PHYSICAL REVIEW D VOLUME 41, NUMBER 6 15 MARCH 1990

Matter, quantum gravity, and adiabatic phase

Roberto Balbinot
Institute of Physics, University of Palermo, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, Bologna, Italy

Antonio Barletta and Giovanni Venturi*
Department of Physics, University of Bologna and Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, Bologna, Italy
(Received 2 August 1988; revised manuscript received 30 November 1989)

Based on the observation that particle masses are much smaller than the Planck mass, a framework for the matter-gravity system in which matter follows gravitation adiabatically is examined in a path-integral approach. It is found that the equations that the resulting gravitational wave function satisfies involve, in addition to the expectation value of the matter stress tensor, an adiabatically induced gauge field which can lead to interesting topological structures in superspace. Such a nontrivial geometric contribution modifies the semiclassical quantization condition and can change the conserved quantities associated with the symmetries of the system.

I. INTRODUCTION

Quantum gravity[1] in path-integral form has been the subject of considerable interest.[2-4] Such formulations were largely stimulated by the superspace approach to the canonical quantization of gravity. In such an approach the space-time dynamical variables are the three-geometries of spacelike surfaces together with their conjugate momenta. As a consequence, Einstein's equations become geodesic equations in the manifold of three-geometries[5,6] (superspace) modified by the presence of a "force term." Canonical quantization then leads to a Schrödinger-type equation (Wheeler-DeWitt) and a corresponding wave function satisfying it.

An alternative way of quantizing a system is of course through the path-integral formulation in which one sums over all paths joining two states of a system and associates to each path a suitable weight related to the action.[7] For the case of gravitation one must consider a transition amplitude from one three-geometry to another. In the path integral one must then sum over all possible paths connecting the two given three-geometries, which are separated by a given local proper time, and integrate over all possible local proper-time separations. This in effect corresponds to a sum over all four-geometries which match the initial and final three-geometries on the initial and final spacelike surfaces, respectively.

As a consequence of the invariance of the action under arbitrary space-time transformations, time does not appear in the Hamiltonian form of gravity and we expect that such a feature will be maintained in a quantum formulation. It has been observed, however, that the introduction of matter allows one to introduce the concept of time; time parametrizes how matter follows gravity.[8,9] In particular it has been noted that the semiclassical wave function for gravity provides a parametrization for the evolution of matter in which the latter follows the former adiabatically.

That one may consider an adiabatic approximation to the matter-gravity system is a consequence of the mass scale of matter going much less than the Planck mass. Indeed for the matter-gravity system one may consider the coordinates associated with the former as being the "fast" variables and those associated with the latter as being the "slow" variables. After resolving the dynamics of the "fast" variables one is left with an effective action governing the "slow" variables.

The adiabatic approximation has been studied in a number of different contexts. In the Born-Oppenheimer studies of molecules one has a Hamiltonian which contains "slow" and "fast" degrees of freedom and in particular the internuclear distance, which is "frozen" in the adiabatic process, is regarded as a dynamical variable.[10] On the other hand, if one considers a Hamiltonian depending on a slowly varying external parameter and closed loops in parameter space, a wave function acquires an extra phase, and associated "gauge" connection, with respect to the conventional dynamical one.[11] In both approaches such an additional phase arises which is a function of the slowly varying heavy degrees of freedom and which cannot be eliminated in the presence of nontrivial mappings of the "gauge" connection into the heavy parameter space. This will then give rise to generalized Aharonov-Bohm effects. The purpose of this paper is to examine the consequences of the above observations for the case of the matter-gravity system and in particular for the case of the three-geometry, corresponding to heavy degrees of freedom, considered as a dynamical variable. We shall examine the physical consequences of the above and illustrate its connection to the topological properties of superspace.

With the above aim in mind, in the next section we shall discuss the path-integral description of the matter-gravity system in the adiabatic approximation, exhibit the change in the effective action due to the presence of an induced gauge connection and obtain and discuss the correspondingly modified equations satisfied by the system.

In Sec. III we discuss the semiclassical approximation to the wave function describing the system and illustrate how time arises in the semiclassical limit for the gravitational wave function. The case of closed orbits in superspace is then discussed and the modification to the semiclassical quantization condition due to a nontrivial topology induced in superspace by the introduction of matter is illustrated. Further the possible consequences on the topology of superspace of the degeneracy of the matter energy levels at some point during a closed orbit is briefly illustrated together with the relation of the fluctuations neglected as a consequence of the adiabatic approximation, to a quantum metric tensor.

Lastly in Sec. IV our results are summarized and discussed.

II. A PATH-INTEGRAL FORMULATION

Before obtaining a path-integral description of the interaction between gravity and matter we briefly illustrate the Hamiltonian constraints one would expect for the classical system.[6] If we write the line element in the form[12]

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu =(-\beta^2+\beta_i\beta^i)dt^2+2\beta_i dx^i dt+\gamma_{ij}dx^i dx^i \tag{2.1}$$

corresponding to the standard $3+1$ decomposition of space-time, one has the total Hamiltonian (gravity plus matter) is given by

$$H=H^G+H^M=\int d^3x(\beta\mathcal{H}_0^T+\beta^i\mathcal{H}_i^T) =\sum_x \epsilon^3(\beta\mathcal{H}_0^T+\beta^i\mathcal{H}_i^T)\equiv\sum_x H_x \ , \tag{2.2}$$

where in the second line of the above we have replaced the integral over three-space by a sum over all points x of a spatial lattice of volume ϵ^3, thus obtaining a sum of Hamiltonians at each point. The introduction of such a lattice, in addition to being relevant for our subsequent path-integral discussion and manipulations involving functional derivatives, is the most natural approach to superspace which is the topological product of the six-dimensional space of "points" $\{\gamma^{ij}\}$ with itself over all points of three-space x. We note that the lapse and shift functions β and β^i are not true degrees of freedom but play the role of Lagrange multipliers and we have chosen the so-called "proper time" gauge conditions[13]

$$\frac{\partial\beta}{\partial t}=\frac{\partial\beta^i}{\partial t}=0 \ . \tag{2.3}$$

From the above one has that the Hamiltonian constraints for the classical system are given by

$$\begin{aligned}\mathcal{H}_0^T&=\mathcal{H}_0^G+\mathcal{H}_0^M\\ &=\frac{1}{8\pi G}\left[\frac{(8\pi G)^2}{2}\gamma^{-1/2}(\gamma_{ik}\gamma_{jl}+\gamma_{il}\gamma_{jk}-\gamma_{ij}\gamma_{kl})\pi^{ij}\pi^{kl}+V\right]+\mathcal{H}_0^M\\ &\equiv\frac{1}{8\pi G}[(8\pi G)^2G^{RS}\pi_R\pi_S+V]+\mathcal{H}_0^M=0 \ ,\end{aligned} \tag{2.4}$$

where V is a gravitational potential, G is Newton's constant, and

$$\begin{aligned}\mathcal{H}_i^T&=\mathcal{H}_i^G+\mathcal{H}_i^M\\ &=-\frac{1}{8\pi G}\gamma_{ip}[2\pi^{pl}{}_{,l}+\gamma^{pl}(2\gamma_{ml,n}-\gamma_{mn,l})\pi^{mn}]+\mathcal{H}_i^M\\ &=-\frac{1}{4\pi G}\gamma_{ip}\pi^{pl}{}_{;l}+\mathcal{H}_i^M=0 \ .\end{aligned} \tag{2.5}$$

In the above we have identified pairs of indices into single indices according to[6]

$$\begin{aligned}&\gamma_{11}=\gamma^1, \quad \gamma_{22}=\gamma^2,\\ &\gamma_{33}=\gamma^3, \quad \gamma_{23}=2^{-1/2}\gamma^4,\\ &\gamma_{31}=2^{-1/2}\gamma^5, \quad \gamma_{12}=2^{-1/2}\gamma^6 \ ,\end{aligned} \tag{2.6}$$

and we note that $\gamma=|\gamma_{ij}|$, π^{ij} is the momentum conjugate to γ_{ij} and is given by

$$\pi^{ij}=-\frac{1}{8\pi G}(K^{ij}-\gamma^{ij}K) \tag{2.7}$$

with

$$K^{ij}=\gamma^{im}\gamma^{il}\tfrac{1}{2}\beta^{-1}(\beta_{m;l}+\beta_{l;m}-\gamma_{ml,0}) \ , \tag{2.8}$$

where K^{ij} is the extrinsic curvature and $K=\gamma_{ij}K^{ij}$. Lastly $\mathcal{H}_0^M$ and $\mathcal{H}_i^M$ are the matter-field stress-energy tensor projected in a direction normal to the three-dimensional spacelike surface and with one component normal and one tangential, respectively. Further we shall assume that $\mathcal{H}_0^M$ depends only on the matter field variables and the spatial metric tensor (γ^L) and not on its conjugate momenta.

The object we shall be interested in is the transition amplitude from one three-geometry and matter state to another. These states are associated with two generic spacelike surfaces each point of which has a proper time and in the path-integral formulation one must integrate over all possible histories which connect the two given states and over all possible local proper time separations or relative positions of the spacelike surfaces. The latter in effect corresponds to a sum over all four-geometries which match the initial and final three-geometries on the initial and final spacelike surfaces, respectively.[2-4]

In order to obtain the desired gravitational wave function let us consider the following Hamiltonian operator in Hilbert space:

$$\hat{H}=\sum_x \epsilon^3\left[\beta\left[\hat{\mathcal{H}}_0^T+\frac{\hbar^2}{12}R\right]+\beta^i\hat{\mathcal{H}}_i^T\right]\equiv\sum_x \hat{H}_x \ , \tag{2.9}$$

where R is the curvature scalar associated with the manifold of all three-metrics and has been introduced in order to take into account operator ordering.[14]

If we now denote the matter-gravity state vectors by $|n(\gamma,x),\gamma_{ij}(x)\rangle(\equiv|n,\gamma\rangle)$ where $\gamma_{ij}(x)$ and $n(\gamma,x)$ describe gravity and matter, respectively, we may write the transition matrix element between two states as[15,16]

$$\langle n'',\gamma''|n',\gamma'\rangle=\int\langle n'',\gamma'',\beta'',\beta''^i,t''|n',\gamma',\beta',\beta'^i,t'\rangle\prod_{x,t,i}d[\beta(t''-t')]d[\beta^i(t''-t')]\delta\left[\frac{\partial\beta}{\partial t}\right]\delta\left[\frac{\partial\beta^i}{\partial t}\right]$$
$$=\int\langle n'',\gamma''|e^{-(i/\hbar)\hat{H}t}|n',\gamma'\rangle\prod_{x,i}d\tau\,d\tau^i$$
$$=\int\langle n'',\gamma''|\left[\exp\left[-\frac{i\hat{H}t}{N\hbar}\right]\right]^N|n',\gamma'\rangle\prod_{x,i}d\tau\,d\tau^i\ , \tag{2.10}$$

where $\hat{H}$ is the Hamiltonian operator in Hilbert space and, in addition to having introduced a spatial lattice, we have also separated the interval $\beta''t''-\beta't'(\beta''^it''-\beta'^it')$ into N equal segments of size $\epsilon(\epsilon^i)$ such that $N\epsilon=\beta''t''-\beta't'(N\epsilon^i=\beta''^it''-\beta'^it')$. The constraint equation (2.3) then allows us to introduce a local proper time $\tau=\beta(t''-t')$ with $\beta=\beta'=\beta''$ at each spatial point (similarly for τ^i).

A detailed derivation of expression Eq. (2.10) requires the introduction of ghost fields in order to obtain a result independent of the gauge condition, that is, the determination of the correct measure; this has been previously discussed for the constraints, Eq. (2.3) (Ref. 13). Let us further remark that the integrals over all possible values of τ and τ^i for each x reflect the fact that one must obtain the quantum-mechanical version of the constraint equations (2.4) and (2.5). These integrals of course eliminate the dependence on τ and τ^i in the final matrix element.

We have previously mentioned that, since the mass scale associated with gravity (Planck mass) is much larger than that associated with usual matter, one may expect that the latter follows the former adiabatically. Therefore we assume that the coupling of matter to gravity involves the three-metric, but not its conjugate momentum, and that matter only undergoes transitions between states having the same quantum numbers. We shall later comment on the corrections to this.

On writing our states as $|n,\gamma\rangle=|n(\gamma)\rangle\otimes|\gamma\rangle$ and using the completeness relations for γ^L and its conjugate momentum π_L we obtain, from Eq. (2.10),

$$\langle n'',\gamma''|n',\gamma'\rangle=\int\langle n'',\gamma''|e^{-(i/\hbar)\hat{H}(N)t/N}|\pi(N)\rangle\langle\pi(N)|\gamma(N-1)\rangle$$
$$\times\cdots\langle\gamma(1)|e^{-(i/\hbar)\hat{H}(1)t/N}1\pi(1)\rangle\langle\pi(1)|n'\gamma'\rangle\prod_{x,i,L}d\tau^i d\tau\frac{d\pi_L(N)}{2\pi\hbar}\prod_{k=1}^{N-1}d\gamma^L(k)\frac{d\pi_L(k)}{2\pi\hbar}\ , \tag{2.11}$$

where $\hat{H}(k)$ denotes the Hamiltonian at a point $k\epsilon$ on the path going from γ' to γ''.

The transition matrix element then becomes, in the limit $N\to\infty$,

$$\langle n'',\gamma''|n',\gamma'\rangle=\int e^{(i/\hbar)S_G}P(n'',n')\prod_L\mathcal{D}\pi_L\mathcal{D}\gamma^L\prod_{x,i}d\tau^i d\tau\ , \tag{2.12}$$

where S_G is the gravitational action

$$S_G=\frac{i}{\hbar}\int d^3x\left[\int_{\gamma'}^{\gamma''}\pi_L\delta\gamma^L-\int_{\tau'}^{\tau''}\delta\tau\left[\mathcal{H}_0^G-\frac{\hbar^2}{12}R\right]-\int_{\tau'^i}^{\tau''^i}\delta\tau^i\mathcal{H}_i{}^G\right] \tag{2.13}$$

and $P(n'',n')$ is the path-ordered matter transition amplitude[17,18]

$$P(n'',n')=\langle n''|e^{-(i/\hbar)\hat{H}^M(N)t/N}\cdots e^{-(i/\hbar)\hat{H}^M(k)t/N}\cdots e^{-(i/\hbar)\hat{H}^M(1)t/N}|n'\rangle\ , \tag{2.14}$$

where $\hat{H}^M(k)$ denotes the matter Hamiltonian at a point $k\epsilon$ on the path going from γ'^L to γ''^L.

We may now use the completeness relation for the matter states at each point k on the path obtaining

$$P(n'',n')=\sum_{n(1),\ldots,n(N-1)}\langle n''|e^{-(i/\hbar)\hat{H}^M(N)t/N}|n(N-1)\rangle\langle n(N-1)|\cdots|n(1)\rangle\langle n(1)|e^{-(i/\hbar)\hat{H}^M(1)t/N}|n'\rangle\ . \tag{2.15}$$

As we have mentioned before in the adiabatic approximation we need only consider transitions between matter states with the same quantum number $n(=n'=n'')$. Then on introducing the energy density associated with an adiabatic level n at $\gamma^L=\gamma^L(k)$:

$$\mathcal{H}_0^M(k)|n(k)\rangle=\mathcal{E}^n(k)|n(k)\rangle \tag{2.16}$$

we have

$$P(n,n)=\exp\left[-\frac{i}{\hbar}\sum_x\sum_k\epsilon^3[\epsilon\mathcal{E}^n(k)+\epsilon^i\langle n|\mathcal{H}_i^M|n\rangle]\right]\prod_{k=1}^N\langle n(k)|n(k-1)\rangle\ . \tag{2.17}$$

If we now consider the matrix element in Eq. (2.17) we observe that it refers to two infinitesimally separated points; hence, one may expand

$$\langle n(k)|n(k-1)\rangle \simeq 1-\sum_x \epsilon^3\left\langle n(k)\left|\frac{\delta}{\delta\gamma^L}\right|n(k)\right\rangle\delta\gamma^L \simeq \exp\left[i\sum_x \epsilon^3\left\langle n(k)\left|i\frac{\delta}{\delta\gamma^L}\right|n(k)\right\rangle\delta\gamma^L\right] \tag{2.18}$$

and Eq. (2.17) then becomes

$$P(n,n)=\exp\left[-\frac{i}{\hbar}\int d^3x\left[\int_{\tau'}^{\tau''}\delta\tau\langle n|\mathcal{H}_0^M|n\rangle+\int_{\tau'^i}^{\tau''^i}\delta\tau^i\langle n|\mathcal{H}_i^M|n\rangle-\int_{\gamma'}^{\gamma''}\delta\gamma^L\left\langle n\left|i\hbar\frac{\delta}{\delta\gamma^L}\right|n\right\rangle\right]\right] . \tag{2.19}$$

The final expression for the transition matrix element Eq. (2.10) is then

$$\langle n,\gamma''|n,\gamma'\rangle=\int \exp\left[\frac{i}{\hbar}\left[S_n+\int d^3x\int_{\gamma'}^{\gamma''}\delta\gamma^L\left\langle n\left|i\hbar\frac{\delta}{\delta\gamma^L}\right|n\right\rangle\right]\right]\prod_L \mathcal{D}\pi_L\mathcal{D}\gamma^L\prod_{x,i}d\tau^i d\tau \tag{2.20}$$

with

$$\begin{aligned} S_n&=\int d^3x\left[\int_{\gamma'}^{\gamma''}\pi_L\delta\gamma^L-\int_{\tau'}^{\tau''}\delta\tau\left[\mathcal{H}_0^G-\frac{\hbar^2}{12}R+\mathcal{E}^N\right]-\int_{\tau'^i}^{\tau''^i}\delta\tau^i(\mathcal{H}_i^G+\langle n|\mathcal{H}_i^M|n\rangle)\right]\\ &=S_G-\int d^3x\left[\int_{\tau'}^{\tau''}\delta\tau\,\mathcal{E}^n+\int_{\tau'^i}^{\tau''^i}\delta\tau^i\langle n|\mathcal{H}_i^M|n\rangle\right] ,\end{aligned} \tag{2.21}$$

where S_n is the adiabatic action function.

One may now verify that the transition amplitude satisfies the quantum-mechanical version of the constraint equations (2.4) and (2.5). This can be done in a straightforward way by introducing normal coordinates[19] and one then obtains

$$\left[-8\pi G\hbar^2 G^{LM}\left[\frac{\delta}{\delta\gamma^L}+\left\langle n\left|\frac{\delta}{\delta\gamma^L}\right|n\right\rangle\right]\left[\frac{\delta}{\delta\gamma^M}+\left\langle n\left|\frac{\delta}{\delta\gamma^M}\right|n\right\rangle\right]+V+\mathcal{E}^n\right]\langle n,\gamma|n,\gamma'\rangle=0 \tag{2.22}$$

which is the Wheeler-DeWitt equation modified by the presence of matter and

$$\frac{i\hbar}{4\pi G}\gamma^{jk}\left[\left[\frac{\delta}{\delta\gamma^{ij}}+\left\langle n\left|\frac{\delta}{\delta\gamma^{ij}}\right|n\right\rangle\right]\langle n,\gamma|n,\gamma'\rangle\right]_{;k}+\langle n|\mathcal{H}_i^M|n\rangle\langle n,\gamma|n,\gamma'\rangle=0 \tag{2.23}$$

which is associated with reparametrization invariance on the spacelike three-surface in the presence of matter.

Let us comment on our results. As a consequence of the Born-Oppenheimer (adiabatic) approximation, off-diagonal matrix elements have been ignored (we shall later comment on this). This means that during the motion of the system in superspace matter remains in the same eigenstate which will be a function only of the three-metric γ^L. In addition to the expected result the presence of a "gauge" field $\langle n|i\delta/\delta\gamma^L|n\rangle$ is worth noting. The presence of such a term leads to some changes in the interpretation of Eqs. (2.22) and (2.23).

In particular it is clear that reparametrization invariance on the spacelike three-surface can be realized in a nontrivial way if one also allows for a gauge transformation of the connection $\langle n|i\delta/\delta\gamma^L|n\rangle$ (Ref. 20). Further one may have nontrivial topological structures in the manifold of three-metrics (superspace) analogous to θ vacua in Yang-Mills theories.[21] One then has a topological charge and different homotopy classes. This will lead to a modification of the quantization conditions and we shall return to this in the next section.

Let us end this section by observing that we have not discussed the introduction of ghosts in order to implement our gauge choice or the consequences of the condition $\gamma>0$ since these are not necessary in order to illustrate the effect of the adiabatic treatment of matter in the gravity-matter system.

III. SEMICLASSICAL APPROXIMATION, PHASES, AND FLUCTUATIONS

In the previous section we have seen that as a consequence of the adiabatic approximation matter enters in the functional integral through the expectation value of the stress tensor. Let us now examine the semiclassical approximation for the gravitational field.

The semiclassical approximation is obtained through the stationary-phase approximation to Eq. (2.20). One may first expand around (barred) three-metrics satisfying the Hamilton-Jacobi equations

$$\langle n,\gamma''|n,\gamma'\rangle=\int \exp\left[\frac{i}{\hbar}\left[\bar{S}_n+\int d^3x\int_{\gamma'}^{\gamma''}\delta\bar{\gamma}^L\left\langle n\left|i\hbar\frac{\delta}{\delta\bar{\gamma}^L}\right|n\right\rangle\right]\right]\prod_{x,i}(2\pi i\hbar)^{-3}\left[\det\frac{-\partial^2\bar{S}}{\partial\gamma'^R(x)\partial\gamma''^S(x)}\right]^{1/2}d\tau^i d\tau , \tag{3.1}$$

where the determinant on the right-hand side (RHS) is the Van Vleck–DeWitt determinant[14] and is interpreted as the density of paths. The barred quantities are evaluated for three-metrics which are solutions to the Hamilton-Jacobi equations and we note that the presence of a term associated with reparametrization invariance in Eq. (2.20) requires that the quantities appearing in the Van Vleck determinant be just functions of the three-geometry. In Eq. (3.1) let us emphasize the presence of the additional phase factor, with respect to the expected result, corresponding to an adiabatically induced gauge field. If one replaces the Van Vleck–DeWitt determinant by its modulus one will acquire a further phase factor in the exponent of Eq. (3.1). Such a phase factor is related to the number and multiplicity of the determinant's singularities (Morse index).

In the semiclassical limit, Eq. (3.1) for $\hbar \to 0$, one may perform a further steepest-descent approximation which leads to the conditions[22]

$$\frac{\delta \bar{S}_G}{\delta \tau} - \mathcal{E}^n + \frac{\delta \bar{\gamma}^L}{\delta \tau} \left\langle n \left| i\hbar \frac{\delta}{\delta \bar{\gamma}^L} \right| n \right\rangle = 0 , \quad (3.2)$$

$$\frac{\delta \bar{S}_G}{\delta \tau^i} - \langle n | \mathcal{H}_i^M | n \rangle + \frac{\delta \bar{\gamma}^L}{\delta \tau^i} \left\langle n \left| i\hbar \frac{\delta}{\delta \bar{\gamma}^L} \right| n \right\rangle = 0 , \quad (3.3)$$

which in general for a given classical path will choose values of $\tau(=\tau^*)$ and $\tau^i(=\tau^{*i})$. This means that in general we shall have introduced a classical (local) time as a consequence of our semiclassical and adiabatic approximations. We observe that in general both from Eqs. (2.22) and (3.2) one obtains conditions for either the gravitational wave function or the matter eigenvalue.[6]

From Eq. (3.1) we finally obtain

$$\langle n, \gamma'' | n, \gamma' \rangle = \exp\left[\frac{i}{\hbar} \left[\bar{S}_n^* + \int d^3x \int_{\gamma'}^{\gamma''} \delta \bar{\gamma}^L \left\langle n \left| i\hbar \frac{\delta}{\delta \bar{\gamma}^L} \right| n \right\rangle \right] \right]$$
$$\times \prod_x (2\pi i \hbar)^{-3} \left[\det \frac{-\partial^2 \bar{S}^*}{\partial \gamma'^R(x) \partial \gamma''^S(x)} \right]^{1/2} \left[\frac{1}{(2\pi\hbar)^4} \frac{\partial^2 \bar{S}^*}{\partial \tau^{*2}(x)} \det \frac{\partial^2 \bar{S}^*}{\partial \tau^{*i}(x) \partial \tau^{*j}(x)} \right]^{-1/2} , \quad (3.4)$$

where the asterisk denotes that the action is evaluated for values of τ, τ_i satisfying Eqs. (3.2) and (3.3) and we have just considered one classical path otherwise one would have to sum over all possible classical paths. We note that we have finally obtained the semiclassical gravitational wave function in the adiabatic approximation and including the back reaction of matter.

One may now examine the semiclassical quantization of our system by considering closed orbits in superspace (for example, a closed sequence of classical solutions). Let us assume, for the sake of simplicity, that there is only one isolated closed orbit (just as before we considered a single classical path) and perform only a single circuit. According to our result, Eq. (3.3), the wave function will acquire a phase factor associated with the change of action $\Delta(\bar{S}_n^*)$ during a circuit plus a multiple (Maslov index) of $-\pi/2$ equal to the number of caustics encountered during the circuit. In order that the wave function be single valued (we shall later mention the case of degeneracy) one must have that the phase factor be a multiple of 2π; hence,

$$\frac{1}{\hbar} \left[\Delta \bar{S}_n^* + \int d^3x \oint \delta \bar{\gamma}^L \left\langle n \left| i\hbar \frac{\delta}{\delta \bar{\gamma}^L} \right| n \right\rangle \right] - \alpha\pi/2 = \frac{1}{\hbar} \int d^3x \left[\oint \delta \bar{\gamma}^L \bar{\pi}_L + \oint \delta \bar{\gamma}^L \left\langle n \left| i\hbar \frac{\delta}{\delta \bar{\gamma}^L} \right| n \right\rangle \right] - \alpha\pi/2$$
$$= 2n\pi \quad (3.5)$$

and we immediately observe that our topological phase factor modifies the semiclassical quantization condition.

At this point before further discussing the phase factor let us note that in obtaining the path-ordered matter transition amplitude equation (2.17) in the adiabatic approximation we have only considered states with the same quantum number. Clearly the lowest correction to this will actually be to allow just one intermediate transition to another eigenstate then subsequently returning to the original eigenstate. This suggests we consider the matrix element

$$B_{RS}^n = \sum_{l \neq n} \left\langle n \left| \frac{\delta}{\delta \gamma^R} \right| l \right\rangle \left\langle l \left| \frac{\delta}{\delta \gamma^S} \right| n \right\rangle = \left\langle n \left| \left[\frac{\delta}{\delta \gamma^R} - \left\langle n \left| \frac{\delta}{\delta \gamma^R} \right| n \right\rangle \right] \left[\frac{\delta}{\delta \gamma^S} - \left\langle n \left| \frac{\delta}{\delta \gamma^S} \right| n \right\rangle \right] \right| n \right\rangle$$
$$= -\sum_{l \neq n} \frac{1}{(\mathcal{E}_n - \mathcal{E}_l)^2} \left\langle n \left| \frac{\delta \mathcal{H}_0^M}{\delta \gamma^R} \right| l \right\rangle \left\langle l \left| \frac{\delta \mathcal{H}_0^M}{\delta \gamma^S} \right| n \right\rangle , \quad (3.6)$$

where B_{RS}^n is Hermitian and we note the opposite sign of the connection appearing in the covariant derivative acting on the matter eigenstate. This reflects the fact that the matter eigenfunction acquires an equal but opposite adiabatic phase to the gravitational wave function.[10]

The matrix element in Eq. (3.6) may be decomposed into real and imaginary parts:

$$\mathrm{Re} B_{RS}^n = \tfrac{1}{2}(B_{RS}^n + B_{SR}^n) \quad (3.7)$$

and

$$\mathrm{Im} B_{RS}^n = \tfrac{1}{2}(B_{RS}^n - B_{SR}^n) . \quad (3.8)$$

The first term Eq. (3.7), provides a possible means of measuring distances along paths in parameter space: it corresponds to a metric tensor on a manifold of quantum states.[23] Further we see that the quantum metric tensor for a given eigenstate of the energy is related to the energy fluctuations about the state and that such a term occurs if one allows states other than $|n\rangle$ to contribute in one of the intermediate state sums in Eq. (2.15). Such terms are of course absent in the adiabatic approximation.

The second term, Eq. (3.8) can be related to the connection. Indeed it is straightforward to see that

$$\frac{\delta}{\delta\gamma^R}\left\langle n\left|\frac{\delta}{\delta\gamma^S}\right|n\right\rangle-\frac{\delta}{\delta\gamma^S}\left\langle n\left|\frac{\delta}{\delta\gamma^R}\right|n\right\rangle=2\,\mathrm{Im}B^n_{RS}$$
$$=-\sum_{l\neq n}\frac{1}{(\mathcal{E}^l-\mathcal{E}^n)^2}\left[\left\langle n\left|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^R}\right|l\right\rangle\left\langle l\left|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^S}\right|n\right\rangle-\left\langle n\left|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^S}\right|l\right\rangle\left\langle l\left|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^R}\right|n\right\rangle\right] \tag{3.9}$$

which is a phase two-form in parameter space. It is worth noting that if for some values of the parameter γ some states $|l\rangle$ become almost degenerate with $|n\rangle$ then they will dominate the sum in Eq. (3.9) and in the absence of such singularities, if the matter Hamiltonian is real, Eq. (3.9) is zero (the eigenfunctions can then be made real). Further if we consider a closed cycle C in parameter space (superspace), we have using Eq. (3.9) that the additional phase factor in Eq. (2.20) becomes

$$\int d^3x\int_C\left\langle n\left|i\frac{\delta}{\delta\gamma^L}\right|n\right\rangle\delta\gamma^L=\int d^3x\int\int_C\delta\sigma^{RS}\mathrm{Im}\sum_{l\neq n}\frac{1}{(\mathcal{E}^l-\mathcal{E}^n)^2}\left\langle n\left|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^R}\right|l\right\rangle\left\langle l\left|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^S}\right|n\right\rangle, \tag{3.10}$$

where as before the integrations in superspace are only over the three-geometries because of the reparametrization invariance on the spacelike three-surface and $\delta\sigma^{RS}$ denotes an element of a two-dimensional surface in superspace bounded by the circuit C.

The above result is the gravity and matter system analogue of a previously obtained result.[11] When during the circuit C one passes near a point where the state n becomes degenerate, the sum in Eq. (3.10) is dominated by the "degenerate states" and for the case of three dimensions and a double degeneracy this leads to a monopole-like singularity.[11] Clearly our case, since our expression involves an integration over all three-space (or a sum over all points of a spatial lattice) and a six-dimensional superspace, is much more complicated. However it may lead to correspondingly very interesting results. Indeed both from the study of a model having N-dimensional rotational symmetry and an adiabatically induced non-Abelian gauge symmetry[24] and from a study of the topology of superspace it has been speculated that the total angular momentum can have nonintegral values.[25,26]

IV. CONCLUSIONS

The purpose of this paper has essentially been to examine the consequences of the introduction of matter in quantum gravity and to examine some of its general consequences. The framework in which we work is essentially provided by the adiabatic approximation, or Born-Oppenheimer method, because of the extremely small ratio of the usual particle masses to the Planck mass.

In order to illustrate the approach we have examined a path-integral formulation of the matter-gravity system. We have constructed, in terms of a path integral, a solution to the Wheeler-DeWitt equation for gravitation in the presence of matter. This was done through a suitable choice of gauge (time independence) for the lapse parameter which allowed us to introduce a local time. This local time is allowed to vary from $-\infty$ to ∞ thus ensuring both that the system has only the true physical degrees of freedom and that matter is the source for gravity. The resulting wave function is then shown to satisfy the Wheeler-DeWitt equation with an induced gauge field other than the matter-field stress-energy tensor.

An analogous (time independence) choice of gauge is also made for the shift function and again one obtains a local time-related parameter. This parameter, as before, is also made to vary from $-\infty$ to ∞ in order to ensure the reduction to the physical degrees of freedom. One then finds that reparametrization invariance on a spacelike three-surface for the wave function can be realized nontrivially owing to the presence of a gauge connection other than the usual matter term.

We then performed a semiclassical approximation to the gravitational wave function and noted how a local time is introduced along a classical trajectory. The results we have obtained agree with previous ones[8,9] in which matter follows gravity adiabatically. We have however included both the back reaction of matter and the adiabatically induced connection. Concerning the adiabatic approximation we have seen that it consists of the neglect of terms associated with fluctuations and have noted that the energy fluctuations about an energy eigenstate are related to a quantum metric tensor.

In addition to the quantum metric tensor we have seen that our equations also involve a gauge connection and an associated phase which cannot be gauged away in the presence of gravitational configurations having nontrivial homotopy.[27] The presence of such a connection is of particular interest since one can then obtain nontrivial topological structures in the manifold of all three-metrics (superspace), some of them analogous to the θ-vacua structure in Yang-Mills theories. Such structures can lead to a modification of the constants of motion associated with a symmetry of a system; this of course is of particular interest for physically observable quantities. Further reparametrization invariance on the spacelike three-surface can be realized in a nontrivial way if one also allows for a gauge transformation of the connection.

Finally let us comment on the induced phase. A time evolution in which the state of a system returns to its original state is of particular interest in physics. An example of this for a quantum system in adiabatic evolution is when a Hamiltonian returns to its original value and the state evolves as an eigenstate of the Hamiltonian and returns to the original state. Other examples are periodic particle motions or the splitting and recombination of a beam so that the system may be regarded as going backward in time along one beam and returning along the other beam to its original state at the same time. In all the above cases of cyclic evolution the initial and final states can differ by a phase factor which is the holonomy transformation due to the parallel transport in parameter space of the state vector with respect to a connection. Such a geometrical phase factor can lead to observable consequences such as the modification of the energy levels of particles executing periodic motions of slowly varying three-geometries.

To summarize, the consequences of the framework we have examined for the study of the matter-gravity system are numerous and are associated with a rich topological structure.[28]

ACKNOWLEDGMENTS

One of us (G.V.) wishes to thank R. Brout for stimulating discussions and suggestions.

*Postal address: Scientific Affairs Division, NATO, B-1110 Brussels, Belgium.

[1]See, for example, C. J. Isham, in *Proceedings of the 28th Scottish Universities Summer School in Physics,* St. Andrews, Scotland, 1985, edited by A. T. Davies and D. G. Sutherland (NATO Advanced Study Institute) (SUSSP, Edinburgh, 1986).

[2]C. Teitelboim, Phys. Rev. D **25**, 3159 (1982); **28**, 297 (1983); **28**, 310 (1983).

[3]J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[4]A. O. Barvinsky and V. N. Ponomariov, Phys. Lett. **167B**, 289 (1986); A. O. Barvinsky, Phys. Lett. B **175**, 401 (1986); **195**, 345 (1987).

[5]J. A. Wheeler, in *Relativity Groups and Topology,* 1963, Les Houches Lectures, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964).

[6]B. S. DeWitt, Phys. Rev. **160**, 113 (1967).

[7]See, e.g., R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965) for a discussion of quantum mechanics from this point of view.

[8]T. Banks, Nucl. Phys. **B249**, 332 (1985).

[9]R. Brout, Found. Phys. **17**, 603 (1987); Z. Phys. B **68**, 339 (1987).

[10]C. A. Mead and D. G. Truhlar, J. Chem. Phys. **70**, 2284 (1979); C. A. Mead, Chem. Phys. **49**, 23, 33 (1980).

[11]M. V. Berry, Proc. R. Soc. London **A392**, 45 (1984).

[12]Greek indices run from 0 to 3, small latin indices from 1 to 3, and capital latin indices from 1 to 6. Differentiation is denoted by a comma, covariant differentiation by a semicolon and we use units for which $c=1$.

[13]See C. Teitelboim, in *Quantum Structure of Space and Time,* edited by M. J. Duff and C. J. Isham (Cambridge University Press, Cambridge, England, 1982). For a more recent discussion, see J. J. Halliwell, Phys. Rev. D **38**, 2468 (1988).

[14]B. S. DeWitt, Rev. Mod. Phys. **29**, 377 (1957).

[15]We expect a state normalization of the form

$$\lim_{\substack{t' \to t'' \\ \beta' \to \beta' \\ \beta'' \to \beta''}} \langle n',\gamma'',t'',\beta'',\beta''|n',\gamma',t',\beta',\beta''\rangle \propto \prod_x \frac{(-|G|)^{1/2}}{\epsilon^3}\delta(\gamma''-\gamma')\delta_{n'',n'} ,$$

where $|G|$ is the determinant of the supermetric. In general we shall not be concerned with overall normalization factors.

[16]The analogy with the proper time formulation (where, however, the integration goes from 0 to ∞) should be noted. See also Ref. 2.

[17]P. Pechukas, Phys. Rev. **181**, 174 (1969).

[18]H. Kuratsuji and S. Iida, Prog. Theor. Phys. **74**, 439 (1985).

[19]See L. Parker, in *Recent Developments in Gravitation,* 1978 Cargèse Lectures, edited by M. Levy and S. Deser (Plenum, New York, 1979).

[20]For analogous situations, see, e.g., R. Jackiw, Int. J. Mod. Phys. A **3**, 285 (1988).

[21]C. J. Isham, Phys. Lett. **106B**, 188 (1981).

[22]One may alternatively consider a steepest-descent approximation associated with the limit $G \to 0$ (extreme adiabatic case). However one then loses the contribution of the back reaction of matter (see, for example, Ref. 8). One then obtains for Eq. (3.4) a purely gravitational wave function multiplied by the amplitude for matter to go from $|n,\gamma'\rangle$ to $|n,\gamma''\rangle$.

[23]J. P. Provost and C. Vallee, Commun. Math. Phys. **26**, 286 (1982).

[24]H.-Z. Li, Phys. Rev. Lett. **58**, 539 (1987).

[25]J. L. Friedman and R. D. Sorkin, Phys. Rev. Lett. **44**, 1100 (1980).

[26]D. Finkelstein and C. W. Misner, Ann. Phys. (N.Y.) **6**, 230 (1959).

[27]Concerning this, see, also, J. J. Halliwell, Phys. Rev. D **36**, 3626 (1988), and the following reference.

[28]Of course one does not have any such structures in minisuperspace which is one dimensional. For such a case it is however extremely interesting to examine fluctuations and the emergence of time. See R. Brout and G. Venturi, Phys. Rev. D **39**, 2436 (1989).