d \nabla_c) n_a \\ &= n^d \nabla_c K_{da} - \nabla_n K_{ac} \\ &= K_{da} K^d{}_c - \mathcal{L}_n K_{ac} \ ; \end{aligned} \tag{2.12}$$

hence,

$${}^{(3)}R_{ac} = h^b{}_a h^d{}_c R_{bd} + \mathcal{L}_n K_{ac} + KK_{ac} - 2K_{ab}K^b{}_c \ . \tag{2.13}$$

From (2.12) we may also deduce

$$R_{ac} n^a n^c = -\mathcal{L}_n K - K_{ac}K^{ac} \tag{2.14}$$

and similarly that

$$\begin{aligned} R_{ab} n^a h^b{}_e &= R_{cadb} n^a h^b{}_e g^{cd} \\ &= (\nabla_d K_{bc} - \nabla_b K_{cd}) h^b{}_e g^{cd} \\ &= D_c K^c{}_e - D_e K \ . \end{aligned} \tag{2.15}$$

Finally, from (2.13) and (2.14) we may deduce that

$${}^{(3)}R = R + K_{ac}K^{ac} + K^2 + 2\mathcal{L}_n K \ . \tag{2.16}$$

Having now split the curvature tensor into its perpendicular and parallel components, we may use Einstein's equation and expression (2.3) for the stress-energy tensor to find

$$\begin{aligned} \frac{\partial K_{ab}}{\partial z} = {}^{(3)}R_{ab} + 2K_{ac}K^c{}_b - KK_{ab} - 8\pi D_a\phi D_b\phi \\ - 8\pi\lambda(\phi^2 - \eta^2)^2 h_{ab} \ . \end{aligned} \tag{2.17}$$

Equation (2.2) becomes

$$\frac{\partial^2 \phi}{\partial z^2} + K\frac{\partial \phi}{\partial z} + D_a D^a \phi - 4\lambda\phi(\phi^2 - \eta^2) = 0 \ . \tag{2.18}$$

Equations (2.11), (2.17), and (2.18) are equivalent to the Einstein-scalar equations.

Define the variable u and the field X by

$$u \equiv z/\epsilon \ , \tag{2.19}$$

$$X \equiv \phi/\eta \ . \tag{2.20}$$

Note that for the flat nongravitating wall, ϕ/η is a smooth function of z/ϵ even in the $\epsilon \to 0$ limit. Since the curved wall should locally resemble the flat wall, we expect that X is a smooth function of u even as $\epsilon \to 0$. Define the *zero-thickness limit* of the wall to be the limit $\epsilon \to 0$ with σ remaining fixed. We assume that X, h_{ab}, and K_{ab} are smooth functions of u even in the zero-thickness limit. Writing Eq. (2.11), (2.17), and (2.18) in terms of ϵ and quantities that are smooth in the zero-thickness limit we obtain

$$\frac{\partial h_{ab}}{\partial u} = 2\epsilon K_{ab} \ , \tag{2.21}$$

$$\begin{aligned} \frac{\partial K_{ab}}{\partial u} = &-3\pi\sigma(X^2-1)^2 h_{ab} \\ &+ \epsilon({}^{(3)}R_{ab} + 2K_{ac}K^c{}_b - KK_{ab}) \\ &- 6\pi\epsilon^2\sigma D_a X D_b X \ , \end{aligned} \tag{2.22}$$

$$\frac{\partial^2 X}{\partial u^2} - 2X(X^2-1) + \epsilon K\frac{\partial X}{\partial u} + \epsilon^2 D_a D^a X = 0 \ . \tag{2.23}$$

For any quantity S let S_0 denote $S|_{\epsilon=0}$ and let $\dot{S}$ denote $\partial S/\partial\epsilon|_{\epsilon=0}$. Then for small ϵ, S is well approximated by $S_0 + \epsilon\dot{S}$. By evaluating Eq. (2.21)-(2.23) and their derivatives at $\epsilon=0$ we hope to obtain approximate equations for h_{ab}, K_{ab}, and X.

III. ZEROTH- AND FIRST-ORDER EQUATIONS

We begin by evaluating Eq. (2.21)–(2.23) at $\epsilon=0$:

$$\frac{\partial h_{0ab}}{\partial u}=0 , \tag{3.1}$$

$$\frac{\partial K_{0ab}}{\partial u}=-3\pi\sigma(X_0^2-1)^2 h_{0ab} , \tag{3.2}$$

$$\frac{\partial^2 X_0}{\partial u^2}-2X_0(X_0^2-1)=0 . \tag{3.3}$$

Equation (3.1) has solution

$$h_{0ab}(u)=h_{0ab}(0) . \tag{3.4}$$

Thus, to zeroth order the intrinsic metric is constant across the wall. This is what we would expect, since in the zero-thickness limit, the metric is continuous across the wall.

Equation (3.3) is solved by

$$X_0=\tanh u . \tag{3.5}$$

This solution has the same form as the flat wall solution in Eq. (2.6). However, the interpretation is somewhat different. Though X_0 is a function of u alone, this does not imply any symmetry of the wall. On the contrary, since the $u=0$ surface in general has no symmetries, a field configuration that depends only on u has, in general, no symmetries. Instead Eq. (3.5) simply expresses the fact that the curved gravitating wall locally resembles the flat nongravitating wall.

Using Eq. (3.4) and (3.5) in Eq. (3.2) we find

$$K_{0ab}=K_{0ab}(0)-\pi\sigma h_{0ab}(0)\frac{\sinh u(2\cosh^2 u+1)}{\cosh^3 u} . \tag{3.6}$$

In the zero-thickness limit, all points at finite values of u are equivalent to $z=0$. We therefore need to be more careful in interpreting this solution. In the Israel formalism the quantity $[K_{ab}]$ was defined by

$$[K_{ab}]=K_{ab}|_{0^+}-K_{ab}|_{0^-}=\lim_{\zeta\to 0}(K_{ab}|_{\zeta}-K_{ab}|_{-\zeta}) ,$$

the difference in extrinsic curvature across the wall. Since $u=z/\epsilon$, clearly in the limit $\epsilon\to 0$, the correct quantity to consider is

$$[K_{ab}]=\lim_{u\to\infty}[K_{0ab}(u)-K_{0ab}(-u)]$$
$$=-4\pi\sigma h_{0ab}(0) . \tag{3.7}$$

This is just Israel's equation for a zero-thickness domain wall. Thus the solutions of our zeroth-order equations reproduce the usual results of the Israel formalism for domain walls.

Now let us consider the energy-momentum tensor. Though T_{ab} becomes singular as $\epsilon\to 0$, the quantity ϵT_{ab} is smooth. Define the quantities S and Q_{ab} by

$$S\equiv\tfrac{1}{3}\epsilon h^{ab}T_{ab} , \tag{3.8}$$

$$Q_{ab}\equiv\epsilon T_{ab}-Sh_{ab} . \tag{3.9}$$

Thus S is the part of ϵT_{ab} proportional to h_{ab} and Q_{ab} is the remainder. Using Eq. (2.3), we see that in terms of the wall fields

$$S=-\tfrac{3}{8}\sigma\left[\left(\frac{dX}{du}\right)^2+(X^2-1)^2\right]$$
$$-\tfrac{1}{8}\epsilon^2\sigma D_a X\, D^a X , \tag{3.10}$$

$$Q_{ab}=\tfrac{3}{8}\sigma n_a n_b\left[\left(\frac{dX}{du}\right)^2-(X^2-1)^2\right]$$
$$+\tfrac{3}{4}\epsilon^2\sigma\left[D_a X\, D_b X-\tfrac{1}{6}D_c X\, D^c X(2h_{ab}+3n_a n_b)\right.$$
$$\left.+\frac{dX}{du}(n_a D_b X+n_b D_a X)\right] . \tag{3.11}$$

Evaluating these quantities at $\epsilon=0$, using Eq. (3.5) we find

$$S_0=-\frac{3}{4}\frac{\sigma}{\cosh^4 u} , \tag{3.12}$$

$$Q_{0ab}=0 . \tag{3.13}$$

It then follows that as a distribution T_{ab} is (to lowest order in ϵ) equal to $-\sigma h_{ab}\delta(z)$. This is just the usual thin-wall expression for the stress-energy tensor of a domain wall. Therefore the results of our zeroth-order equations reproduce all of the usual results of the thin-wall formalism applied to domain walls.

We now consider the first-order corrections to this thin-wall limit. Taking the derivative of Eq. (2.21)–(2.23) with respect to ϵ at $\epsilon=0$ gives

$$\frac{\partial \dot h_{ab}}{\partial u}=2K_{0ab} , \tag{3.14}$$

$$\frac{\partial \dot K_{ab}}{\partial u}=-3\pi\sigma(X_0^2-1)^2\dot h_{ab}-12\pi\sigma X_0(X_0^2-1)\dot X h_{0ab}$$
$$+{}^{(3)}R_{0ab}+2K_{0ac}K^c_{0b}-K_0K_{0ab} , \tag{3.15}$$

$$\frac{\partial^2 \dot X}{\partial u^2}-\dot X(3X_0^2-1)+K_0\frac{\partial X_0}{\partial u}=0 . \tag{3.16}$$

Using Eq. (3.5) we find that Eq. (3.16) can be written as

$$\frac{\partial}{\partial u}\left[\cosh^{-4}u\frac{\partial}{\partial u}(\cosh^2 u\dot X)\right]=-\frac{K_0}{\cosh^4 u} . \tag{3.17}$$

In order for $\dot X$ to have the appropriate behavior for large $|u|$, it follows from Eq. (3.17) that $\int_{-\infty}^{\infty}K_0\cosh^{-4}u\, du=0$. Using the expression for K_{0ab} in Eq. (3.6) we find that $K_0(0)=0$. [The condition that $K_0(0)=0$ can also be derived from Eq. (3.7).] Then integration of Eq. (3.17) using Eq. (3.6) yields

$$\dot X=-\frac{\pi}{2}\sigma\frac{3u+\tanh u}{\cosh^2 u} . \tag{3.18}$$

Thus to first order, the solution for the scalar field is given by

$$X=\tanh u-\frac{\epsilon\pi\sigma}{2}\frac{3u+\tanh u}{\cosh^2 u} .$$

The zeroth-order solution is only slightly modified and the expansion remains consistent. It is interesting to note the corrections to the stress-energy tensor. From Eqs. (3.10), (3.11), and (3.18) it follows that

$$\dot{S}=-\frac{3\pi}{8}\sigma^2\frac{5\tanh^2 u+12u\tanh u-4}{\cosh^4 u} , \qquad (3.19)$$

$$\dot{Q}_{ab}=-\frac{3\pi}{8}\sigma^2\frac{3\cosh^2 u+1}{\cosh^6 u}n_a n_b , \qquad (3.20)$$

and hence

$$\epsilon T_{ab}=-\tfrac{3}{4}\sigma\,\mathrm{sech}^4 u\,h_{ab}\left[1+\frac{\pi\sigma\epsilon}{2}(5\tanh^2 u+12u\tanh u-4)\right]-\frac{3\pi\epsilon\sigma^2}{8}\frac{3\cosh^2 u+1}{\cosh^6 u}n_a n_b .$$

Thus although the stress-energy tensor is predominantly proportional to the induced metric of the wall, it does acquire a piece orthogonal to the wall which vanishes smoothly in the zero-thickness limit. This shows that, as far as the matter fields are concerned, the zero-thickness limit is a physically consistent approximation.

We find $\dot{h}_{ab}$ by integrating Eq. (3.14) using Eq. (3.6):

$$\dot{h}_{ab}=\dot{h}_{ab}(0)+2uK_{0ab}(0)-\pi\sigma h_{0ab}(0)(4\ln\cosh u+\tanh^2 u) . \qquad (3.21)$$

When our expansion scheme is valid, h_{ab} is well approximated by $h_{0ab}+\epsilon\dot{h}_{ab}$, i.e.,

$$h_{ab}=h_{0ab}(0)\left[1-\epsilon\pi\sigma\left[4\ln\cosh\frac{z}{\epsilon}+\tanh^2\frac{z}{\epsilon}\right]\right]+\epsilon\dot{h}_{ab}(0)+2zK_{0ab}(0) . \qquad (3.22)$$

We expect the approximation to be good only for those values of z where the first-order corrections are much less than the zeroth-order values. Thus this equation tells us that for the validity of the approximation, we require that $z \ll (\sigma)^{-1}$ and $z \ll L$ where L is the length scale associated with K_{ab}. This latter cutoff we expect anyway, since it merely expresses the constraint on our coordinate system being well defined. The former provides a cutoff associated with the mass of the domain wall, and is indicative of the horizon structure associated with a domain wall. In analogy with $[K_{ab}]$ in the Israel formalism, we will need to define $[K_{ab}]$ and $[h_{ab}]$ for the thick-wall formalism. Since $[K_{ab}]$ was defined in terms of a limiting process as one approached each side of the wall, a natural way to generalize this for thick walls would seem to be

$$[K_{ab}]=K_{ab}|_{z=\epsilon\Delta}-K_{ab}|_{z=-\epsilon\Delta} ,$$

where $\epsilon\Delta$ is chosen to be sufficiently large so that the matter fields take their vacuum values, but smaller than other relevant length scales. We define $[h_{ab}]$ similarly. It then follows from Eq. (3.22) that

$$[h_{ab}]=4\epsilon\Delta K_{0ab}(0) . \qquad (3.23)$$

We now evaluate $[K_{ab}]$. Since K_{ab} is well approximated by $K_{0ab}+\epsilon\dot{K}_{ab}$ we find that $[K_{ab}]$ is well approximated by

$$[K_{ab}]=K_{0ab}(\Delta)-K_{0ab}(-\Delta)+\epsilon\int_{-\Delta}^{\Delta}\frac{\partial\dot{K}_{ab}}{\partial u}du . \qquad (3.24)$$

Then using Eqs. (3.6), (3.15), (3.18), and (3.21) and the fact that $\Delta \gg 1$, we find that $[K_{ab}]$ is well approximated by

$$[K_{ab}]=-4\pi\sigma[h_{0ab}(0)+\epsilon\dot{h}_{ab}(0)]+2\epsilon\pi^2(\tfrac{29}{5}-8\ln 2)\sigma^2 h_{0ab}(0)+2\epsilon\Delta[{}^{(3)}R_{0ab}(0)+2K_{0ac}(0)K^{c}_{0b}(0)-4\pi^2\sigma^2 h_{0ab}(0)] . \qquad (3.25)$$

Now define the quantity $\tilde{K}_{ab}$ to be $\frac{1}{2}(K_{ab}|_{z=\epsilon\Delta}+K_{ab}|_{z=-\epsilon\Delta})$ and define $\tilde{h}_{ab}$ analogously. Then, to zeroth order in ϵ,

$$\tilde{K}_{ab}=K_{0ab}(0) , \qquad (3.26)$$

and, to first order in ϵ,

$$\tilde{h}_{ab}=h_{0ab}(0)+\epsilon\dot{h}_{ab}(0) . \qquad (3.27)$$

Then evaluating Eqs. (3.23) and (3.25) to first order in ϵ we find

$$[h_{ab}]=4\epsilon\Delta\tilde{K}_{ab} , \qquad (3.28)$$

$$[K_{ab}]=-4\pi\sigma\tilde{h}_{ab}[1+\pi\epsilon\sigma(4\ln 2-\tfrac{29}{10}+2\Delta)]+2\epsilon\Delta(\tilde{R}_{ab}+2\tilde{K}_{ac}\tilde{K}^{c}_{\ b}) . \qquad (3.29)$$

Here $\tilde{R}_{ab}$ is the Ricci tensor associated with $\tilde{h}_{ab}$. Equations (3.28) and (3.29) are the modified version of the Israel equations for thin domain walls with corrections due to the thickness of the wall.

IV. WALL MOTION

We now apply the modified Israel equations to the motion spherical and planar walls. We first find two vacuum spacetimes with boundary, representing, respectively, the $z \geq \epsilon\Delta$ and the $z \leq -\epsilon\Delta$ portions of the wall spacetime. We then impose Eqs. (3.28) and (3.29) and thus find the motion of the wall.

Equation (3.28) is somewhat more difficult to apply than the corresponding Israel condition $[h_{ab}]=0$. This is because when the Israel condition is applied both surfaces often have the same metric and this yields a natural way to identify them, whereas in our case we are comparing the surfaces $z=\pm\epsilon\Delta$, which are at a finite proper distance apart, thus the identification of corresponding points in the two surfaces will not in general be so obvious. In the case of reflection symmetry (such as the conventional planar wall) this difficulty is circumvented, since $[h_{ab}]=\tilde{K}_{ab}=0$, and Eq. (3.29) reduces to

$$K_{ab} = -2\pi\sigma h_{ab}[1+\pi\epsilon\sigma(4\ln 2 - \tfrac{29}{10} + 2\Delta)] + \epsilon\Delta R_{ab} \ , \tag{4.1}$$

on the $z=\epsilon\Delta$ surface. However, even here we are left with the problem of finding K_{ab} at $z=\epsilon\Delta$ rather than $z=0$. We therefore need to find a way of rewriting (3.28) and (3.29) in terms of quantities evaluated at $z=0$; in fact, this turns out to be quite straightforward.

To model the thick domain wall, we first glue together two vacuum spacetimes along a surface Σ in such a way that Σ has induced metric H_{ab} and extrinsic curvatures k_{ab}^{+} on one side and k_{ab}^{-} on the other side. We then realize the $z=\pm\epsilon\Delta$ surfaces of the wall as the surfaces at geodesic distances $\pm\epsilon\Delta$ from the surface σ in that spacetime just introduced. Since Eq. (3.28) and (3.29) are only correct to first order in ϵ we need only calculate quantities to first order in ϵ, and to find these, we simply expand in a Taylor series around $z=0$. Equations (3.28) and (3.29) then become conditions on H_{ab}, k_{ab}^{+}, and k_{ab}^{-}.

Using this procedure for the induced metric gives, via Eq. (2.11),

$$h_{ab}|_{z=\pm\epsilon\Delta} = H_{ab} \pm 2\epsilon\Delta k_{ab}^{\pm} \ . \tag{4.2}$$

For the extrinsic curvature, we note that in a vacuum spacetime K_{ab} satisfies the relation

$$\mathcal{L}_n K_{ab} = {}^{(3)}R_{ab} + 2K_{ac}K^{c}{}_{b} - KK_{ab} \ , \tag{4.3}$$

where ${}^{(3)}R_{ab}$ is the Ricci tensor of the intrinsic metric of the surface. In our particular coordinate system $\mathcal{L}_n = \partial_n$, and therefore we have

$$K_{ab}|_{z=\pm\epsilon\Delta} = k_{ab}^{\pm} \pm \epsilon\Delta(R_{ab} + 2k_{ac}^{\pm}k^{\pm c}{}_{b} - k^{\pm}k_{ab}^{\pm}) \ . \tag{4.4}$$

From Eqs. (4.2) and (4.4) we can now see that Eq. (3.28) is automatically satisfied (to first order in ϵ).

Now define the quantity $[k_{ab}]$ by

$$[k_{ab}] \equiv k_{ab}^{+} - k_{ab}^{-} \ . \tag{4.5}$$

Then using Eqs. (4.2) and (4.4) in Eq. (3.29) and keeping only terms to order ϵ we obtain

$$[k_{ab}] = -4\pi\sigma H_{ab}[1+\pi\epsilon\sigma(4\ln 2 - \tfrac{29}{10} + 2\Delta)] + 2\epsilon\Delta(\tilde{K}\tilde{K}_{ab} + \tfrac{1}{4}[k][k_{ab}] - \tfrac{1}{2}[k_{ac}][k^{c}{}_{b}]) \ . \tag{4.6}$$

Recall in Sec. III we showed that $\tilde{K}=0$. Using this result, Eq. (4.6) reduces to an equation on $z=0$. We can now see that $[k_{ab}] = -4\pi\sigma H_{ab}$ to zeroth order in ϵ, which is just Israel's equation for the wall surface, as would be expected. Substituting back in Eq. (4.6) we obtain the first-order correction

$$[k_{ab}] = -4\pi\sigma H_{ab}[1+\pi\epsilon\sigma(4\ln 2 - \tfrac{29}{10})] \ . \tag{4.7}$$

Defining the quantity $\tilde{\sigma}$ by

$$\tilde{\sigma} = \sigma[1+\pi\epsilon\sigma(4\ln 2 - \tfrac{29}{10})] \ , \tag{4.8}$$

we see that Eq. (4.7) is identical to the usual Israel equation with the quantity $\tilde{\sigma}$ playing the role of σ. Thus the finite-width corrections simply produce corrections to σ, and leave the form of the equations of motion unchanged. Since $4\ln 2 < \frac{29}{10}$, it follows that $\tilde{\sigma} < \sigma$. Thus the effect of finite thickness is to substitute a lower effective surface energy in the Israel equations. We can now apply these results to the cases of spherical and planar domain walls.

In the case of the reflection and plane-symmetric domain wall (the case treated by Ipser and Sikivie and by Widrow), $k_{ab}^{+} = -k_{ab}^{-}$, and hence Eq. (4.7) reduces to

$$k_{ab}^{+} = -2\pi\tilde{\sigma}H_{ab} \ . \tag{4.9}$$

Thus we find a plane-symmetric surface satisfying Eq. (4.9) in a plane-symmetric vacuum spacetime. There are only two different plane-symmetric vacuum spacetimes,[8] (called class I and class II by Ipser and Sikivie). The class II spacetime is singular and we will not consider it here. The class I spacetime has a metric of the form

$$ds^2 = 2\,dv\,dr + r^2(dx^2+dy^2) \ , \tag{4.10}$$

which is simply the metric of Minkowski spacetime written in an unusual set of coordinates. A plane-symmetric surface in this spacetime is given by a relation of the form $r=R(\tau)$ where τ is the proper time measured by an observer in the surface whose four-velocity is orthogonal to the planes of symmetry. Let u^a be the four-velocity of this observer and define the tensor q_{ab} by

$$q_{ab} \equiv r^2(\nabla_a x \nabla_b x + \nabla_a y \nabla_b y) \ . \tag{4.11}$$

Then the intrinsic metric of the surface is given by

$$h_{ab} = -u_a u_b + q_{ab} \ . \tag{4.12}$$

Using Eq. (4.10) a straightforward calculation shows that the extrinsic curvature of the surface is

$$K_{ab} = s\left[\frac{\ddot{R}}{\dot{R}}u_a u_b - \frac{\dot{R}}{R}q_{ab}\right] \ . \tag{4.13}$$

Here an overdot denotes a derivative with respect to τ and $s=\pm 1$ with the value of s depending on the direction of the normal to the surface (without loss of generality $s=1$).

It now follows that the tensor equation (4.9) reduces to the following pair of ordinary differential equations: equations:

$$\frac{\ddot{R}}{\dot{R}} = 2\pi\tilde{\sigma} \ , \tag{4.14a}$$

$$-\frac{\dot{R}}{R} = -2\pi\tilde{\sigma} \ . \tag{4.14b}$$

Clearly the solution to these equations is

$$R = ce^{2\pi\tilde{\sigma}\tau} \ , \tag{4.15}$$

where c is some constant.

The Ipser-Sikivie wall solution has the same form as Eq. (4.15) but with σ instead of $\tilde{\sigma}$. Thus our solution simply substitutes an "effective energy per unit area" with finite-thickness corrections for the quantity σ; otherwise our solution is identical to the Ipser-Sikivie solution. The Widrow solution in the zero-thickness limit reduces to the Ipser-Sikivie solution. Our solution can be regarded as the Widrow solution expanded to first order in thickness.

We now apply the equations of motion to the collapse of a spherical wall in a vacuum. In this case, by Birkhoff's theorem, we know that the spacetime exterior to the wall must be Schwarzschild spacetime, and we will take the interior to be flat. The motion of a spherical wall is given by a relation of the form $r=R(\tau)$ where r is the "radius" of the wall and τ is the proper time of a radially moving observer in the wall. Using the Ipser-Sikivie equation of motion for the wall[6] and substituting $\tilde{\sigma}$ for σ we find that the equation of motion for the wall is

$$(1+\dot{R}^2)^{1/2}-\left[1+\dot{R}^2-\frac{2M}{R}\right]^{1/2}=4\pi\tilde{\sigma}R\ . \qquad (4.16)$$

Let R_m be the maximum radius of the wall. Then it follows from Eq. (4.16) that

$$M=4\pi\tilde{\sigma}R_m{}^2(1-2\pi\tilde{\sigma}R_m)\ . \qquad (4.17)$$

Using Eq. (4.17) in Eq. (4.16) we find

$$\dot{R}^2=\left[\frac{R_m}{R}\right]^3\left[1-\frac{2\pi\tilde{\sigma}}{R_m}(R_m{}^3-R^3)\right]^2-1\ . \qquad (4.18)$$

Thus for a given R and R_m smaller $\tilde{\sigma}$ leads to a larger $\dot{R}^2$. Therefore the finite-thickness corrections lead to a slightly faster collapse of the wall.

V. CONCLUSIONS

In this paper we have developed an expansion procedure for the equations describing the motion of a domain wall. Our formalism has a well defined zero-thickness limit which is just the Israel formalism. Perhaps most importantly, we have shown that for a general domain wall, the stress-energy tensor tends uniformly to the standard distributional form in the zero-thickness limit.

We applied our equations of motion to the case of domain walls in vacuo, and showed that the effect of finite (as opposed to zero) thickness was to substitute a smaller effective surface energy in the Israel equations. Thus finite-thickness vacuum domain walls have exactly the same qualitative behavior as their zero-thickness cousins. In particular, the general result that domain walls are gravitationally repulsive still holds, although the repulsive force is slightly weaker than their zero-thickness counterparts.

Whilst it is generally thought that grand-unified-theory scale walls are cosmologically disastrous, it has recently been suggested[9] that very thick domain walls could have played a part in the formation of structure in the Universe. It is therefore important to obtain a good understanding of the gravitational properties of such walls. It is therefore encouraging that many of the properties of zero-thickness walls persist in the case of thick walls.

We applied our results to the cases of planar and spherical domain walls to obtain the first-order corrections to the metric. For the planar wall, our metric was clearly consistent with the zero thickness results of Ipser and Sikivie[6] and Vilenkin,[5] and the thick-wall treatment of Widrow. We then turned to the cosmologically more interesting case of a spherical collapsing domain wall. Here the smaller effective energy per unit area produces qualitatively the same behavior as with the zero-thickness wall, but the result of having a "lighter" wall is to speed up collapse. This latter result is suggestive of the rigidifying effect of gravity on domain walls. The zero-thickness limit corresponds to scaling out all other physics except gravity. Thus, we would expect the gravitational effects on the dynamics of the wall to be the strongest in the zero-thickness limit. Since this limit produces the slowest collapse rate for a spherical wall, the effect of gravity is to resist regimes of large curvature. We can therefore think of gravity as rigidifying domain walls. The effect of gravity and wall thickness upon the dynamics of domain walls is currently under investigation.

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*Present address: Department of Physics, University of California, Santa Barbara, CA 93106.

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