Wave functions on a minisuperspace of higher-dimensional geometries

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We consider minisuperspace quantum models of the universes with spatial sections being a product of two maximally symmetric spaces. If neither of the spaces has negative curvature, no Lorentzian solutions exist. If one of the subspaces is flat, and the other has negative curvature, the Lorentzian Hartle-Hawking ground state is the Minkowski space with a specific parametrization chosen by the boundary term in the action. We analyze the propagation of wave packets in the minisuperspace of the model with both subspaces of negative curvature. Both the wave packets and classical trajectories oscillate if the number of space-time dimensions is less than ten. However, the wave packets do not follow classical trajectories, although they are (in principle) distinguishable even after a large number of oscillations, in contrast with the gravity–scalar-field model.

I. INTRODUCTION

The action of classical gravity can be written in a canonical form very similar to the action of a particle moving in a (generally, curved) space-time. This is the basis of canonical quantum gravity, which applies the standard Dirac procedure to quantize gravity.[1–3] The space of positive-definite, compact three-dimensional metrics—the superspace, which is the configuration space in this formalism—is infinite-dimensional. This is one of the very problems of canonical quantum gravity. Almost all but a small number of results were obtained for truncated models, with a finite number of degrees of freedom in the superspace. Here we present yet another model of this type. We consider pure-gravity higher-dimensional universes for two reasons.

First, many similar, but more complex systems have been already quantized.[4–8] By comparing our results with what is known about various nonvacuum models one can investigate the importance of the matter content for the compactification process. The classical solutions for the $R \times S^n \times R^N$ models with maximally symmetric both "internal" and "physical" spaces were first found by Sahdev.[9] They have been subject to a detailed analysis.[10–15] The result is that, in the vacuum case, both scale factors of the two spatial sections vary monotonically, one starting from zero and tending in a finite time to infinity, and the other one starting from infinity and tending to zero. Quantum gravity was one of the effects supposed to stop the collapse of the internal space, thus leading to the Universe observed today—a long time after a period of inflation of the physical space and a corresponding shrinking of the internal space.[10,11,16–19] Here we consider a vacuum model, as opposed to those of Refs. 4–8, which have the desired stabilizing effect already at the classical level, due to a specific choice of matter field or cosmological constant.

It is shown that the evolution of vacuum geometries is qualitatively different from the evolution of geometries coupled on a classical level to various matter fields.[4–9] This raises a question to what extent it is justified to truncate the superspace by fixing the matter fields by geometry.

Second, the pure-gravity model with only two degrees of freedom provides an interesting example of wave packets in superspace. The issue of measurements in quantum gravity is not clear. In a particular case of a scalar field coupled to Robertson-Walker geometry, it seems likely that no well-peaked wave packets can survive a large number of semiclassical oscillations of the scalar field, thus making the issue even less clear. In the case of vacuum higher-dimensional universes there is no arbitrariness in the coupling coefficients, and the model seems to indicate that, under certain circumstances, there may exist well-peaked wave packets, even though the semiclassical evolution is certainly very unrealistic.

This paper is organized as follows. After deriving in Sec. II the Wheeler-DeWitt equation for the model, we obtain the classical solutions with one flat spatial section as an immediate result of our choice of coordinates in the minisuperspace. We solve the Wheeler-DeWitt equation for the Hartle-Hawking ground state of the Universe with a topology of $R \times S^n \times R^N$. The wave function has no Lorentzian regime. Then we analyze the geometries with one subspace flat, and the other of negative curvature. The Hartle-Hawking proposal singles out the maximally symmetric Minkowski space-time.

In Sec. IV we consider the wave functions of spatial geometries with two subspaces of negative curvature. If the number of space-time dimensions is not less than ten, one may apply the Wentzel-Kramers-Brillouin (WKB) approximation. We obtain a general solution in terms of an asymptotic series. If the number of space-time dimensions is less than ten, caustics develop and one may not use the WKB approximation. However, we find approximate solutions in the adiabatic approximation for a harmonic oscillator with a complex frequency. Because no Born-Oppenheimer approximation is valid, both degrees of freedom are nontrivially coupled. The group velocity of wave packets in the minisuperspace is different from the phase velocity. Generically, one would expect the wave packets to smear out rapidly in such a case. However, we construct a well-peaked wave packet which can survive a large number of classical oscillations. Section V contains discussion and conclusions.

II. THE MODEL

Throughout this paper we restrict the infinite number of the degrees of freedom to be quantized to two—the scale factor the "physical" (three-dimensional) space, and the scale factor of the "internal" space. Namely, we restrict the geometry to be homogenous and isotropic separately in both the "physical" and the "internal" spaces. In a useful system of coordinates the model has the metric

$$ds^2=-\mathcal{N}^2(t)dt^2+r^2\frac{\sum_{i=1}^{n}(dx^i)^2}{\left[1+\frac{k_n}{4}\sum_{i=1}^{n}(x^i)^2\right]^2}+R^2\frac{\sum_{j=1}^{N}(dx^j)^2}{\left[1+\frac{k_N}{4}\sum_{j=1}^{N}(x^j)^2\right]^2} . \quad (1)$$

Here, both n and N are arbitrary integers, such that $n>1<N$. The constants k_n and k_N are equal to ± 1 or 0, depending on whether the respective subspace has positive, negative, or zero curvature.

The action functional of canonical quantum gravity[3] for the model is

$$S=\kappa^2\int dt\,\mathcal{N}^{-1}r^nR^N\{n(n-1)r^{-2}\dot{r}^2+N(N-1)R^{-2}\dot{R}^2+2nNr^{-1}R^{-1}\dot{r}\dot{R}-\tfrac{1}{2}\mathcal{N}^2\sigma^2[k_nn(n-1)r^{-2}+k_NN(N-1)R^{-2}]\} , \quad (2)$$

where

$$\kappa^2=\frac{\sigma^2}{8\pi}\int d^{(n+N)}x\left[1+\frac{k_n}{4}\sum_{i=1}^{n}(x^i)^2\right]^{-1}\times\left[1+\frac{k_N}{4}\sum_{j=1}^{N}(x^j)^2\right]^{-1} \quad (3)$$

and σ^2 (of dimensions ML^{-n-N+2}) depends on the gravitational constant G appropriate for $n+N$ spatial dimensions. If $k_n\cdot k_N=0$ the value of κ^2 has no meaning unless one specifies the volume of the flat space(s), which should be regarded as a torus (tori). We will show that in this case no physical predictions depend on the value of κ^2.

The classical equations of motion are obtained by varying the action (2) with respect to r, R, and $\mathcal{N}$. This leads to

$$\dot{h}=-h(nh+NH)-k_n(n-1)r^{-2} , \quad (4)$$

$$\dot{H}=-H(nh+NH)-k_N(N-1)R^{-2} , \quad (5)$$

and to the Hamiltonian constraint

$$n(n-1)(h^2+k_nr^{-2})+N(N-1)(H^2+k_NR^{-2})+2nNhH=0 , \quad (6)$$

where $h=\dot{r}r^{-1}$, $H=\dot{R}R^{-1}$, and the overdot denotes differentiation with respect to the cosmic time $(\sigma/\sqrt{2})\mathcal{N}dt$.

The wave function of canonical quantum gravity[3] is defined over a two-dimensional minisuperspace with a natural metric

$$ds^2=\kappa^2\mathcal{N}^{-1}r^nR^N[n(n-1)r^{-2}dr^2+N(N-1)R^{-2}dR^2+2nNr^{-1}R^{-1}dr\,dR] . \quad (7)$$

The wave function is a solution to the Wheeler-DeWitt equation, being an operator version of the Hamiltonian constraint (6). Let

$$U=\left[\frac{n}{n-1}\right]^{1/2}r^{n-1}R^{N(1+1/a)} , \quad (8)$$

$$V=-\left[\frac{n}{n-1}\right]^{1/2}r^{n-1}R^{N(1-1/a)} , \quad (9)$$

where

$$a=\left[\frac{nN}{n+N-1}\right]^{1/2} . \quad (10)$$

Then the metric (7) expressed in coordinates U and V becomes manifestly conformally flat:

$$ds^2=-\kappa^2\mathcal{N}^{-1}r^{2-n}R^{-N}dU\,dV . \quad (11)$$

With a natural choice[20] of the kinetic energy operator as the Laplacian in the metric (11) the Wheeler-DeWitt equation is

$$\left[-\frac{\partial^2}{\partial U\partial V}+\frac{M^2}{4}(k_n+\alpha r^2R^{-2})\right]\psi(U,V)=0 , \quad (12)$$

where $M^2=n(n-1)\kappa^4$ and $\alpha=k_NN(N-1)n^{-1}\times(n-1)^{-1}$. In the foregoing section we discuss the case $\alpha=0$.

III. WAVE FUNCTION OF A GEOMETRY WITH ONE FLAT SUBSPACE

If $k_N=0$ the wave equation (12) is a Klein-Gordon equation for a particle moving in a two-dimensional Minkowski space-time with null coordinates U and V.[21] Therefore, the classical trajectories are timelike straight lines in the fourth quadrant of the UV plane [since $U\geq 0$ and $V\leq 0$ from Eqs. (8) and (9)]. With a convenient choice of constants the trajectories satisfy

$$A^{-1/a}U-A^{1/a}V=2\left[\frac{n}{n-1}\right]^{1/2}B . \quad (13)$$

If $k_n=+1$ the first integral (6) of the equations of motion (4) and (5) suggests the following parametrization:

$$U=2\left[\frac{n}{n-1}\right]^{1/2} A^{1/a}B\sin^2\frac{(n-1)\tau}{2}\ , \tag{14}$$

$$V=-2\left[\frac{n}{n-1}\right]^{1/2} A^{-1/a}B\cos^2\frac{(n-1)\tau}{2}\ , \tag{15}$$

if $B\neq 0$ or $UV^{-1}=$const otherwise. If $B\neq 0$ Eq. (6) gives $(\sigma/\sqrt{2})\mathcal{N}dt=r\,d\tau$, and Eqs. (14) and (15) are exactly the solutions of Sahdev.[9] In the form used in Ref. 12,

$$R^N=A\tan^a\frac{(n-1)\tau}{2}\ , \tag{16}$$

$$r^{n-1}=BR^{-N}\sin(n-1)\tau\ . \tag{17}$$

A general solution to the wave equation (12) can be written in several equivalent ways, for example,

$$\psi=\int d\omega\, a(\omega)\exp[iM(\omega U+\omega^{-1}V)]$$
$$=\int dk\left[-\frac{U}{V}\right]^k[b(k)I_k(M\sqrt{-UV}) + c(k)K_k(M\sqrt{-UV})]\ , \tag{18}$$

where $I_k(\ )$ and $K_k(\ )$ are the modified Bessel functions of order k, and $a(\ )$, $b(\ )$, and $c(\ )$ are arbitrary functions.

The null lines $UV=0$ form a Cauchy surface for the hyperbolic equation (12). Thus one can impose boundary conditions on the wave function ψ by specifying the characteristic data on these lines.

On the other hand, in extending the Hartle-Hawking ground-state proposal to the higher-dimensional case new features arise. It was pointed out by Hawking and Halliwell[4] that, in order to obtain a nontrivial wave function at an $(n+N)$-dimensional surface, which (for $n+N\geq 3$) is not necessarily cobordant to zero, one has to evaluate the path integral over all compact Euclidean $(n+N+1)$ manifolds which connect the given surface and other $(n+N)$-dimensional surfaces with the same characteristic numbers. However, in the case of a truncated model with a global topology of a product of a three-dimensional manifold and another manifold, the spacelike sections always form a boundary of an $(n+N+1)$-dimensional manifold with no other boundaries. Thus we need not consider any modifications of the no-boundary proposal for the model.[19]

The Euclidean path integral between two $(n+N)$-geometries should be dominated by the Euclidean action along the classical manifold, joining those geometries. The Euclidean version of the action (2) is zero when evaluated along a classical path which is null with respect to the minisuperspace metric (11).[22,23] Therefore, one may argue (compare Refs. 4, 5, and 20) that the path integral is approximately constant along the null characteristic surface $UV=0$, and can be normalized so that

$$\psi|_{UV=0}=1\ . \tag{19}$$

Equation (19) provides the characteristic data for the Cauchy problem of Eq. (12). Then, the Hartle-Hawking wave function for the model is

$$\psi=I_0(M\sqrt{-UV})\ . \tag{20}$$

The wave function (20) is not normalizable, but this is not the main problem.[21] More importantly, the solution (20) does not oscillate and therefore it does not admit any Lorentzian semiclassical interpretation.[1]

Generally, since in no region of the minisuperspace the term $-M^2k_n$ in Eq. (12) is positive, any wave function is a tunneling one on the entire superspace, rather than a propagating one. In particular, the solutions of Sahdev and their discussion in Refs. 9–12 describe spacelike trajectories in the Euclidean regime.

Now we consider the case of one of the subspaces having negative curvature, $k_n=-1$. The classical trajectories are timelike straight lines in the UV plane, and can be obtained by replacing the trigonometric functions in Eqs. (14) and (15) by their hyperbolic counterparts,

$$U=2\left[\frac{n}{n-1}\right]^{1/2} A^{1/a}B\sinh^2\frac{(n-1)\tau}{2}\ , \tag{21}$$

$$V=-2\left[\frac{n}{n-1}\right]^{1/2} A^{-1/a}B\cosh^2\frac{(n-1)\tau}{2}\ , \tag{22}$$

where, as before,

$$\frac{\sigma}{\sqrt{2}}\mathcal{N}dt=r\,d\tau\ . \tag{23}$$

Also, the straight lines passing through the origin,

$$V=-CU,\quad C>0\ , \tag{24}$$

are classical trajectories extremizing the action (2).

If $k_n=-1$ a general solution to the Wheeler-DeWitt equation (12) is

$$\psi=\int d\omega\, a(\omega)\exp[iM(\omega U-\omega^{-1}V)]$$
$$=\int dk\left[-\frac{U}{V}\right]^k[b(k)J_k(M\sqrt{-UV}) + c(k)N_k(M\sqrt{-UV})]\ , \tag{25}$$

where $J_k(\ )$ and $N_k(\ )$ are the Bessel functions of the kth order [for other representations of the solutions (25) see Ref. 24].

The Hartle-Hawking ground-state proposal implies the boundary condition (19), which specifies the ground state as

$$\psi=J_0(M\sqrt{-UV})\ . \tag{26}$$

The wave function (26) admits a semiclassical interpretation at $-UV=n(n-1)^{-1}r^{2n-2}R^{2N}>W\gg M^{-2}$. The wave fronts are orthogonal to the WKB trajectories, which are given by Eq. (24). Thus the Hartle-Hawking proposal picks a particular one-parameter set of classical solutions out of the two-parameter family given by Eqs. (21)–(24).

The evolution of the scale factors along the trajectories of this one-parameter family is given by

$$\mathcal{N}^{-1}\frac{dR}{dt}=0\ , \tag{27}$$

$$\mathcal{N}^{-1}\frac{dr}{dt}=\frac{\sigma}{\sqrt{2}}\ . \tag{28}$$

Actually, any solution to Eqs. (27) and (28) depends on two parameters. However, one of them should be fixed by shifting the origin of time t so that the initial singularity occurs at $t=0$, as was implicitly assumed in Eqs. (14), (15), (21), and (22).

In this model the scale factor of the flat space is exactly constant, whereas the scale factor of the curved space increases linearly with time, starting from initial singularity. The time dependence of the size of the "internal" space is supposed to determine the possible variations of the Newton's gravitational constant G. If one identifies the physical space with the n-dimensional expanding hyperboloid (and therefore the "internal" space is flat) the gravitational constant G, proportional to $\sigma^{-2}R^{-N}$, is exactly constant in the classical regime, contrary to the results of Refs. 25 and 26.

One can use the wave function (26) to evaluate the probability density P for the scale factors being within a given range. If one fixes

$$W=-UV=n(n-1)^{-1}r^{2n-2}R^{2N}$$

to be within a given range dW in the classical regime $W+dW>W\gg M^{-2}$, then

$$P(r)dr \propto dr \ , \tag{29}$$

or

$$P(R)dR \propto R^{-(n+N-1)/(n-1)}dR \ , \tag{30}$$

the coefficients of proportionality depending on the value of W and its range dW. Note that W is invariant under the rescalings of the flat N-dimensional subspace, as one would expect if the probabilities are physically meaningful. As a consequence of Eqs. (29) and (30) it is more likely to have a very small scale factor of the "internal" space and a (relatively) large scale factor of the "physical" space early in the classical evolution, and their respective ratio is ever decreasing along any classical trajectory.

However, if Eqs. (27) and (28) are satisfied, the metric (1) defines a flat, $(n+N+1)$-dimensional "Rindler" space, for which the splitting of the spatial sections into two spaces on different footage appears to be artificial from the classical point of view. Note, that even though the Einstein-Hilbert action is zero in this case, the action (2) does not vanish because it includes a boundary term,[3] vanishing only in the limit $t\to\infty$.

Moreover, one may not consistently consider the flat "internal" space as a limit of a curved space with a large scale factor. In such a case (i.e., if $k_N\neq 0$) its curvature term in the action (2), or in the wave equation (12), where it is equal to $\alpha^2 r^2 R^{-2}\psi$, would eventually dominate over the curvature of the "physical" space. This is discussed in the next section.

IV. WAVE FUNCTIONS OF GEOMETRIES WITH TWO CURVED SUBSPACES

If both the n- and N-dimensional subspaces have positive curvature, one should not expect the solutions to the Wheeler-DeWitt equation (12) to exhibit semiclassical oscillatory behavior. The classical trajectories correspond to the Euclidean universes rather than to the Lorentzian ones. We will not discuss this case. From now on we assume that both spatial submanifolds have negative curvatures, $k_n=k_N=-1$, unless explicitly specified otherwise. Then the classical trajectories lie entirely in the region $ab'c'$ and $a'bc$ in Fig. 1 of Ref. 12. Without losing generality we consider only the former region; i.e., we choose a particular direction of time. Then there exists only one focal point in the phase space (it is denoted by e on Fig. 1 of Ref. 12), and all trajectories asymptotically tend to the one given by

$$r^2R^{-2}=\left[\frac{n}{n-1}\right]^{-1/(n-1)} \qquad U^{\mu}(-V)^{\nu}=\frac{n-1}{N-1} \ , \tag{31}$$

where

$$\mu=\frac{1}{n-1}\left[1-\frac{n}{a}\right] \ , \tag{32}$$

and

$$\nu=\frac{1}{n-1}\left[1+\frac{n}{a}\right] \ . \tag{33}$$

Along this trajectory $dr\propto R\propto \mathcal{N}t^{-1/2}dt$, where t is the cosmic time of Eq. (1).

One can determine the evolution of linear perturbations about the solution (31) of Eqs. (4)–(6), or equivalently, the geodesic deviations along the timelike geodesic (31) in the superspace with the metric (11).[22] If $n+N<9$ the perturbations satisfy

$$\ln r=\tau-a\exp\left[-\frac{n+N-1}{2}\tau\right]\times\sin\left[\pm\frac{(n+N-1)^{1/2}(9-n-N)^{1/2}}{2}\tau\right] \ , \tag{34}$$

and

$$\ln R=\tau+\frac{1}{2}\ln\frac{N-1}{n-1}+b\exp\left[-\frac{n+N-1}{2}\tau\right]\times\sin\left[\pm\frac{(n+N-1)^{1/2}(9-n-N)^{1/2}}{2}\tau\pm\zeta\right] \ , \tag{35}$$

where

$$\zeta=\arctan\left[2\frac{(n+N-1)^{1/2}(9-n-N)^{1/2}}{3N+n-3}\right] \ , \tag{36}$$

and a and b are small constants:

$$b=2n[(3N+n-3)^2+(9-n-N)(n+N-1)]^{1/2}a \ . \tag{37}$$

If $n+N\geq 9$, the perturbations are

$$\ln r = \tau + a(e^{\omega_1 \tau} - e^{\omega_2 \tau}) , \tag{38}$$

and

$$\ln R = \tau + \frac{1}{2}\ln\frac{N-1}{n-1} + a(k_1 e^{\omega_1 \tau} - k_2 e^{\omega_2 \tau}) , \tag{39}$$

where

$$\omega_{1,2} = -\tfrac{1}{2}[n+N-1 \pm (n+N-9)^{1/2}(n+N-1)^{1/2}] , \tag{40}$$

and

$$k_{1,2} = -n(2N+n-2+\omega_{1,2})^{-1} . \tag{41}$$

If $n+N=9$ there exists also another solution different from the one given by Eqs. (38) and (39). It dominates at late τ and can be obtained from Eqs. (38) and (39) by replacing $a \to a\tau$.

On the other hand, one can obtain approximate solutions far away from the focal point e [i.e., at $r^2R^{-2} \gg (n-1)(N-1)^{-1}$ or $r^2R^{-2} \ll (n-1)(N-1)^{-1}$] by using the results of the preceding section. In particular, close to the line $V=0$ in the minisuperspace $r^2R^{-2} \ll 1$, and one may neglect the curvature of the N-dimensional subspace. The trajectories in that regime are straight lines in the UV plane, and the solutions (21) and (22) yield

$$R^N = A\,\tanh^a\frac{(n-1)\tau}{2} , \tag{42}$$

$$r^{n-1} = BR^{-N}\sinh\frac{(n-1)\tau}{2} . \tag{43}$$

Close to the $U=0$ line in the UV plane, where $r^2R^{-2} \gg 1$, one can neglect the curvature of the n-dimensional subspace. The solutions in this regime are

$$R^{N-1} = Br^{-n}\sinh\frac{(N-1)\tau'}{2} , \tag{44}$$

$$r^n = A\,\tanh^a\frac{(N-1)\tau'}{2} , \tag{45}$$

with τ' defined by

$$\frac{\sigma}{\sqrt{2}}\mathcal{N}\,dt = R\,d\tau' . \tag{46}$$

To obtain a more complete approximation, one should match the solutions (34) and (35) or (38) and (39) with either Eqs. (42) and (43) or (44) and (45), depending on the number of spatial dimensions and the initial conditions for the trajectory, respectively.

As far as the Wheeler-DeWitt equation (12) is concerned, no exact solutions are readily at hand. However, if the number of space-time dimensions is not less than ten, the classical trajectories do not oscillate and one may hope to obtain useful solutions to Eq. (12) by applying the Wentzel-Kramers-Brillouin (WKB) approximation. We will proceed a little further and find a general solution to the wave equation as a series of terms with increasing power of M^{-1}, for the moment assumed to be a small parameter in Eq. (12).

Let

$$\xi = \ln[n(n-1)^{-1}r^{2(n-1)}R^{2N}] = \ln(-UV) , \tag{47}$$

and

$$\eta = 2\ln(rR^{-1}) = \ln\left[\frac{n}{n-1}\right]^{-1/(n-1)} U^{\mu}(-V)^{\nu} , \tag{48}$$

with μ and ν defined in Eqs. (32) and (33). The Wheeler-DeWitt equation (12) is now

$$\left[\frac{\partial^2}{\partial\xi^2} + (\mu+\nu)\frac{\partial^2}{\partial\xi\partial\eta} + \mu\nu\frac{\partial^2}{\partial\eta^2} + \frac{M^2}{4}e^{\xi}(1+|\alpha|e^{\eta})\right]\psi = 0 . \tag{49}$$

Assume that the wave function may be represented as a series

$$\psi = \exp(iMS_0 + S_1 + M^{-1}S_2 + \cdots + M^{-p}S_{p+1} + \cdots) . \tag{50}$$

If one demands that all terms of order M^{-p}, up to some $p=l>1$, vanish separately in Eq. (49), one gets the following system of equations:

$$4\left[\left[\frac{\partial S_0}{\partial\xi}\right]^2 + (\mu+\nu)\frac{\partial S_0}{\partial\eta}\frac{\partial S_0}{\partial\xi} + \mu\nu\left[\frac{\partial S_0}{\partial\eta}\right]^2\right] - e^{\xi}(1+|\alpha|e^{\eta}) = 0 , \tag{51}$$

$$\frac{\partial^2 S_0}{\partial\xi^2} + \mu\nu\frac{\partial^2 S_0}{\partial\eta^2} + (\mu+\nu)\frac{\partial^2 S_0}{\partial\xi\partial\eta} + 2\frac{\partial S_0}{\partial\xi}\frac{\partial S_1}{\partial\xi} + 2\mu\nu\frac{\partial S_0}{\partial\eta}\frac{\partial S_1}{\partial\eta} + (\mu+\nu)\left[\frac{\partial S_0}{\partial\eta}\frac{\partial S_1}{\partial\xi} + \frac{\partial S_0}{\partial\xi}\frac{\partial S_1}{\partial\eta}\right] = 0 , \tag{52}$$

and $l-1$ equations

$$\frac{\partial^2 S_p}{\partial\xi^2} + \mu\nu\frac{\partial^2 S_p}{\partial\eta^2} + (\mu+\nu)\frac{\partial^2 S_p}{\partial\eta\partial\xi} + \sum_{k=1}^{p}\left[\frac{\partial S_k}{\partial\xi}\frac{\partial S_{p+1-k}}{\partial\xi} + \mu\nu\frac{\partial S_k}{\partial\eta}\frac{\partial S_{p+1-k}}{\partial\eta} + (\mu+\nu)\left[\frac{\partial S_k}{\partial\xi}\frac{\partial S_{p+1-k}}{\partial\eta} + \frac{\partial S_k}{\partial\eta}\frac{\partial S_{p+1-k}}{\partial\xi}\right]\right] + i\left[2\frac{\partial S_0}{\partial\xi}\frac{\partial S_{p+1}}{\partial\xi} + 2\mu\nu\frac{\partial S_0}{\partial\eta}\frac{\partial S_{p+1}}{\partial\eta} + (\mu+\nu)\left[\frac{\partial S_0}{\partial\xi}\frac{\partial S_{p+1}}{\partial\eta} + \frac{\partial S_0}{\partial\eta}\frac{\partial S_{p+1}}{\partial\xi}\right]\right] = 0 \tag{53}$$

for each p up to $p=l$.

Equation (51) is the Hamilton-Jacobi equation for the classical action S_0, and Eq. (52) determines the WKB prefactor $\exp(S_1)$. Equations (52) and (53) are linear in the highest-index function S_{p+1}. Once Eq. (51) is solved, one can (in principle, of course) solve consecutively all of Eqs. (53), starting with Eq. (52), provided appropriate initial conditions are imposed on each $S_p, l>p>0$. A particular example is presented later.

Now we discuss the solutions to the Hamilton-Jacobi equation (51). One can guess a one-parameter family of solutions (as opposed to a two-parameter general solution) of the form

$$S_0=e^{\xi/2}f(\eta) \; . \tag{54}$$

Then the function $f(\eta)$ satisfies

$$-4\frac{n+N}{N(n-1)}f'^2+4\frac{1}{n-1}f'f+f^2-(1+|\alpha|e^{\eta})=0 \; . \tag{55}$$

Asymptotically, $f(\eta)$ behaves like

$$f(\eta)\to\pm 1, \quad \eta\ll -1 \; , \tag{56}$$

and

$$f(\eta)\sim\pm\frac{N}{n}e^{\eta/2}, \quad \eta\gg 1 \; . \tag{57}$$

This behavior is what one would expect for the Hartle-Hawking wave function, which should be approximately given by Eq. (26) for $\eta\ll -1$, and by

$$\psi=J_0[M'U^{\mu+1}(-V)^{\nu+1}] \; , \tag{58}$$

with

$$M'^2=|\alpha|\left[\frac{n-1}{n}\right]^{1/(n-1)}(\mu+1)^{-1}(\nu+1)^{-1}M^2 \; , \tag{59}$$

at $\eta\gg 1$.

No general solutions to Eq. (55) are known to the author. This equation can be brought into the form

$$y'-\frac{F'}{F}y^3+\left[\pm 1-3\frac{F'}{F}\right]y^2+\left[\pm 2-2\frac{F'}{F}\right]y=0 \; , \tag{60}$$

by using the substitutions

$$y=\coth\left\{\operatorname{arcosh}\left[\exp\left[-\frac{N\eta}{2(n+N)}\right]fF^{-1}\right]\right\}-1 \tag{61}$$

and

$$F=-\frac{(n-1)(n+N)+N^2}{4(n+N)^2}\times(e^{N\eta/(n+N)}+|\alpha|e^{[(2N+n)/(n+N)]\eta}) \; , \tag{62}$$

with prime denoting differentiation with respect to $2[N^2+(n+N)(n-1)]^{1/2}(n+N)^{-1}\eta$. Equation (60) is an Abel's equation of the first kind.[27,28] There are several specific cases in which a solution to this equation is known. Unfortunately, Eq. (60) does not belong to any of these classes. From now on we assume that the solution to Eq. (55) is known, obtained either by numeric integration or by choosing a particular specific solution.

If the initial conditions for Eqs. (52) and (53) are

$$S_k(\xi=\xi_0,\eta)=\phi_k(\eta) \; , \tag{63}$$

then one may solve Eqs. (52) and (53) recursively,

$$\begin{aligned}S_k(\xi,\eta)&=L_k(\xi-K(\eta);\eta)\\&\quad -L_k[\xi-K(\eta);K^{-1}(\xi_k-\xi+K(\eta))]\\&\quad +\phi_k[K^{-1}(\xi_k-\xi+K(\eta))] \; ,\end{aligned} \tag{64}$$

along the classical trajectories of the Hamilton-Jacobi equation (55):

$$K(x)\equiv\int_{\eta_0}^{x}d\eta\frac{f+(\mu+\nu)f'}{2\mu\nu f'+\frac{1}{2}(\mu+\nu)f}=\xi(x)-\xi_0 \; . \tag{65}$$

The function $L(\;;\;)$ in Eq. (64) is defined as

$$L_k(x;y)\equiv\int_{K(\eta_0)}^{K(y)}R_k(x+K(\theta);\theta)d\theta \tag{66}$$

and $R_{k+1}(\;,\;)$ is determined recursively by all S_p, $p<k+1$,

$$\begin{aligned}R_{k+1}(\xi,\eta)=i[f+(\mu+\nu)f']^{-1}e^{-\xi/2}\Bigg\{&\frac{\partial^2 S_k}{\partial\xi^2}+\mu\nu\frac{\partial^2 S_k}{\partial\eta^2}+(\mu+\nu)\frac{\partial^2 S_k}{\partial\xi\partial\eta}\\&+\sum_{p=1}^{k}\Bigg[\frac{\partial S_p}{\partial\xi}\frac{\partial S_{k+1-p}}{\partial\xi}+\mu\nu\frac{\partial S_p}{\partial\eta}\frac{\partial S_{k+1-p}}{\partial\eta}\\&+(\mu+\nu)\left[\frac{\partial S_p}{\partial\xi}\frac{\partial S_{k+1-p}}{\partial\eta}+\frac{\partial S_p}{\partial\eta}\frac{\partial S_{k+1-p}}{\partial\xi}\right]\Bigg]\Bigg\} \; ,\end{aligned} \tag{67}$$

with

$$R_1(\xi,\eta)=-\frac{\frac{1}{4}f+\mu\nu f''+\frac{1}{2}(\mu+\nu)f'}{f+(\mu+\nu)f'} \; . \tag{68}$$

Now one can investigate the role of the assumption that the number of dimensions is not less than ten. It can be proved that, if $n+N<9$ and $\alpha\neq 0$, any function $f(\eta)$ solving Eq. (55) is necessarily multivalued. Thus neither $K(\;)$ nor $L_k(\;;\;)$ are single valued. Also, at the caustics, the integrand in Eq. (65) is singular, and $K(\;)$ is not defined. This is exactly because the WKB approximation breaks down near the caustics—all terms in the exponent in Eq. (50) are infinite and one may not consider the higher-order terms as small corrections.

If $n+N\geq 9$ all solutions to Eq. (55) are single valued. In this case one might consider the radius of convergence of the series (50). For simplicity, we compare the series with the known solutions to Eq. (12) for $\alpha=0$. Then one may choose a solution to Eq. (55):

$$f(\eta)=1\ . \tag{69}$$

Let $\phi_k(\eta)=0$ for all $k>0$. Then Eq. (64) gives

$$S_p=\frac{2}{1-p}R_p'(e^{[(1-p)/2]\xi}-e^{[(1-p)/2]\xi_0})\ , \tag{70}$$

with recursively defined R_p': $R_1'=-\frac{1}{4}$, and

$$R_p'=i\left[\frac{2-p}{2}R_{p-1}'+\sum_{k=1}^{p-1}R_k'R_{p-k}'\right]\ . \tag{71}$$

One can compare the limit for $\xi_0\to-\infty$ of the expression in Eq. (50) with S_p given by Eq. (70), to the exact solution, corresponding to the same boundary conditions,

$$\psi=H_0^1(Me^{\xi/2})\ , \tag{72}$$

where $H_0^1(\ )$ is the first Hankel function of zeroth order. Then, using formula 8.451 of Ref. 29 for the expansion of the Hankel functions one can show that the series (50) is, at $\xi\gg 1$, an asymptotic series of the solution (72) to Eq. (49). The optimum number of terms in the series of Eq. (50), giving the smallest approximation error, is roughly $p\approx M^{-1}e^{\xi/2}+2$. Then the smallest error between the function and its series is, very roughly,

$$\Delta=2p^{-1}i\left[\frac{1-p}{2}R_p'+\sum_{k=1}^{p}R_k'R_{p+1-k}'\right]e^{-p\xi/2}\ . \tag{73}$$

Also, by comparing term by term the series (50) and the asymptotic expansion of the solution (72) (again, formula 8.451 of Ref. 29) one can prove a nontrivial identity for the coefficients R_k',

$$\sum_{m=0}^{k}\frac{1}{m!}\sum_{\substack{q_1+\cdots+q_p=m\\ q_1+2q_2+\cdots+pq_p=k}}C_{q_1,\ldots,q_p}^{m,p}\prod_{j=1}^{p}\left[-\frac{2}{j}R_{j+1}'\right]^{q_j}=\left[\frac{-1}{2i}\right]^k\frac{\Gamma(k+1/2)}{\Gamma(-k+1/2)}\ , \tag{74}$$

where $C_{q_1,\ldots,q_p}^{m,p}$ are defined as the p-nomial coefficients:

$$\left[\sum_{j=1}^{p}x_j\right]^m=\sum_{q_1+\cdots+q_p=m}C_{q_1,\ldots,q_p}^{m,p}\prod_{j=1}^{p}(x_j)^{q_j}\ . \tag{75}$$

One can repeat an analogous discussion for the other independent solution of Eq. (49), $H_0^2(Me^{\xi/2})$, using $f(\eta)=-1$ instead of Eq. (69). A linear combination

$$\psi=\tfrac{1}{2}[H_0^1(Me^{\xi/2})+H_0^2(Me^{\xi/2})] \tag{76}$$

is the Hartle-Hawking ground-state solution (26). Thus, in the general case $k_N=-1$, one could argue that the ground-state solution has an asymptotic series which is a linear combination of two solutions (50), each evaluated with a different boundary condition on $f(\eta)$ allowed by Eq. (56) or, equivalently, by Eq. (57).

Unfortunately, the series (50) is not useful if $n+N<9$. From now on, we consider this case. Let

$$\xi'=\xi-\frac{\mu+\nu}{2\mu\nu}\eta\ , \tag{77}$$

$$\eta'=\frac{1}{\mu\nu}\eta\ , \tag{78}$$

and

$$\eta_0'=\frac{1}{\mu\nu}\ln\frac{n-1}{N-1}\ , \tag{79}$$

so that $\eta'=\eta_0'$ is the classical equilibrium trajectory (31). The Wheeler-DeWitt equation (12) or (49) now takes the form

$$\left[\left[1-\frac{(\mu+\nu)^2}{4\mu\nu}\right]\frac{\partial^2}{\partial\xi'^2}+\frac{1}{\mu\nu}\frac{\partial^2}{\partial\eta'^2}+\frac{M^2}{4}e^{\xi'}(e^{[(\mu+\nu)/2]\eta'}+|\alpha|e^{[(\mu+\nu)/2+\mu\nu]\eta'})\right]\psi=0\ . \tag{80}$$

The potential term in Eq. (80) considered as a function of η' has a minimum at $\eta'=\eta_0'$. This suggests a simple harmonic-oscillator potential approximation, valid for $|\eta'-\eta_0'|\ll 1$,

$$\left[\frac{\partial^2}{\partial\xi'^2}+C_3\frac{\partial^2}{\partial\eta'^2}+e^{\xi'}[C_1+C_2(\eta'-\eta_0')^2]\right]\psi=0\ , \tag{81}$$

with

$$C_1=\frac{M^2}{4}\frac{n+N}{n}\left[\frac{N-1}{n-1}\right]^{N/(n+N)}\left[1-\frac{(\mu+\nu)^2}{4\mu\nu}\right]^{-1}\ , \tag{82}$$

$$C_2=\frac{n}{2N(n-1)^2}C_1\ , \tag{83}$$

and

$$C_3=(\mu\nu)^{-1}\left[1-\frac{(\mu+\nu)^2}{4\mu\nu}\right]^{-1}\ . \tag{84}$$

At this point one[30,31] usually makes the Born-Oppenheimer approximation, neglecting the variations of the wave function as a function of one of the variables, solving the wave equation and then correcting for the changes of the variable initially fixed. In the pure-gravity case of Eq. (81) one may not neglect the dependence of ψ

on any variable, which makes the Born-Oppenheimer approximation useless. However, if we assume a quasiadiabatic separation $\psi = g(\xi')h(\xi',\eta')$, without assuming that h, compared to g, varies slowly with ξ', but requiring that h considered as a function of $(\eta' - \eta_0')$ is the standard harmonic-oscillator wave function, its frequency and the function $g(\xi')$ being fixed by Eq. (81), we get an approximate solution

$$\psi_l = N_l \exp[-\tfrac{1}{2}\Omega e^{\xi'/2}(\eta' - \eta_0')^2] H_l[\sqrt{\Omega} e^{\xi'/4}(\eta' - \eta_0')] \times \exp\left[i \left[2\sqrt{C_1} e^{\xi'/2} - \frac{C_3}{\sqrt{C_1}}(l + \tfrac{1}{2})\Omega\xi' + O(e^{-\xi'/2}) \right] \right] . \quad (85)$$

Here

$$N_l = (\pi^{1/2} 2^l l!)^{-1/2} , \quad (86)$$

$H_l(\,)$ are the Hermite polynomials and

$$\Omega = \tfrac{1}{4}\sqrt{C_1}\mu\nu \left[1 - \frac{(\mu+\nu)^2}{4\mu\nu} \right] \left[i \pm \left[\frac{9-n-N}{n+N-1} \right]^{1/2} \right] . \quad (87)$$

The solution (85) has the same accuracy as one would get if the Born-Oppenheimer approximation were appropriate. However, the frequency of the harmonic-oscillator wave function is not equal to the frequency of oscillations for fixed ξ'. Instead, it is given by Eq. (87). In the Born-Oppenheimer approximation the imaginary part of the frequency would have been suppressed by a large factor ($m_{\rm Planck}^{-2}$ in the gravity-scalar field model[30]).

Finally, the solutions (85) can be used to construct wave packets, for example, a generalization of the coherent (or classical) states of a harmonic oscillator,

$$\begin{aligned}\Psi &= \sum_{l=0}^{\infty} (l!)^{-1/2} a^l \psi_l \\ &= \pi^{1/4} \exp\left[-i\frac{C_3}{\sqrt{C_1}}\Omega\xi' + 2i\sqrt{C_1}e^{\xi'/2} - \tfrac{1}{2}\Omega e^{\xi'/2}(\eta'-\eta_0')^2 \right. \\ &\qquad \left. + 2\sqrt{\Omega} a \exp\left[\frac{\xi'}{4} - i\frac{C_3}{\sqrt{C_1}}\Omega\xi' \right](\eta'-\eta_0') - a^2 \exp\left[-2i\frac{C_3}{\sqrt{C_1}}\Omega\xi' \right] \right] ,\end{aligned} \quad (88)$$

where a is an arbitrary constant. The probability density $\Psi^*\Psi$ of the wave (88) considered as a function of η' is a Gaussian with a standard deviation

$$\sigma = \sqrt{2} |\Omega|^{-1} (\cos\phi e^{-\xi'/4})^{-1} , \quad (89)$$

centered at η' such that

$$\eta' - \eta_0' \propto a \cos\left[\frac{1}{4} \left[\frac{9-n-N}{n+N-1} \right]^{1/2} \xi' - \frac{\phi}{2} \right] , \quad (90)$$

where $\phi = \pm\arctan[(n+N-1)/(9-n-N)]^{1/2}$.

The results of (89) and (90) may be compared with the classical solutions (34) and (35), which give

$$\eta' - \eta_0' \propto a e^{-\xi'/4} \cos\left[\frac{1}{4} \left[\frac{9-n-N}{n+N-1} \right]^{1/2} \xi' + \phi' \right] , \quad (91)$$

ϕ' being a constant. Thus the center of the wave packet (88) oscillates with the same frequency as the classical trajectories do. However, the oscillations of the wave packet have a constant amplitude, whereas the amplitude of the classical perturbations vanishes exponentially.

The difference between the behavior of the wave packet and the classical solutions is due to the nonvanishing imaginary part of the frequency Ω in Eq. (87), and therefore is a result of the fact that the Born-Oppenheimer approximation may not be applied. In many cases the approximation is valid[31] and the frequency is usually real. But the second correction for the motion of the "heavier particles" as in Eq. (85) inevitably yields a complex frequency of oscillations. This causes the difference between the group velocity and the phase velocity even for a Gaussian packet such as Eq. (88). The difference becomes extreme for the pure gravity model, (90) and (91).

Although the concept of measurements in quantum gravity is far from being clear, it is likely that the results of observations are determined by semiclassical wave packets such as Eq. (88). Then the ratio of the scale factors should oscillate,

$$\frac{r}{R} \approx \left[\frac{n-1}{N-1} \right]^{1/2} \left\{ 1 + a' \cos\left[\tfrac{1}{2}\sqrt{(9-n-N)(n+N-1)} \ln\left[\int \mathcal{N} t^{-1/2} dt \right] + \text{const} \right] \right\} , \quad (92)$$

rather than tend to a limit of $[(n-1)/(N-1)]^{1/2}$ given by the purely classical considerations (91). The constant a' in Eq. (92) should be small in order not to break down the approximation (81) of Eq. (80). Then, the width of the wave packet (89) decreases exponentially, so that the trajectory of the wave packet should be easily distinguishable for large t, even after a large number of oscillations.

The oscillations can be traced back to early times where, however, our approximation (81) breaks down. Still, it might happen that even if this approximation is not valid, one of the two potential terms in Eq. (12) dominates over the other, and one may roughly use the analysis of the preceding section.

Again, one may investigate the role of the number of spatial dimensions. If this number were greater than nine, the effective frequency Ω would be purely imaginary, corresponding to an upside-down harmonic oscillator.[32]

If both spatial sections have curvature of opposite signs, the classical trajectory must lie within the dotted region of Fig. 1 of Ref. 12. In this case, comparing Figs. 1 and 2 of Ref. 12, and using the results of the preceding section, one may argue that only the trajectories from b' to b on Fig. 1 correspond to the Lorentzian solutions picked out by the Hartle-Hawking proposal. These trajectories have one classical turning point, corresponding to the maximum of the scale factor the "physical" space and minimum scale factor of the "internal" space.

V. CONCLUSIONS

We have analyzed a pure-gravity cosmological model with spatial sections being products of two maximally symmetric spaces. If neither of the subspaces has negative curvature, there exist no Lorentzian solutions.

If one of the subspaces is flat, and the other has negative curvature, the Hartle-Hawking ground-state proposal picks out a classical solution with maximal possible symmetry, which is the Minkowski space. The space-time is split into a product of constant curvature and flat spaces because of the specific choice of the boundary term in the action of canonical gravity.[3] This space can be considered as the "initial" state for the models with both spacial sections of negative curvature. In this case we found the classical trajectories, which, to our surprise, oscillate only if the number of space-time dimensions is less than ten—the number of dimensions already called critical for various other reasons, which seem to have little connection with our model.

We found approximate solutions to the Wheeler-DeWitt equation and (for $n+N<9$) used them to construct wave packets. The trajectories of the wave packets oscillate about a classically stable solution. The frequencies of oscillations of the classical trajectories and the trajectory of the packet are equal, but the time-dependent amplitudes are not. The wave packets can be distinguished even after a large number of oscillations as they remain well peaked for a large number of periods. This is exactly opposite to what happens in the gravity-scalar field model,[30] and is a direct consequence of the equal strength of coupling for all gravitational degrees of freedom. Concluding, the quantum-mechanical evolution of empty higher-dimensional Universes is qualitatively different from both their classical evolution and the evolution of the universes coupled to various matter fields. Thus, it is very unlikely that specific results do not depend on the minisuperspace truncation chosen for the model.

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[1] J. J. Halliwell, University of California Report No. NSF-ITP-88-131 (unpublished).
[2] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).
[3] J. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[4] J. J. Halliwell, Nucl. Phys. **B266**, 228 (1986).
[5] J. J. Halliwell, Nucl. Phys. **B286**, 729 (1987).
[6] Z. C. Wu, Phys. Lett. **146B**, 307 (1984).
[7] X. M. Hu and Z. C. Wu, Phys. Lett. **155B**, 237 (1985).
[8] U. Carow-Watamura, T. Inami, and S. Watamura, Class. Quantum Grav. **4**, 23 (1985).
[9] D. Sahdev, Phys. Rev. D **30**, 2495 (1984).
[10] E. Kolb, D. Lindley, and D. Seckel, Phys. Rev. D **30**, 1205 (1984).
[11] R. B. Abbott, S. M. Barr, and S. D. Ellis, Phys. Rev. D **30**, 720 (1984).
[12] P. Turkowski and K. Maślanka, Gen. Relativ. Gravit. **19**, 611 (1987).
[13] D. L. Wiltshire, Phys. Rev. D **36**, 1634 (1984).
[14] K. Maeda, Phys. Lett. B **186**, 33 (1987).
[15] L. M. Sokolowski and Z. A. Golda, Phys. Lett. B **195**, 349 (1987).
[16] P. G. O. Freund, Nucl. Phys. **B209**, 146 (1982).
[17] E. Alvarez and M. B. Gavela, Phys. Rev. Lett. **51**, 931 (1983).
[18] Q. Shafi and C. Wetterich, Nucl. Phys. **B289**, 787 (1987).
[19] Z. C. Wu, *Quantum Cosmology* (Cambridge University Press, Cambridge, England, in press).
[20] D. N. Page and S. W. Hawking, Nucl. Phys. **B264**, 185 (1986).
[21] D. N. Page, in *Quantum Concepts in Space and Time,* edited by R. Penrose and C. J. Isham (Clarendon, Oxford, 1976).
[22] B. S. DeWitt, in *Relativity,* proceedings of the Relativity Conference in the Midwest, edited by M. Carmeli, S. I. Fickler, and L. Witten (Plenum, New York, 1970).
[23] R. Gowdy, Phys. Rev. D **2**, 2774 (1970).
[24] D. N. Page (in preparation).
[25] O. Klein, Z. Phys. **37**, 895 (1926).
[26] P. Candelas and S. Weinberg, Nucl. Phys. **B237**, 397 (1984).
[27] H. Lemke, Publ. Math. (Belgrade) **4**, 201 (1935).
[28] R. Liouville, Acta Math. **27**, 55 (1903).
[29] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products,* 4th ed., edited by Yu. V. Geronimus and M. Yu. Tseytlin (Academic, New York, 1965).
[30] C. Kieffer, Phys. Rev. D **38**, 1761 (1988).
[31] R. Brout, G. Horwitz, and D. Weil, Phys. Lett. B **192**, 318 (1987).
[32] J. J. Halliwell, Phys. Rev. D **36**, 3626 (1987).