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Radiative corrections and semileptonic B decays

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A prescription for approximating electroweak radiative corrections to weak decays is given. The method is illustrated for $\tau \to e\nu\bar{\nu}$ and a simplified (structureless) model of $B \to Me\bar{\nu}$, $M=D$ or π, where the complete $O(\alpha)$ corrections are known. Our procedure is shown to provide a proper description of radiation damping near the electron's end-point energy and a reasonable estimate of radiative corrections for much of the spectrum as well as the integrated rate. As a practical application, it is applied to the semileptonic decays $B \to Xe\bar{\nu}$, where an exact $O(\alpha)$ treatment of radiative corrections is very difficult, but an estimate of their effect is important for the extraction of V_{ub} and leptonic branching ratios. We also discuss an 18% enhancement of $\Upsilon(4S) \to B^+B^-$ relative to $B^0\bar{B}^0$ due to large Coulomb corrections near threshold.

As weak-decay measurements reach high precision, it becomes important to include electroweak radiative corrections in the comparison of theory and experiment. For purely leptonic[1] and a few very special semileptonic decays (such as superallowed nuclear β decays[2]), a full analysis of the one-loop electroweak corrections in the standard model and QED bremsstrahlung is straightforward, but nontrivial. However, for most semileptonic weak decays, hadronic structure and other strong-interaction complications make a reliable lowest-order decay-rate analysis difficult and a complete $O(\alpha)$ calculation of electroweak corrections virtually impossible. Given that situation, we have devised a simple prescription for approximating some of the most important aspects of electroweak radiative corrections for the generic weak decay $Y \to Xe\bar{\nu}$. Here, we describe that technique, illustrate it by examples, and apply our prescription to semileptonic B decays. We also take this opportunity to discuss B^+B^- vs $B^0\bar{B}^0$ production at the $\Upsilon(4S)$ and show that the former is enhanced by about 18% due to large Coulomb-threshold corrections.

Before giving our prescription, we outline some of its virtues and shortcomings. (i) It incorporates leading-log short-distance loop corrections via a simple enhancement factor. (ii) Soft virtual- and real-photon corrections are summed to all orders following the well-known exponentiation formalism of Yennie, Frautschi, and Suura.[3] In that way, radiation damping near the electron spectrum end point as well as some QED modifications of the spectrum shape are properly described. (iii) An infrared exponentiation factor is introduced which helps normalize QED corrections to the high-energy electron spectrum and total decay rate. What our prescription does not do is attempt to incorporate hard-photon bremsstrahlung which mainly modifies the low-energy electron spectrum. That contribution is, in any case, very uncertain for semileptonic B decays due to hadronic structure effects. Also, we do not consider strong-interaction uncertainties at all, even though they are clearly important. Instead, we assume that a lowest-order decay rate which incorporates strong interactions via form factors or perturbative QCD is given, and our task is to include electroweak radiative corrections.

We begin our discussion by considering the short-distance leading-log electroweak corrections to the generic decay $Y \to Xe\bar{\nu}$, where Y and X may be hadrons or leptons and X may be a single- or multiparticle state. By short distance, we mean virtual loop effects coming from high frequencies $\gtrsim m_Y$. In the standard $SU(2)_L \times U(1)$ model, such corrections are finite and calculable. Employing a current-algebra approach, Sirlin[4] has carried out a general analysis. He showed that when weak decay rates are expressed in terms of the muon decay constant G_μ defined by the muon lifetime[5,6]

$$\tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3}\left(1-8\frac{m_e^2}{m_\mu^2}\right)\left[1+\frac{\alpha(m_\mu)}{2\pi}\left(\frac{25}{4}-\pi^2\right)\right], \tag{1a}$$

$$G_\mu = (1.166\,37 \pm 0.000\,02)\times 10^{-5}\ \mathrm{GeV}^{-2}, \tag{1b}$$

they are enhanced by a short-distance correction factor[4]

$$1+\frac{3\alpha}{2\pi}(1+2\bar{Q})\ln\frac{m_Z}{m_Y}, \tag{2}$$

where $\bar{Q}$ is the average quark or lepton doublet electric charge (of the fundamental decay isodoublet pair) and $m_Z = 91$ GeV. For standard quark doublets $\bar{Q} = \frac{1}{2}(\frac{2}{3} - \frac{1}{3}) = \frac{1}{6}$, while for leptons $\bar{Q} = -\frac{1}{2}$. One can go beyond the $\alpha\ln(m_Z)$ correction in (2) by summing up all leading logs of the form $\alpha^n \ln^n m_Z$ via the renormalization group[7] and even include perturbative QCD corrections to that factor.[2] However, those additional modifications are not significant at the level of our considerations, so we do not include them. The first step in our prescription is, therefore, to reexpress all lowest-order weak-decay rates in terms of G_μ and for semileptonic processes to include an enhancement factor

$$1+\frac{2\alpha}{\pi}\ln\frac{m_Z}{m_Y}. \tag{3}$$

For purely leptonic decays such as $\tau \to e\nu\bar{\nu}$, which we subsequently consider, $\bar{Q} = -\frac{1}{2}$ in (2), so there is no enhancement factor.

Below the scale m_Y, additional electroweak corrections are of QED origin and nominally $O(\alpha)$; but they can be significantly enhanced by infrared logs and collinear singularities. The first of these arise from low-frequency (-energy) photonic virtual loop corrections and bremsstrahlung effects. Following the Bloch-Nordsieck[8] prescription of adding real and virtual soft-photon contributions, infrared divergences cancel order by order in perturbation theory. In fact, as shown by Yennie, Frautschi, and Suura,[3] virtual and real soft-photon corrections separately exponentiate and completely cancel at that level. Exponentiation, thereby, eliminates infrared divergences and introduces a residual electron-spectrum radiation-damping factor of the form[3,9,10]

$$(E_{\max}-E_e)^{(2\alpha/\pi)\{(1/2\beta)\ln[(1+\beta)/(1-\beta)]-1\}}, \tag{4}$$

where $E_{\max}$ is the electron end-point energy and $\beta=|P_e|/E_e$ is the electron velocity. Missing from (4) is a mass scale. That quantity is not specified by exponentiation and is thus ambiguous. We choose to employ a sliding energy scale via the following radiation-damping factor in the differential decay rate (and use $\beta \simeq 1-\frac{1}{2}m_e^2/E_e^2$):

$$\left(\frac{E_{\max}-E_e}{cE_e}\right)^{(2\alpha/\pi)[\ln(2E_e/m_e)-1]}, \tag{5}$$

where some arbitrariness still resides in a constant c, to be subsequently specified. By introducing the electron energy E_e in the denominator of (5), rather than a fixed mass such as m_Y, we suppress the high-energy electron spectrum but actually enhance the low-energy decay rate. That properly mimics the fact that bremsstrahlung effects tend to shift high-energy electrons to lower energy and thus change the spectrum shape without significantly modifying the total integrated decay rate. The second step in our prescription, therefore, entails multiplying the lowest-order differential decay rate $d\Gamma^0/dE_e$ by the factor in (5).

To be more specific, we need to fix the constant c in (5). That parameter does not significantly affect the electron spectrum shape, but does give an overall normalization. If possible, it should be fixed experimentally by a best fit to the data. Otherwise, we propose to pick c such that the factor in (5) becomes unity at the average electron energy $\bar{E}_e$ as determined by the lowest-order spectrum

$$\bar{E}_e=\int E_e\,d\Gamma^0\Big/\int d\Gamma^0. \tag{6}$$

That requirement gives

$$c=\frac{E_{\max}-\bar{E}_e}{\bar{E}_e}, \tag{7}$$

which means the spectrum is suppressed for $E_e>\bar{E}_e$ and enhanced for $E_e<\bar{E}_e$, but the total decay rate is not significantly modified by (5).

Before illustrating our prescription, we mention one further correction factor that should be applied if the initial state Y in $Y\to Xe\bar{\nu}$ is neutral. In that case, there is an additional static Coulomb interaction between the charged X and e final states.[11] At the high-energy end of the electron spectrum, that long-distance effect gives rise to an overall Coulomb-correction factor of approximately

$$1+\pi\alpha, \tag{8}$$

which represents about a 2.3% enhancement. It differentiates for example B^0 and $B^{\pm}$ semileptonic decays. If a decay ensemble contains an admixture of B^0 and $B^{\pm}$ decays as, for example,[12] in the case of data taken from $\Upsilon(4S)\to B^0\bar{B}^0$ or B^+B^-, the correction factor in (8) should be weighted by the percentage of $B^0\bar{B}^0$. We later briefly discuss what fraction is expected on theoretical grounds.

As a first illustration, we consider the decay $\tau\to e\nu\bar{\nu}$. For an unpolarized τ decaying at rest, the lowest-order differential decay rate is given by (neglecting m_e^2/m_τ^2 and m_τ^2/m_W^2 effects)

$$\frac{d\Gamma(\tau\to e\nu\bar{\nu})}{dE_e}=\frac{G_F^2 m_\tau}{12\pi^3}(3m_\tau E_e^2-4E_e^3), \tag{9}$$

or, employing the dimensionless variable $x=E_e/E_{\max}$, $E_{\max}=m_\tau/2$,

$$\frac{d\Gamma^0}{dx}=\frac{G_F^2 m_\tau^5}{192\pi^3}x^2(6-4x). \tag{10}$$

Following our prescription, we estimate the electroweak corrections as follows. G_F is replaced by G_μ (there is no short-distance enhancement). The radiation-damping factor in (5) is applied with $c=\frac{3}{7}$, since $\bar{x}=0.7$ for the spectrum in (10). In that way we obtain the "approximate" decay spectrum

$$\frac{d\Gamma}{dx}=\frac{G_\mu^2 m_\tau^5}{192\pi^3}x^2(6-4x)\left(\frac{1-x}{3x/7}\right)^{(2\alpha/\pi)[\ln(m_\tau x/m_e)-1]}. \tag{11}$$

To see how well our prescription works, we can compare with the complete $O(\alpha)$ calculation of radiative corrections by Kinoshita and Sirlin and Berman:[5,13]

$$\frac{d\Gamma}{dx}=\frac{G_\mu^2 m_\tau^5}{192\pi^3}x^2\left[6-4x+\frac{\alpha}{\pi}f(x)\right],$$

$$f(x)=(6-4x)R(x)+(6-6x)\ln x+\frac{1-x}{3x^2}\left[(5+17x-34x^2)\ln\left(\frac{m_\tau}{m_e}x\right)-22x+34x^2\right], \tag{12}$$

$$R(x)=-2\mathrm{Li}(x)-\frac{\pi^2}{3}-2+\left[\frac{3}{2}+2\ln\frac{1-x}{x}\right]\ln\left(\frac{m_\tau}{m_e}\right)-(2\ln x-1)\ln x+\left[3\ln x-1-\frac{1}{x}\right]\ln(1-x).$$

Despite the difference in appearance, the spectra in (11) and (12) are actually quite similar. That is illustrated in Fig. 1, where both are compared with the lowest-order result. Our approximation in (11) actually gives a better representation of the corrections near $x=1$ since it contains an all-orders summation. Of course, one can overcome the shortcomings of (12) near $x=1$ (it diverges) by exponentiating the singular part of the correction.[6,9,14]

As a further illustration of the good agreement between (11) and (12), we give in Fig. 2 a plot of their ratio as a function of x. Our method fails at small x because it misses hard-photon contributions that populate the low-energy spectrum; otherwise, the agreement is very good.

In the case of $\tau \to \mu\nu\bar{\nu}$, radiative corrections are not as important. To illustrate that point, we compare in Fig. 3 the lowest-order decay rate in (10) with the corrected spectrum in (12) as well as the corrected spectrum for $\tau \to \mu\nu\bar{\nu}$ obtained by replacing m_e with m_μ. It is clear that care must be taken in comparing electron and muon spectra, since the latter is not as sensitive to radiative corrections.

As our next example, we consider the decays $B \to Me\bar{\nu}$, where M is either a π or D meson. Since this exercise is meant only to test our prescription, we study a somewhat simplified model in which the initial- and final-state mesons are considered pointlike. That neglect of hadronic structure is not a bad approximation near the electron end point, but it clearly misrepresents the low-energy spectrum.

Adopting that approximation, the lowest-order differential decay rate for $B \to Me\bar{\nu}$ is given by (neglecting electron mass effects)[15]

$$\frac{d\Gamma^0(B \to Me\bar{\nu})}{dx} = \frac{G_F^2 m_B^5}{32\pi^3} |V_{ib}|^2 \eta^5 \frac{x^2(1-x)^2 |f_+^M|^2}{1-\eta x}, \tag{13}$$

where $x = E_e/E_{\max}$, $E_{\max} = (m_B^2 - m_M^2)/2m_B$, $\eta = 1 - m_M^2/m_B^2$, V_{ib} is the Cabibbo-Kobayashi-Maskawa matrix element (V_{ub} for $M=\pi$, V_{cb} for $M=D$), and f_+^M is a form factor which should have x dependence, but we take it to be constant (structureless). Our notation and normalization are borrowed from K_{e3} decays where $f_+(0) \simeq 0.98$. Estimates of f_+^π for $B \to \pi e\bar{\nu}$ give a smaller value[16] $\simeq 0.27$.

Our prescription for approximating radiative corrections is to multiply (13) by the correction factors in (3) (with $m_Y = m_B$) and (5). In addition, for $\eta \simeq 1$ our condition on c gives $c \simeq \frac{2}{3}$, so we have[17]

$$\frac{d\Gamma(B \to Me\bar{\nu})}{dx} = \frac{G_\mu^2 m_B^5}{32\pi^3} |V_{ib}|^2 |f_+^M|^2 \eta^5 \frac{x^2(1-x)^2}{1-\eta x} \left(1 + \frac{2\alpha}{\pi} \ln\frac{m_Z}{m_B}\right) \left(\frac{1-x}{2x/3}\right)^{(2\alpha/\pi)[\ln(m_B x/m_e) - 1]} \tag{14}$$

for charged B decays, while for neutral B decays, an additional $1+\pi\alpha$ correction factor should be appended.[11] For comparison, we can use a complete $O(\alpha)$ calculation of radiative corrections for this simple model by Ginsberg.[15] Such a comparison is illustrated in Fig. 4 where Ginsberg's result has been (arbitrarily) normalized to agree with (14) at $x=0.6$. One formula does quite well in correctly describing the high-energy electron spectrum shape. Because of strong-interaction uncertainties we do not worry about the low-x regime.

Having used two examples to illustrate and test our scheme, we now tackle a practical problem, radiative corrections to inclusive semileptonic decays $B \to X_c e\bar{\nu}$ and $B \to X_u e\bar{\nu}$, where X_q represents an inclusive hadron state containing q. A precise knowledge of the electron spectrum shape, particularly near the end point, is important for extracting V_{ub}/V_{cb} and measuring semileptonic branching ratios. In our approach, the radiative corrections are approximated by the same correction factors as

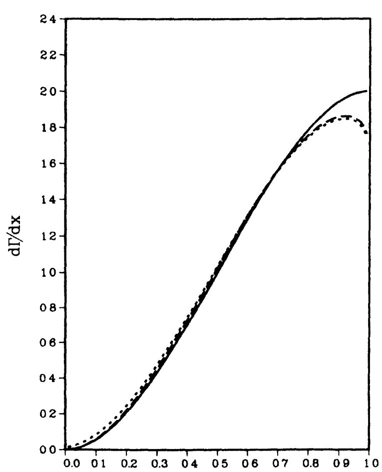

FIG. 1. $d\Gamma/dx$ is shown for Eq. (10) (solid curve), Eq. (11) (long dashes), and Eq. (12) (short dashes). The results are given in units of $G_\mu^2 m_\tau^5/192\pi^3$.

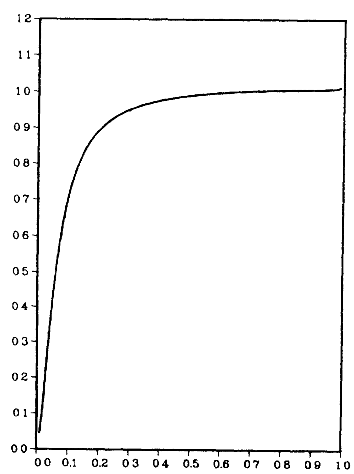

FIG. 2. The ratio between Eqs. (11) and (12) is shown as a function of x.

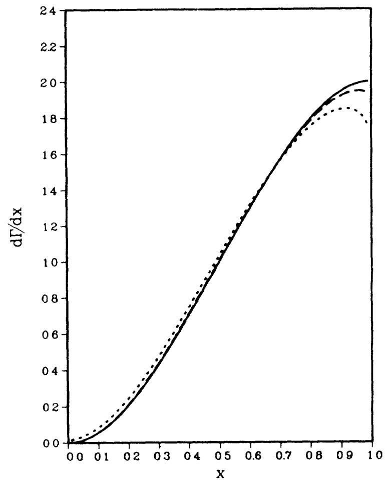

FIG. 3. $d\Gamma/dx$ is shown as a function of x for the Born term of $\tau \to e\bar{\nu}_e\nu_\tau$ (solid curve); the first-order correction given by Eq. (12) (short dashes) and the similar first-order correction applied to $\tau \to \mu\bar{\nu}_\mu\nu_\tau$ (long dashes). The results are in units of $G_\mu^2 m_\tau^5/192\pi^3$.

in (14), but with c as yet undetermined,

$$\left[1+\frac{2\alpha}{\pi}\ln\frac{m_Z}{m_B}\right]\left[\frac{1-x}{cx}\right]^{(2\alpha/\pi)[\ln(m_B x/m_e)-1]}, \qquad (15)$$

and an additional factor of $(1+\pi\alpha)$ for neutral-B decays.

One strategy for employing (15) is to take $c \simeq \frac{2}{3}$ and apply (15) (actually its inverse) as a correction to the data. The corrected data can then be compared with lowest-order model predictions for the spectra. Alternatively, theoretical predictions can be modified by the factor in (15) and then compared with experiment. To illustrate the latter possibility we consider realistic spectra for $B \to X_u e\bar{\nu}$ and $B \to X_c e\bar{\nu}$.

For the case $B \to X_u e\bar{\nu}$, the free-quark-model calculation of Altarelli *et al.*,[18] $b \to u e\bar{\nu}$ with QCD and Fermi-motion corrections seems most appropriate, particularly near the electron end point. On the other hand, semileptonic B decays into charm are known to be dominated by D, D^*, and D^{**} resonances.[12] So, for $B \to X_c e\bar{\nu}$ we use the Isgur *et al.*[19] resonance model as a good representation of the lowest-order spectrum. Correcting those lowest-order differential decay rates with the factor in (15) leads to the modifications illustrated in Figs. 5 and 6.

Radiative corrections are particularly important for $B \to X_c e\bar{\nu}$ near the electron spectrum end point. Since that region can be a background for extracting V_{ub} from $B \to X_u e\bar{\nu}$ data, it is imperative that radiative corrections be included. To be more specific, the decay $B \to \pi e\bar{\nu}$ has an electron end-point energy of 2.64 GeV, while $B \to De\bar{\nu}$ ends at 2.31 GeV. However, if we restrict the data to 2.31–2.64 GeV, the statistics are quite limited. In addition, detector resolution can lead to $B \to X_c e\bar{\nu}$ background even in that region. Therefore, the experimenter must have a good estimate of both spectra near their end points to work with. Also, to precisely determine the semileptonic branching ratios for $B \to De\bar{\nu}$, $B \to D^* e\bar{\nu}$, or $B \to D^{**} e\bar{\nu}$, data must be fitted near the high-energy electron spectra to eliminate secondary electrons from charm decays. Accurate measurements necessitate inclusion of radiative corrections.

Precision B-decay studies require high statistics. They are, therefore, generally carried out at e^+e^- facilities on the $\Upsilon(4S)$ resonance. Its mass, 10580 MeV, is just above $B\bar{B}$ threshold, so decays into B^+B^- and $B^0\bar{B}^0$ are kinematically allowed. In fact, the relatively large $\Upsilon(4S)$ width of 24 MeV suggests that those two modes essentially saturate its decays. To analyze and interpret the data often requires knowledge of the relative B^+B^- and $B^0\bar{B}^0$

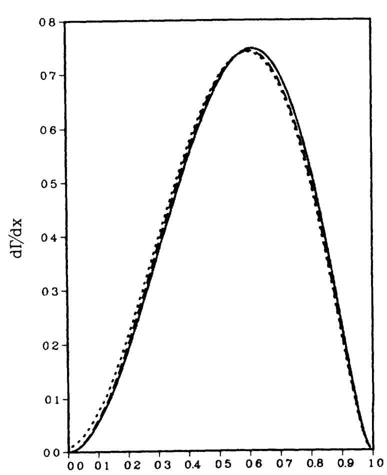

FIG. 4. $d\Gamma/dx$ for $B \to Me\bar{\nu}_e$ given by Eq. (13) is shown (solid curve) as well as the estimated correction of Eq. (14) (long dashes) and the calculated correction from Ref. 15 (short dashes). The results are given in units of $(G_\mu^2 m_B^5/192\pi^3) \times |V_{ib}|^2 |f_+^M|^2$.

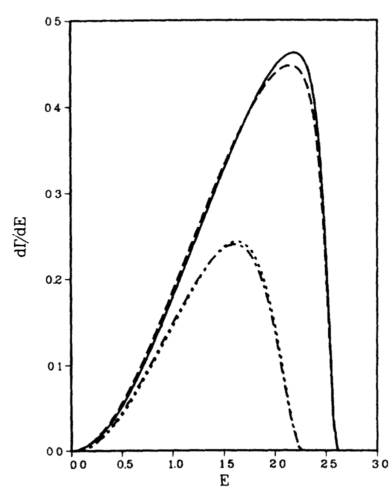

FIG. 5. $d\Gamma/dE$ is shown for $B \to X_u e\bar{\nu}$ using the model of Ref. 8 (solid curve); as well as the corresponding radiative corrected spectrum (long dashes). $d\Gamma/dE$ is also shown for $B \to X_c e\bar{\nu}$ using the model of Ref. 12, (short dashes) and the corresponding corrected spectrum (dashed-dotted curve). All curves are in units of $(G_\mu^2 m_B^4/24\pi^3)\,|V_{ib}|^2$.

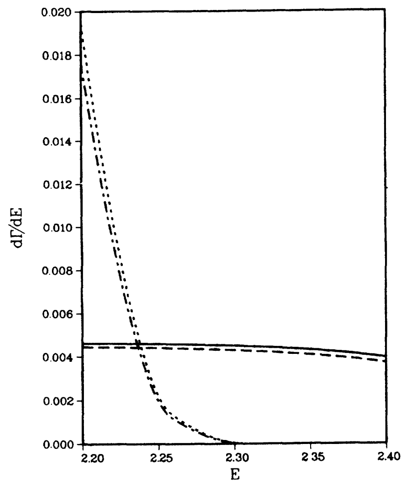

FIG. 6. As an illustrative example, we show $d\Gamma/dE$ near the end point for $B \to X_u e\nu$ (solid curve, uncorrected; dashed curve, corrected) and $B \to X_c e\nu$ (short dashes, uncorrected; dashed-dotted curve, corrected) using the same models as in Fig. 5. Here we take $(|V_{ub}|/|V_{cb}| \simeq 0.1)$ and give results in units of $(G_\mu^2 m_B^5/24\pi^3)|V_{ub}|$.

production rates on resonance. For the recently measured mass difference[20] $m_{B^+} - m_{B^0} \simeq -0.4 \pm 0.6$ MeV, which is consistent with zero, one expects both decays to have similar kinematic and P-wave suppressions and, hence, approximately equal branching ratios. However, there is an important difference which seems to have been overlooked. Coulomb corrections to B^+B^- production near threshold are quite large. In fact, they enhance that $\Upsilon(4S)$ decay rate relative to $B^0\bar{B}^0$ by a factor[21]

$$\left[1+\frac{\pi\alpha}{2\beta}\right], \tag{16}$$

where

$$\beta = \left[1-\frac{4m_B^2}{m_{\Upsilon(4S)}^2}\right]^{1/2} \simeq 0.065\,. \tag{17}$$

Numerically, (16) implies an 18% enhancement of $\Upsilon(4S) \to B^+B^-$. That means, if there are no other important differences (such as B^+-B^0 mass splittings[20] or isospin-violating form factors in the decay amplitude),[22]

$$\frac{\Gamma(\Upsilon(4S)\to B^+B^-)}{\Gamma(\Upsilon(4S)\to B^0\bar{B}^0)} \simeq 1.18\,, \tag{18}$$

which implies branching ratios of approximately 54% and 46% for B^+B^- and $B^0\bar{B}^0$, respectively.

In conclusion, we note that B physics has become a mature subject. High-precision requirements mandate the inclusion of electroweak radiative corrections both for B production as well as decay. Fortunately, the dominant effects are easily estimated and, if necessary, can be further improved. They should, therefore, not hinder the unraveling of b phenomenology and whatever surprises it may hold.

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[22] We do expect additional isospin violations in the $\Upsilon(4S)$ decay form factors from Coulomb effects and they are likely to suppress B^+B^- relative to $B^0\bar{B}^0$. We do not, however, expect them to be large. Also, an updated coupled-channel analysis including Coulomb effects should be undertaken. See E. Eichten, K. Gottfried, K. Lane, T. Kinoshita, and T.-M. Yan, Phys. Rev. D **17**, 3090 (1980); **21**, 313(E) (1980); A. D. Martin and C-K. Ng, Z. Phys. C **40**, 133 (1988).