Hamiltonian theory of self-gravitating perfect fluid and a method of effective deparametrization of Einstein's theory of gravitation

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The Hamiltonian formulation of the theory of a gravitational field interacting with a perfect fluid is considered. There is a natural gauge related to the mechanical and thermodynamical properties of the fluid, which enables us to describe 2 degrees of freedom of the gravitational field and 4 degrees of freedom of the fluid (together with 6 conjugate momenta) by nonconstrained data (g,P) where g is a 3-dimensional metric and P is the corresponding Arnowitt-Deser-Misner momentum. The Hamiltonian of the theory, numerically equal to the entropy of the fluid, generates uniquely the evolution of the data. The Hamiltonian vanishes on the data satisfying the vacuum constraint equations and tends to infinity elsewhere as the amount of the matter tends to zero. In this way the vacuum theory with constraints is obtained as a limiting case of a "deep potential well" theory.

I. INTRODUCTION

The Hamiltonian formulation of general relativity[1,2] may be sketched as follows. Let Σ be a 3-dimensional initial-value surface. Cauchy data for the gravitational field are described by the Riemannian metric g_{kl} on Σ (latin indices run from 1 to 3) and by the so-called Arnowitt-Deser-Misner (ADM) momentum P^{kl}, where

$$P^{kl}=\sqrt{\det g}\,(Kg^{kl}-K^{kl}) . \tag{1}$$

K_{kl} is the second fundamental form of Σ (extrinsic curvature) and $K=K_{kl}g^{kl}$. Four of ten Einstein equations do not contain time derivatives of p^{kl} and g_{kl}:

$$R(g)-\frac{1}{\det g}(P^{kl}P_{kl}-\tfrac{1}{2}P^2)=0 \tag{2}$$

(P denotes again the trace of the tensor P^{kl}) and

$$P^{l}_{k|l}=0 , \tag{3}$$

where a vertical bar denotes the covariant derivative with respect to the metric connection generated by g. They can be treated as Hamiltonian constraints for the generalized Hamiltonian system (P^{kl},g_{kl}). There are 4 (per point) "Lagrange multipliers" N and N^k, canonically conjugate to the four constraints. As for the usual constrained Hamiltonian systems, the dynamics of the data (P^{kl},g_{kl}) resulting from the remaining six Einstein equations

$$G_{kl}=\kappa T_{kl} \tag{4}$$

(by $\kappa=8\pi G$ we denote the gravitational constant) is not uniquely defined unless the parameters (N,N^k) are fixed at each point of Σ and at each instant of time $t=x^0$. Finally, the parameters can be interpreted as the lapse function

$$N=\frac{1}{\sqrt{|g^{00}|}} \tag{5}$$

and the shift vector

$$N_k=g_{kl}N^l=g_{k0} . \tag{6}$$

The freedom in the choice of lapse and shift corresponds to the gauge freedom of the theory with respect to the group of space-time diffeomorphisms.

In the theory of constrained Hamiltonian systems it is usually possible to replace constraints by a "deep potential well" without changing essentially the behavior of the system (one could even argue that all the constrained systems known in classical physics are idealized situations with a deep potential well[3]). The goal of this paper is to show that general relativity coupled to a perfect fluid is indeed a theory with constraints (2) and (3) replaced by a potential well. We show that the phase space of both gravitational and thermomechanical degrees of freedom (2+4 per point) can be described by the same mutually conjugate objects (P^{kl},g_{kl}) as in the vacuum case. The zero on the right-hand side of Eqs. (2) and (3) is replaced by the corresponding components of the matter energy-momentum tensor. However, the equations can no longer be considered as constraints. They enable us to calculate uniquely the lapse and the shift in terms of the data (P^{kl},g_{kl}). Finally, the time evolution of the system is uniquely generated by a regular, nonconstrained Hamiltonian $H=H(P^{kl},g_{kl})$. Therefore, our formulation is well adapted to numerical simulations.

The Hamiltonian vanishes on constraints (2) and (3) and—as the amount of the matter tends to zero—tends to infinity outside the constraints. This shows that our description of the dynamics may also be used for numerical analysis of the vacuum equations, since the constraints can be considered as an "infinitely deep potential well." Replacing it by a deep, but finite well does not change substantially the dynamics.

The specific form of the Hamiltonian function depends on the specific Gibbs functional equation of the liquid:

$$e=e(V,S) \tag{7}$$

(internal energy per mole expressed as a function of the molar volume V and the molar entropy S). The function (7) uniquely determines the properties of the fluid, according to the Gibbs relation[4]

$$de=-p\,dV+T\,dS\ , \tag{8}$$

where p is the pressure and T is the absolute temperature. In our paper we also need another description of the fluid, given by the so-called "entropy picture." To get it we divide Eq. (8) by T:

$$dS=\frac{1}{T}de+\frac{p}{T}dV\ . \tag{9}$$

Here, the entropy S has to be expressed in terms of the molar internal energy and the molar volume,

$$S=S(e,V)\ , \tag{10}$$

and Eq. (9) gives the values of remaining two parameters T and p. Given a function (10) we construct a nonconstrained Hamiltonian $H=H(P^{kl},g_{kl})$ such that the evolution equations derived from H are precisely the Einstein-Euler equations for the self-gravitating perfect fluid whose mechanical and thermodynamical properties are described by (10).

II. RELATIVISTIC HYDRODYNAMICS AS A LAGRANGIAN FIELD THEORY

In the present section we show how to derive the relativistic Euler equations for a barotropic perfect fluid from the first-order variational formula.[5] This formulation of the hydrodynamics is essential for our purpose. We assume therefore that the energy is a function of the specific volume V only and formula (8) reduces to $de=-p\,dV$. In the present section the space-time metric g is given *a priori*.

Consider an abstract 3-dimensional "matter space" Z equipped with an appropriate geometric structure which will be specified later. Points of Z correspond to particles of the matter. A configuration of the fluid is described completely if for each point $x=(x^\mu)$ of the physical space-time X a particle $z=\zeta(x)$ whose world line passes through x is specified. The configuration can thus be represented as a mapping ζ: $X\rightarrow Z$. Given a coordinate system (z^a), $a=1,2,3$, the mapping is described by three functions $z^a=\zeta^a(x)$. Physical laws governing the evolution of the system will be formulated in terms of partial differential equations for the three unknown functions $\zeta^a(x)$. The equations will be derived from the variational formula

$$\delta\int L\,d^4x=0\ , \tag{11}$$

where the Lagrangian L is equal to minus the rest energy density of the fluid:

$$L=-\sqrt{\bar{g}}\,\frac{1}{V}e(V)\ . \tag{12}$$

Here $\bar{g}=|\det g_{\mu\nu}|=|g^{00}|^{(-1)}(\det g_{kl})$ is the absolute value of the determinant of the 4-dimensional metric tensor with the signature $(-,+,+,+)$, and V is the specific volume (per mole) of the fluid in its rest frame. It is convenient to introduce the rest matter density (moles per volume): $\rho=1/V$. In this way we have

$$L=-\sqrt{\bar{g}}\,\rho e(1/\rho)\ . \tag{13}$$

The quantity ρ (or V) has to be expressed in terms of unknown functions ζ^a and their derivatives. This is possible because Z is equipped with the volume structure (scalar density or a differential three-form)

$$r=h(z)dz^1\wedge dz^2\wedge dz^3 \tag{14}$$

which enables us to assign to each volume D in Z the number $n(D)$ of moles (or the number of particles) of the matter contained in D:

$$n(D):=\int_D r\ . \tag{15}$$

The pull back of r from Z to X via the configuration ζ is a differential three-form (i.e., it is a vector density in the physical space-time X):

$$j\equiv\zeta^*r=h(\zeta(x))d\zeta^1\wedge d\zeta^2\wedge d\zeta^3\ . \tag{16}$$

Using the notation $\zeta^a_\mu=\partial\zeta^a/\partial x^\mu$, we have

$$\begin{aligned} j&=h\zeta^1_\alpha\zeta^2_\beta\zeta^3_\gamma dx^\alpha\wedge dx^\beta\wedge dx^\gamma \\ &=-h\frac{1}{3!}\epsilon^{\mu\alpha\beta\gamma}\zeta^1_\alpha\zeta^2_\beta\zeta^3_\gamma\epsilon_{\mu\nu\lambda\sigma}dx^\nu\wedge dx^\lambda\wedge dx^\sigma\ , \end{aligned} \tag{17}$$

where $\epsilon^{\mu\alpha\beta\gamma}$ denotes the completely antisymmetric tensor density (Levi-Civita symbol; we use the convention of Misner, Thorne, and Wheeler[2]). The vector density j can be spanned with respect to basic three-forms:

$$\begin{aligned} j&=j^\mu\frac{\partial}{\partial x^\mu}\ \lrcorner\,(dx^0\wedge dx^1\wedge dx^2\wedge dx^3) \\ &=\frac{1}{3!}\frac{1}{(\bar{g})^{1/2}}j^\mu\epsilon_{\mu\nu\lambda\sigma}dx^\nu\wedge dx^\lambda\wedge dx^\sigma \\ &=\sum_{\mu=0}^{3}j^\mu(-1)^\mu dx^0\wedge\cdots\wedge d\hat{x}^\mu\wedge\cdots\wedge dx^3 \end{aligned} \tag{18}$$

(the factor dx^μ is to be omitted in the μth term). The formulas (17) and (18) enable us to express j^μ in terms of derivatives of the fluid configuration ζ^a:

$$\begin{aligned} j^{\mu} &= -h\sqrt{g}\,\epsilon^{\mu\alpha\beta\gamma}\zeta^{1}_{\alpha}\zeta^{2}_{\beta}\zeta^{3}_{\gamma} \\ &= h(-1)^{\mu}\det \underset{\underset{\mu}{\wedge}}{\begin{vmatrix} \zeta^{1}_{0},\ldots,\zeta^{1}_{3} \\ \zeta^{2}_{0},\ldots,\zeta^{2}_{3} \\ \zeta^{3}_{0},\ldots,\zeta^{3}_{3} \end{vmatrix}} \end{aligned} \tag{19}$$

(again, the μth column of the 3×4 matrix ζ^{a}_{μ} is to be omitted; this is indicated by $\hat{\mu}$). The current j is conserved by virtue of the definition

$$dj = d\zeta^{*}r = \zeta^{*}dr = 0 \tag{20}$$

since $dr=0$ as a differential four-form in the three-dimensional material space Z. In coordinate language this means that the divergence of the current vanishes:

$$dj = \left[\frac{\partial}{\partial x^{\mu}}j^{\mu}\right]dx^{0}\wedge dx^{1}\wedge dx^{2}\wedge dx^{3} = 0\ . \tag{21}$$

Decomposing j^{μ} as a product of a four-dimensional volume density $(g)^{1/2}$, normalized velocity vector field of the fluid u^{μ} $(u^{\mu}u_{\mu}=-1)$, and a rest-frame matter density ρ,

$$j^{\mu} = \sqrt{g}\,\rho u^{\mu}\ , \tag{22}$$

we obtain the formula

$$\rho = \frac{1}{\sqrt{g}}\sqrt{-j^{\mu}j^{\nu}g_{\mu\nu}}\ . \tag{23}$$

Formulas (19) and (23) give us the necessary expression of ρ in terms of ζ^{a}_{μ}. The reader may easily check that the following fundamental identity holds:

$$\frac{\partial j^{\nu}}{\partial\zeta^{a}_{\mu}}\zeta^{a}_{\lambda} = j^{\nu}\delta^{\mu}_{\lambda} - j^{\mu}\delta^{\nu}_{\lambda}\ . \tag{24}$$

The easiest way to prove that the variational formula (11) is equivalent to Euler's equations of motion of the fluid is to calculate the canonical energy-momentum tensor of the above field theory:[6]

$$T^{\mu}_{\lambda} = \frac{\partial L}{\partial\zeta^{a}_{\mu}}\zeta^{a}_{\lambda} - \delta^{\mu}_{\lambda}L\ . \tag{25}$$

Because of formulas (13), (23), and (24) we have

$$\begin{aligned} \frac{\partial L}{\partial\zeta^{a}_{\mu}}\zeta^{a}_{\lambda} &= \frac{\partial L}{\partial\rho}\frac{\partial\rho}{\partial j^{\nu}}\frac{\partial j^{\nu}}{\partial\zeta^{a}_{\mu}}\zeta^{a}_{\lambda} \\ &= \left[e - V\frac{de}{dV}\right]u_{\nu}\frac{\partial j^{\nu}}{\partial\zeta^{a}_{\mu}}\zeta^{a}_{\lambda} \\ &= -\sqrt{g}\left[\rho e - \frac{de}{dV}\right](\delta^{\mu}_{\lambda} + u^{\mu}u_{\lambda})\ . \end{aligned} \tag{26}$$

Finally, using (8) we obtain

$$T^{\mu}_{\lambda} = -\sqrt{g}\,[\rho e u^{\mu}u_{\lambda} + p(\delta^{\mu}_{\lambda} + u^{\mu}u_{\lambda})]\ . \tag{27}$$

Because of the Noether theorem, variational formula (11) implies conservation laws

$$\nabla_{\mu}T^{\mu}_{\lambda} = 0 \tag{28}$$

where ∇_{μ} means the covariant derivative with respect to the space-time metric. Because of the continuity equation (20) only three among the four equations (28) are independent. They are equivalent to three variational equations:[6]

$$\frac{\delta L}{\delta\zeta^{a}} = 0\ . \tag{29}$$

The above formulation of the fluid mechanics as a Lagrangian field theory leads in a natural way to the Hamiltonian formulation[5] (later used also by Künzle and Nester[7]). Momenta canonically conjugate to configurations ζ^{a} are given by $\pi_{a} = p^{0}_{a}$, where

$$p^{\mu}_{a} = \frac{\delta L}{\delta\zeta^{a}_{\mu}}\ .$$

Equation (26) implies that

$$\begin{aligned} \pi_{a} &= -\sqrt{g}\,\rho(e + Vp)u^{0}u_{k}(\zeta^{-1})^{k}_{a} \\ &= -j^{0}\eta u_{k}(\zeta^{-1})^{k}_{a}\ , \end{aligned}$$

where $(\zeta^{-1})^{k}_{a}$ is a 3×3 matrix inverse to (ζ^{a}_{k}) and $\eta = (e + Vp)$ is the enthalpy of the fluid. The above canonical structure is equivalent to the description based on the so-called Clebsch variables,[8] although it is simpler and physically more natural.

Because of the reparametrization freedom of the material space Z the theory may be further reduced. Indeed, any reparametrization of Z which leaves invariant the volume structure (14) is physically equivalent to the previous one. The space of equivalence classes can be parametrized by four parameters (e.g., j^{μ} or ρ and v^{k}). The Poisson brackets between those quantities can be immediately calculated[9] from the canonical structure carried by the variables (ζ^{a}, π_{a}). Some authors introduce those brackets in an axiomatic way as a "noncanonical" Hamiltonian formalism.[10]

The parameters $z^{a} = \zeta^{a}(x)$ can be considered as "potentials" for hydrodynamics: we express physical quantities (ρ and u^{μ}) in terms of the first derivatives of the potentials in such a way that the continuity equation is automatically satisfied. The analogous ansatz in electrodynamics consists in expressing physical quantities (electromagnetic tensor $F_{\mu\nu}$) in terms of the potential A_{μ} in such a way that the first pair of Maxwell equations is automatically satisfied. First-order field equations for physical quantities (Euler equations in hydrodynamics and the second pair of Maxwell equations in electrodynamics) become variational second-order equations for the potentials. In both theories the ansatz is not unique or "unique up to gauge transformations." In hydrodynamics gauge transformations correspond to unimodular (i.e., without changing the volume form r) transformations of the matter space Z.

III. THERMODYNAMICS

To describe thermal properties of the fluid we need one more potential.[11] We add therefore a new dimension $\tau = z^0$ to the matter space Z. This way we obtain the 4-dimensional matter space-time $\underline{Z}$ with τ playing role of a "material time." To describe the configuration of the fluid we need now an additional function $z^0 = \zeta^0(x^\mu)$. The physical interpretation of the new variable will be given in the next section. However, a purely phenomenological point of view is also possible. We prove that the following ansatz for the temperature,

$$T = \beta u^\mu \frac{\partial \zeta^0}{\partial x^\mu} = \beta u^\mu \zeta^0_\mu \tag{30}$$

(where β is a positive constant), together with the choice of minus the free energy $[f(V,T) = e - TS]$ as a Lagrangian of the theory

$$L = -\sqrt{g}\,\frac{1}{V} f(V,T) = -\sqrt{g}\,\rho f(1/\rho, T) \ , \tag{31}$$

leads to correct equations. To prove this statement we calculate again the energy-momentum tensor

$$T^\mu_\lambda = \frac{\partial L}{\partial \zeta^\alpha_\mu} \zeta^\alpha_\lambda - \delta^\mu_\lambda L \ , \tag{32}$$

where $\alpha = 0, 1, 3$. Using the thermodynamical identity

$$df = -p\, dV - S\, dT \tag{33}$$

we obtain

$$\begin{aligned} \frac{\partial L}{\partial \zeta^a_\mu} \zeta^a_\lambda &= -\sqrt{g} \left[\left(f - V \frac{\partial f}{\partial V} \right) \frac{\partial \rho}{\partial j^\nu} + \rho\beta \frac{\partial f}{\partial T} \frac{\partial u^\sigma}{\partial j^\nu} \zeta^0_\sigma \right] \frac{\partial j^\nu}{\partial \zeta^a_\mu} \zeta^a_\lambda \\ &= [(f + Vp)u_\nu + S(\delta^\sigma_\nu + u^\sigma u_\nu)\beta\zeta^0_\sigma] \frac{\partial j^\nu}{\partial \zeta^a_\mu} \zeta^a_\lambda \\ &= -\sqrt{g}\,[(f + Vp)(\delta^\mu_\lambda + u^\mu u_\lambda) + S\beta\zeta^0_\sigma u^\mu(\delta^\sigma_\lambda + u^\sigma u_\lambda)] \ . \end{aligned} \tag{34}$$

Moreover,

$$\frac{\partial L}{\partial \zeta^0_\mu} \zeta^0_\lambda = \sqrt{g}\,\rho(S u^\mu \beta \zeta^0_\lambda) \ . \tag{35}$$

Finally, we obtain formula (27) for T^μ_λ with internal energy defined by $e = f + TS$. Again, the Noether theorem implies the energy-momentum conservation $\nabla_\mu T^\mu_\lambda = 0$, equivalent to four independent variational equations

$$\frac{\delta L}{\delta \zeta^\alpha} = 0 \ . \tag{36}$$

For $\alpha = 0$ this gives the entropy conservation

$$0 = \frac{\delta L}{\delta \zeta^0} = -\beta\partial_\mu(S j^\mu) = -\beta\rho u^\mu \partial_\mu S \ . \tag{37}$$

It is easy to check that four equations (36) are equivalent to the system of three equations of momentum conservation [three independent equations among (28)] plus entropy conservation (37).

In the above formulation the phenomenological constant β may be chosen arbitrarily. The choice $\beta = 1$ is also possible. It gives (time × temperature) for the dimension of the new potential $\tau = z^0$. As we shall see in the next section, it is better to choose the dimension of β equal to the temperature and to measure τ in units of time.

IV. MICROSCOPIC INTERPRETATION OF THE "MATERIAL TIME" τ

Suppose that the liquid is composed of molecules with mass m. If the temperature of the fluid equals T, the molecules move chaotically around the theoretical world lines of the fluid (lines tangent to the vector field u^μ) and the mean kinetic energy of this motion with respect to the rest frame is equal (for low temperature) to $\frac{3}{2}kT = mv^2/2$. Because of this motion the proper time t for the particles is retarded with respect to the physical time x^0 calculated along u^μ. For velocities v much smaller than the velocity of light c this retardation can be calculated from the formula

$$\begin{aligned} t = \int \left(1 - \frac{v^2}{c^2} \right)^{1/2} dx^0 &\approx \left(1 - \frac{v^2}{2c^2} \right) x^0 \\ &= x^0 - x^0 \frac{3kT}{2mc^2} \ . \end{aligned} \tag{38}$$

We identify the parameter τ with the proper time retardation multiplied by an arbitrary dimensionless constant θ:

$$t \equiv \theta(x^0 - t) = \frac{1}{\beta} x^0 T \ , \tag{39}$$

where the constant $\beta = 2mc^2/3\theta k$ has dimensions of temperature. Hence,

$$\beta \frac{d\tau}{dx^0} = T \tag{40}$$

similarly as in formula (30). We interpret therefore the "material time" (up to a multiplicative constant θ) as the "proper time retardation" due to the chaotic motion of the particles. This phenomenon enables us to construct (at least theoretically) a "radium thermometer." We inject a drop of radioactive radium into the fluid. Because of chaotic motion of the particles the lifetime of radium gets lengthened proportionally to the temperature of the fluid. Therefore, measuring the lifetime we measure the temperature.

V. SELF-GRAVITATING FLUID

The theory of self-gravitating fluids will be based on the Lagrangian $L=L_{\mathrm{grav}}+L_{\mathrm{mat}}$, where L_{grav} is the Einstein-Hilbert Lagrangian for the gravitational field and L_{mat} is as in Sec. III. In Hamiltonian formulation, the complete Cauchy data for the theory consists of Cauchy data for both gravitational and hydrothermodynamical field: $(P^{kl}, g_{kl}, \zeta^{\alpha}, p^{0}_{\alpha})$, where $p^{\mu}_{\alpha}=\partial L/\partial\zeta^{\alpha}_{\mu}$ are momenta canonically conjugate to hydrothermodynamical potentials.[5] Using Eqs. (30), (31), and (33), we show that the momentum canonically conjugate to ζ^0 is equal (modulo the factor β) to the entropy current:

$$p^{\mu}_{0}=\sqrt{g}\,\rho\beta S u^{\mu}\ . \tag{41}$$

The Hamiltonian field equations giving the time derivatives of the Cauchy data in terms of the functional derivatives of the Hamiltonian can be briefly written as[2]

$$-\delta H=\frac{1}{16\pi G}\int \dot{P}^{kl}\delta g_{kl}-\dot{g}_{kl}\delta P^{kl}+\int \dot{p}^{\,0}_{\alpha}\delta\zeta^{\alpha}-\dot{\zeta}^{\alpha}\delta p^{0}_{\alpha} \tag{42}$$

(the overdot denotes the "time" derivative), which simply means that the time derivatives of canonical parameters are equal to the variational derivatives of the Hamiltonian with respect to the conjugate parameters. Because of the invariance of the theory with respect to space-time diffeomorphisms, we are allowed to impose the following "gauge conditions": $\zeta^{\alpha}(x^{\mu})=x^{\alpha}$. The three conditions corresponding to $\alpha=1,2,3$ mean that the coordinates x^k are comoving with the fluid (constant on the fluid world lines). In this gauge we have $\zeta^{a}_{k}=\delta^{a}_{k}$ and, according to (1), we have

$$\begin{aligned} j^{\mu}&=-h\sqrt{g}\,\epsilon^{\mu\alpha\beta\gamma}\zeta^{1}_{\alpha}\zeta^{2}_{\beta}\zeta^{3}_{\gamma}\\ &=-h\sqrt{g}\,\epsilon^{\mu 123}=h\,\delta^{\mu}_{0}\ . \end{aligned} \tag{43}$$

Using (23) we obtain

$$\rho=h\frac{\sqrt{-g_{00}}}{\sqrt{g}}=h\frac{\sqrt{g^{00}g_{00}}}{\sqrt{\det g}}=h\frac{\sqrt{1-n^2}}{\sqrt{\det g}}\ , \tag{44}$$

where $\sqrt{\det g}=\sqrt{\det g_{kl}}$ denotes, as usual, the three-dimensional volume density on the Cauchy surface $\Sigma=\{x^0=\mathrm{const}\}$ and

$$n=\frac{1}{N}\sqrt{N^iN^jg_{ij}}\ , \tag{45}$$

where N is the lapse function and N^i is the shift vector. Moreover,

$$u^{\mu}=\frac{1}{\sqrt{-g_{00}}}\delta^{\mu}_{0}=\frac{1}{N\sqrt{1-n^2}}\delta^{\mu}_{0}\ . \tag{46}$$

Because of formula (30), for $\alpha=0$ gauge condition identifying the material time with the time coordinate x^0 implies

$$T=\beta\frac{1}{\sqrt{-g_{00}}}=\beta\frac{1}{N\sqrt{1-n^2}}\ . \tag{47}$$

Our gauge conditions imply $\delta\zeta^{\alpha}=0$, $\dot{\zeta}^{k}=0$ for $k=1,2,3$, and $\dot{\zeta}^{0}=1$. The formula (41) reads $p^{\mu}_{0}=\beta S j^{\mu}=\beta S h\delta^{\mu}_{0}$. These observations enable us to rewrite the Hamiltonian formula (42) as

$$-\delta\left[H-\beta\int Sh\right]=\frac{1}{16\pi G}\int \dot{P}^{kl}\delta g_{kl}-\dot{g}_{kl}\delta P^{kl}\ , \tag{48}$$

where the quantity $\sigma=\int Sh$ is equal to the total entropy of the system. Now, the quantity $\mathcal{H}=H-\beta\sigma$ plays the role of the Hamiltonian of the system described by canonical variables (P^{kl}, g_{kl}) and the evolution is uniquely determined by the Hamiltonian.

For the sake of simplicity let us limit ourselves to the case of spatially compact space-times. In this case the total gravitational energy vanishes identically[2] $(H=0)$ and the global Hamiltonian is determined by the total entropy of the system: $\mathcal{H}=-\beta\sigma$. The quantity $U=\beta hS$ plays the role of minus the Hamiltonian density on the Cauchy surface Σ. To express the Hamiltonian in terms of canonical parameters (P^{kl}, g_{kl}) we consider the constraint equations corresponding to components T^{00} and T^{0}_{k} of the energy-momentum tensor (27):

$$\begin{aligned} X(g,P)&=\frac{1}{16\pi G}\left[R(g)-\frac{1}{\det g}\left(P^{kl}P_{kl}-\tfrac{1}{2}P^2\right)\right]\\ &=\frac{N^2\rho e+pN^iN_i}{N^2-N^iN_i}=\frac{\rho e+pn^2}{1-n^2} \end{aligned} \tag{49}$$

and

$$\begin{aligned} Y_k(g,P)&=\frac{1}{8\pi Gh}P_k{}^{l}{}_{|l}\\ &=-\frac{\sqrt{\det g}}{h}\frac{NN_k(\rho e+p)}{N^2-N^iN_i}\\ &=-\frac{\sqrt{\det g}}{h}\frac{N_k}{N}\frac{\rho e+p}{1-n^2}\ . \end{aligned} \tag{50}$$

We stress that Y is a covector field since $P_k{}^{l}{}_{|l}$ is a covector density and h is a scalar density. For a given fundamental equation (10) of the fluid the pressure p is given as a function of the parameters (e, V). Therefore, the six equations (44), (47), (49), and (50) can be solved with respect to the following six variables: the lapse function N, the shift vector N^i, and the two thermodynamical parameters (e, V). In this way we are able to express these six parameters as functions of five purely geometric variables: two scalars $X, Z:=h^2/\det g$ and a covector Y_k. The simplest way to solve the above system consists in calculating the square of the length of the vector Y_k:

$$Y=g^{kl}Y_kY_l=Z^{-1}\frac{n^2(\rho e+p)^2}{(1-n^2)^2}\ . \tag{51}$$

Equation (44) implies

$$n^2 = 1 - \frac{\rho^2}{Z} = 1 - \frac{1}{V^2 Z} \ . \tag{52}$$

Inserting the above value into (49) and (51) we obtain the following nonlinear equations for two thermodynamical parameters (e, V):

$$e = VX - \sqrt{Y}\sqrt{V^2 Z - 1} \ , \tag{53}$$

$$p(e,V) = \frac{VZ\sqrt{Y}}{\sqrt{V^2 Z - 1}} - X \ , \tag{54}$$

where the function $p(e,V)$ is determined by the state equation (9):

$$p = \frac{\partial S}{\partial V} \Big/ \frac{\partial S}{\partial e} \ . \tag{55}$$

Inserting (53) into (54) we obtain finally a single nonlinear equation for the parameter $V = V(X,Y,Z)$ which has to be solved. Having solved this equation we can express all the thermodynamical parameters together with the lapse and the shift in terms of geometric quantities (X,Y,Z), i.e., in terms of canonical variables P^{kl} and g_{kl}. Moreover, we can find the entropy density

$$S = S(e(X,Y,Z), V(X,Y,Z)) =: F(X,Y,Z) \ . \tag{56}$$

Practically (see Sec. VII) Eq. (54) is very often highly nonlinear and cannot be solved analytically. To calculate the function $F(X,Y,Z)$ (i.e., the Hamiltonian) it is however sufficient to observe that it satisfies the following Hamilton-Jacobi equation:

$$\left(\frac{\partial F}{\partial X} \right)^2 - 4 \frac{\partial F}{\partial Y}\frac{\partial F}{\partial Z} = 0 \ . \tag{57}$$

The proof is given in Appendix A. Observe that the three-dimensional space of parameters (X,Y,Z) has a pseudo-Riemannian structure with the signature $(+,+,-)$ and the characteristics of (57) are straight lines. The initial-value condition for Eq. (57) is given by the following observation: for $Y=0$ Eq. (51) implies $n=0$. Hence, Eq. (49) reduces to $X = e/V$ and Eq. (44) reduces to $1/V = \sqrt{Z}$. Finally, we have $e(X,0,Z) = X/\sqrt{Z}$, $V(X,0,Z) = 1/\sqrt{Z}$, and

$$F(X,0,Z) = S\left(\frac{X}{\sqrt{Z}}, \frac{1}{\sqrt{Z}} \right) , \tag{58}$$

where the function $S = S(e,V)$ on the right-hand side is precisely the one which defines the fundamental equation (10). It is possible to find a thermodynamical condition for the function (58) which prevents the occurrence of caustics (see Appendix C). If the fluid satisfies this condition the Hamilton-Jacobi equation can be solved globally for $Y > 0$, providing the Hamiltonian of the coupled matter and field system. The Hamiltonian is a function of X (depending algebraically on P^{kl} and on derivatives of g_{kl} up to the second order), Y (derivatives of both P^{kl} and g_{kl} up to the first order), and Z (algebraic dependence on g_{kl}). Hamiltonian formula (48) implies the following evolution equations with $U(X,Y,Z) = \beta h F(X,Y,Z)$:

$$\dot{P}^{kl} = 16\pi G \frac{\delta U}{\delta g_{kl}} \ , \tag{59}$$

$$\dot{g}_{kl} = -16\pi G \frac{\delta U}{\delta P^{kl}} \ . \tag{60}$$

VI. CONSTRAINT MANIFOLD AS A BOTTOM OF A "POTENTIAL WELL"

To prove that the free Einstein theory corresponds to a limiting case when the Hamiltonian tends to infinity outside the constraint subspace $\{X=0, Y=0\}$, assume that we change the mass of particles of the fluid: $m' = \theta m$, where θ is a real number (finally, we are going to pass to the limit $\theta \to 0$). For a given kinematical state of the particles we have therefore $T' = \theta T$. Moreover, we assume that the total molar energy changes in the same way: $e' = \theta e$ (this assumption is compatible with the fact that both the rest energy mc^2 and the kinetic energy $\frac{3}{2}kT$ change in the same way). On the other hand, we do not change the density (particle number) of the fluid: $\rho' = \rho$. Hence, $V' = V$. Consequently, according to (8) we have $p' = \theta p$ and $S' = S$. If $p' = p'(e', V')$ gives the pressure as a function of energy and volume for the new liquid and if $p = p(e,V)$ is the corresponding function for the old one, we have

$$p'(e',V') = \theta p(e,V) = \theta p\left(\frac{e'}{\theta}, V' \right) \ . \tag{61}$$

Similarly

$$S'(e',V') = S(e,V) = S\left(\frac{e'}{\theta}, V' \right) \ . \tag{62}$$

To parametrize the dynamics we use the "material time" τ given by the formula (39) with the corresponding value of θ. This way we keep the parameter β unchanged $(\beta' = \beta)$ during the entire operation.

Consider now Eq. (49) for the new fluid (i.e., with ρ, e, and p replaced by corresponding primed quantities). Dividing both sides by θ and using (61) we obtain on the right-hand side the same function of parameters $(e'/\theta, V')$ as we have had for the cold fluid. The same observation is valid for Eq. (51) divided by θ^2. On the other hand, Eq. (52) is the same for both fluids. We conclude that

$$\frac{e'}{\theta} = e\left(\frac{X}{\theta}, \frac{Y}{\theta^2}, Z \right) \tag{63}$$

and

$$V' = V\left(\frac{X}{\theta}, \frac{Y}{\theta^2}, Z \right) , \tag{64}$$

where $e = e(X,Y,Z)$ and $V = V(X,Y,Z)$ are corresponding solutions for the first fluid. Using (62), (63), and (64) we obtain the following expression for the Hamiltonian density of the second fluid:

$$U'(X,Y,Z)=h\beta F'(X,Y,Z)=h\beta S'(e',V')=h\beta S\left[\frac{e'}{\theta},V'\right]=h\beta S\left[e\left[\frac{X}{\theta},\frac{Y}{\theta^2},Z\right],V\left[\frac{X}{\theta},\frac{Y}{\theta^2},Z\right]\right]$$
$$=h\beta F\left[\frac{X}{\theta},\frac{Y}{\theta^2},Z\right]=U\left[\frac{X}{\theta},\frac{Y}{\theta^2},Z\right] . \tag{65}$$

It is easy to see that U' satisfies (57) if U does. We prove in Appendix B that for $X\neq 0$ and $Y\neq 0$ the value of the right-hand side tends to infinity as θ goes to zero. This proves that indeed, the constraint manifold $\{X=0, Y=0\}$ can be considered as a bottom of "a very deep potential well." The dynamics of the free gravitational field can thus be approximated by the nonconstrained dynamics of the self-gravitating fluid. For $\theta \ll 1$ the degrees of freedom transversal to constraints become "fast degrees of freedom" and decouple practically from the dynamics along the constraints. Starting the numerical simulation from the bottom of the well the system will remain for a long time in the vicinity of the bottom.

VII. EXAMPLES

Example 1. A monoatomic ideal gas is described by the fundamental equation

$$S=R\ \ln(e-m)^{3/2}V , \tag{66}$$

where m is the rest mass per mole. It follows that

$$p=\frac{2}{3}\frac{e-m}{V} \tag{67}$$

and Eq. (54) reads

$$\frac{2}{3}\frac{e-m}{V}=\frac{VZ\sqrt{Y}}{\sqrt{V^2Z-1}}-X . \tag{68}$$

Inserting (53) into (68) we obtain

$$\frac{2}{3}\left[X-\frac{\sqrt{Y}}{V}\sqrt{V^2Z-1}-\frac{m}{V}\right]=\frac{VZ\sqrt{Y}}{\sqrt{V^2Z-1}}-X , \tag{69}$$

which can be rearranged into

$$V^4Z(X^2-YZ)-\tfrac{4}{5}V^3mXZ+V^2(\tfrac{4}{5}YZ-X^2+\tfrac{4}{25}m^2Z)$$
$$+\tfrac{4}{5}VmX-\tfrac{4}{25}(m^2+Y)=0 . \tag{70}$$

As the roots of a fourth-order algebraic equation are given by a rather lengthy formula we note only that there exists a unique solution satisfying the condition $V\geq 1/\sqrt{Z}$. The entropy density is given by Eq. (56).

Equation (70) becomes much simpler when the rest mass can be neglected. For such a ultrarelativistic ideal gas, however, the fundamental equation is a special case of a more general class that is described in the next example.

Example 2. Let us consider a class of thermodynamical systems that are described by a fundamental equation

$$S=f(e^{\mu}V) , \tag{71}$$

where f is an increasing concave ($f'>0, f''<0$) function and μ is a positive constant. Moreover, we assume $[(\mu+1)xf''(x)+\mu f'(x)]<0$, which is necessary for (71) to be concave (thermodynamical stability condition). This class includes the ultrarelativistic ideal gas ($\mu=\frac{3}{2}$) and the photon gas [$S=\frac{4}{3}(e^3V\sigma)^{1/4}$, $\mu=3$]. An interesting class of fluids corresponds to $\mu=1$, where the pressure is equal to the energy density ρe. This makes the energy-momentum tensor proportional to the metric. Einstein equations contain therefore a "dynamical cosmological constant" (e.g., if $S=\sqrt{eV}$ we have $T=2\sqrt{e/V}$ and the constant is equal to T^2).

For a general fluid described by (71) we have

$$p=\frac{1}{\mu}\frac{e}{V} \tag{72}$$

and Eq. (54) can be solved:

$$V=\left[\frac{X^2-\frac{2}{\mu+1}YZ+X\left[X^2-\frac{4\mu}{(\mu+1)^2}YZ\right]^{1/2}}{2Z(X^2-YZ)}\right]^{1/2} . \tag{73}$$

The function $F(X,Y,Z)$ is given again by (56), i.e.,

$$F(X,Y,Z)=f\{[V(X,Y,Z)X$$
$$-\sqrt{Y}\sqrt{V^2(X,Y,Z)Z-1}]^{\mu}V(X,Y,Z)\} , \tag{74}$$

with $V(X,Y,Z)$ given by (73). For $S=\sqrt{eV}$ the expression (74) can be somewhat simplified:

$$F(X,Y,Z)=\left[\frac{X(\sqrt{1-YZ/X^2}+1)}{2Z}\right]^{1/2} . \tag{75}$$

APPENDIX A

Take the function

$$F(X,Y,Z)=S(e(X,Y,Z),V(X,Y,Z)) . \tag{A1}$$

At each point (X,Y,Z) the system of equations

$$\frac{\partial F}{\partial X}=\frac{\partial S}{\partial e}\frac{\partial e}{\partial X}+\frac{\partial S}{\partial V}\frac{\partial V}{\partial X} , \tag{A2}$$

$$\frac{\partial F}{\partial Y}=\frac{\partial S}{\partial e}\frac{\partial e}{\partial Y}+\frac{\partial S}{\partial V}\frac{\partial V}{\partial Y} , \tag{A3}$$

$$\frac{\partial F}{\partial Z}=\frac{\partial S}{\partial e}\frac{\partial e}{\partial Z}+\frac{\partial S}{\partial V}\frac{\partial V}{\partial Z} , \tag{A4}$$

can be uniquely solved with respect to $(\partial S/\partial e)$ and $(\partial S/\partial V)$. This proves that there must be two linearly independent equations among the above three. Suppose, those independent equations are (A2) and (A3). Hence,

$$\frac{\partial S}{\partial V}=\frac{\dfrac{\partial F}{\partial Y}\dfrac{\partial e}{\partial X}-\dfrac{\partial F}{\partial X}\dfrac{\partial e}{\partial Y}}{\dfrac{\partial e}{\partial X}\dfrac{\partial V}{\partial Y}-\dfrac{\partial e}{\partial Y}\dfrac{\partial V}{\partial X}}\ , \tag{A5}$$

$$\frac{\partial S}{\partial e}=\frac{\dfrac{\partial F}{\partial X}\dfrac{\partial V}{\partial Y}-\dfrac{\partial F}{\partial Y}\dfrac{\partial V}{\partial X}}{\dfrac{\partial e}{\partial X}\dfrac{\partial V}{\partial Y}-\dfrac{\partial e}{\partial Y}\dfrac{\partial V}{\partial X}}\ . \tag{A6}$$

Using (55) we have

$$p=\frac{\dfrac{\partial S}{\partial V}}{\dfrac{\partial S}{\partial e}}=\frac{\dfrac{\partial F}{\partial Y}\dfrac{\partial e}{\partial X}-\dfrac{\partial F}{\partial X}\dfrac{\partial e}{\partial Y}}{\dfrac{\partial F}{\partial X}\dfrac{\partial V}{\partial Y}-\dfrac{\partial F}{\partial Y}\dfrac{\partial V}{\partial X}}\ . \tag{A7}$$

Using Eqs. (53) and (54) we can eliminate the derivatives of $e(X,Y,Z)$:

$$\frac{\partial e}{\partial X}=-\frac{\partial V}{\partial X}p+V\ , \tag{A8}$$

$$\frac{\partial e}{\partial Y}=-\frac{\partial V}{\partial Y}p-\frac{\sqrt{V^2Z-1}}{2\sqrt{Y}}\ . \tag{A9}$$

Inserting (A8) and (A9) into (A7) and using (54) we finally obtain

$$V\frac{\partial F}{\partial Y}+\frac{\partial F}{\partial X}\frac{\sqrt{V^2Z-1}}{2\sqrt{Y}}=0\ , \tag{A10}$$

or, equivalently,

$$V=\left[\frac{\left[\dfrac{\partial F}{\partial X}\right]^2}{Z\left[\dfrac{\partial F}{\partial X}\right]^2-4Y\left[\dfrac{\partial F}{\partial Y}\right]^2}\right]^{1/2}\ . \tag{A11}$$

Assuming that (A2) and (A4) are independent we obtain in a similar way the result

$$V=\left[\frac{4\left[\dfrac{\partial F}{\partial Z}\right]^2}{4Z\left[\dfrac{\partial F}{\partial Z}\right]^2-Y\left[\dfrac{\partial F}{\partial X}\right]^2}\right]^{1/2}\ . \tag{A12}$$

Finally, taking (A3) and (A4) as independent equations we obtain

$$V=\left[\frac{\dfrac{\partial F}{\partial Z}}{Z\dfrac{\partial F}{\partial Z}-Y\dfrac{\partial F}{\partial Y}}\right]^{1/2}\ . \tag{A13}$$

The solvability of the system (A2)–(A4) implies that at least two formulas among (A11), (A12), and (A13) are satisfied. Equating the right-hand sides of any two of them we obtain (57).

Conversely, we will prove that (57) is sufficient for F being obtained from Eq. (A1). The function satisfying (57) is constant on the characteristic lines generated by the following "Hamiltonian":

$$K(X,Y,Z,P_X,P_Y,P_Z)=(P_X)^2-4P_YP_Z\ . \tag{A14}$$

The "Hamiltonian" (A14) does not depend on the "positions." Hence, the "momenta" are constant on characteristic lines. Of course,

$$P_X=\frac{\partial F}{\partial X},\quad P_Y=\frac{\partial F}{\partial Y},\quad P_Z=\frac{\partial F}{\partial Z}\ . \tag{A15}$$

This proves that $\partial F/\partial X$, $\partial F/\partial Y$, and $\partial F/\partial Z$ are constant on the characteristic lines of (57). The equations generated by (A14) are linear. Therefore, the lines can be parametrized as follows:

$$X=2P_Xs+X_0,\quad Y=-4P_Zs,\quad Z=-\frac{P_X^2}{P_Z}s+Z_0\ , \tag{A16}$$

where s denotes the parameter along the line and X_0, Z_0, P_X, P_Z are initial conditions on the surface $\{Y=0\}$. Let us calculate the value of the (mutually equivalent) expressions under the square root in the formulas (A11), (A12), and (A13). The value equals $1/Z_0>0$. This enables us to define the function $V=V(X,Y,Z)$ by any of the formulas (A11)–(A13). Moreover, define $e=e(X,Y,Z)$ by the formula (53) [again, it is easy to prove the inequality $(V^2Z-1)>0$]. The reader may easily check that both V and e are also constant on the characteristic lines. Hence,

$$\begin{aligned}F(X,Y,Z)=F(X(0),0,Z(0))&=F\left[\frac{e(X(0),0,Z(0))}{V(X(0),0,Z(0))},0,\frac{1}{V^2(X(0),0,Z(0))}\right]\\&=F\left[\frac{e(X,Y,Z)}{V(X,Y,Z)},0,\frac{1}{V^2(X,Y,Z)}\right]=S(e(X,Y,Z),V(X,Y,Z))\ ,\end{aligned} \tag{A17}$$

where the function $S=S(e,V)$ has been defined by the formula (58), i.e.,

$$S(e,V)=F\left[\frac{e}{V},0,\frac{1}{V^2}\right] . \tag{A18}$$

A physical interpretation can be given only to those solutions of (57) which correspond to physically admissible entropy functions (A18) taken as initial condition $F(X,0,Z)$.

APPENDIX B

From Appendix A, Eqs. (A11)–(A13) we conclude that

$$V\left[\frac{X}{\theta},\frac{Y}{\theta^2},Z\right]=V(X,Y,Z) \tag{B1}$$

and from (53) we have

$$e\left[\frac{X}{\theta},\frac{Y}{\theta^2},Z\right]=\frac{e}{\theta}(X,Y,Z) . \tag{B2}$$

Thus, according to (65),

$$U'(X,Y,Z)=h\beta S\left[\frac{e}{\theta}(X,Y,Z),V(X,Y,Z)\right] . \tag{B3}$$

The entropy is necessarily an increasing and concave function of e and V, so when θ goes to zero, S tends to infinity or to a finite constant. The latter possibility corresponds to vanishing of the heat capacity c_V, which can be excluded for physically reasonable systems.

APPENDIX C

We define a mapping ϕ: $(X_0,Z_0)\rightarrow(X,Z)$. To each point (X_0,Z_0) on the plain $Y=0$ we assign the intersection point (X,Z) of the plain $Y=\text{const}\neq 0$ with the characteristic line of Eq. (57) starting from (X_0,Z_0). Because of Eq. (A16) the value of the parameter s corresponding to $Y=\text{const}$ equals $s=-Y/4P_Z$. Denoting $f=P_X/2P_Z$, where $P_X=\partial F/\partial X$, $P_Z=\partial F/\partial Z$ on the initial plain $Y=0$, the mapping ϕ can be written as

$$X(X_0,Z_0)=X_0-Yf(X_0,Z_0) , \tag{C1}$$

$$Z(X_0,Z_0)=Z_0+Yf^2(X_0,Z_0) . \tag{C2}$$

The corresponding Jacobian is equal to

$$J=\det\frac{\partial(X,Z)}{\partial(X_0,Z_0)}$$

$$=\det\left[\begin{matrix} 1-Y\dfrac{\partial f}{\partial X_0}, & -Y\dfrac{\partial f}{\partial Z_0} \\ 2fY\dfrac{\partial f}{\partial X_0}, & 1+2fY\dfrac{\partial f}{\partial Z_0}\end{matrix}\right]=1+AY , \tag{C3}$$

where we have denoted

$$A=2f\frac{\partial f}{\partial Z_0}-\frac{\partial F}{\partial X_0}$$

$$=\left[\frac{2\dfrac{\partial F}{\partial X}\dfrac{\partial F}{\partial Z}\dfrac{\partial^2F}{\partial X\partial Z}-\left[\dfrac{\partial F}{\partial X}\right]^2\dfrac{\partial^2F}{\partial Z^2}-\left[\dfrac{\partial F}{\partial Z}\right]^2\dfrac{\partial^2F}{\partial X^2}}{2\left[\dfrac{\partial F}{\partial Z}\right]^3}\right]_{Y=0} . \tag{C4}$$

Using (58) we express the derivatives of F in terms of thermodynamical quantities of the fluid:

$$\left[\frac{\partial F}{\partial X}\right]_{Y=0}=V\frac{\partial S}{\partial e}=\frac{V}{T},\quad \left[\frac{\partial F}{\partial Z}\right]_{Y=0}=-\frac{1}{2}V^2\left[\frac{\partial S}{\partial e}e+\frac{\partial S}{\partial V}V\right]=-\frac{1}{2}\frac{V^2}{T}(e+pV) ,$$
$$\left[\frac{\partial^2F}{\partial X^2}\right]_{Y=0}=V^2\frac{\partial^2S}{\partial e^2},\quad \left[\frac{\partial^2F}{\partial X\partial Z}\right]_{Y=0}=-\frac{1}{2}V^3\left[e\frac{\partial^2S}{\partial e^2}+V\frac{\partial^2S}{\partial e\partial V}+\frac{1}{T}\right] , \tag{C5}$$
$$\left[\frac{\partial^2F}{\partial Z^2}\right]_{Y=0}=\frac{3}{4}\frac{V^4}{T}(e+pV)+\frac{1}{4}V^4\left[e^2\frac{\partial^2S}{\partial e^2}+2eV\frac{\partial^2S}{\partial e\partial V}+V^2\frac{\partial^2S}{\partial V^2}\right]$$

so that

$$A=\eta^{-3}\left[\eta+V^2T\left[p^2\frac{\partial^2S}{\partial e^2}-2p\frac{\partial^2S}{\partial e\partial V}+\frac{\partial^2S}{\partial V^2}\right]\right] , \tag{C6}$$

where $\eta=e+pV$ denotes the enthalpy function. The expression in parentheses can be further simplified:

$$T\left[p^2\frac{\partial^2 S}{\partial e^2}-2p\frac{\partial^2 S}{\partial e\,\partial V}+\frac{\partial^2 S}{\partial V^2}\right]=-\frac{p^2}{T}\frac{1}{c_V}+\frac{p}{T}\left[\frac{\partial T}{\partial V}\right]_e+\left[\frac{\partial p}{\partial V}\right]_e$$

$$=-p\left[\frac{\partial p}{\partial e}\right]_V+\left[\frac{\partial p}{\partial V}\right]_e=-\frac{\left[\frac{\partial \eta}{\partial V}\right]_p}{\left[\frac{\partial e}{\partial p}\right]_V}=-\frac{c_p}{c_V}\frac{\left[\frac{\partial T}{\partial V}\right]_p}{\left[\frac{\partial T}{\partial V}\right]_p}=-\frac{\left[\frac{\partial S}{\partial V}\right]_p}{\left[\frac{\partial S}{\partial V}\right]_p}=\left[\frac{\partial p}{\partial V}\right]_S . \quad \text{(C7)}$$

Finally we get

$$A=\eta^{-3}\left[\eta+V^2\left[\frac{\partial p}{\partial V}\right]_S\right]$$

$$=\eta^{-3}\left[\eta-V^2\left[\frac{\partial^2 e}{\partial V^2}\right]_S\right], \quad \text{(C8)}$$

where $\eta=e+pV$ denotes the enthalpy function. Observe that only the half-space $Y>0$ is physically meaningful. Equation (C3) implies that $A>0$ is the necessary and sufficient condition for the global existence of the function F within this half-space. The condition is satisfied in all the examples considered in Sec. VII. For a perfect gas we get

$$A=\frac{3}{25(e-m)^2} \quad \text{(C9)}$$

and for fluids described by $S=f(e^{\mu}V)$,

$$A=\left[\frac{1}{(\mu+1)e}\right]^2\mu(\mu-1) . \quad \text{(C10)}$$

We see that for $\mu>1$ the coefficient A is positive. This happens, in particular, for the ultrarelativistic gas ($m=0$) and for the photon gas. For $\mu=1$ we have $A=0$ and (57) can be solved globally, also in nonphysical sector $Y<0$.

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