Chiral phase transitions in QCD for finite temperature and density

A. Barducci
Dipartimento di Fisica, Università di Firenze, I-50125 Firenze, Italy
(INFN), Sezione di Firenze, I-50125 Firenze, Italy and Istituto Nazionale di Fisica Nucleare

R. Casalbuoni
Dipartimento di Fisica, Università di Lecce, I-73100 Lecce, Italy,
Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Lecce, I-73100 Leece, Italy
and Département de Physique de Théorique, Universite de Genève CH-1211 Genève 4, Switzerland

S. De Curtis
Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Firenze, I-50125 Firenze, Italy

R. Gatto
Département de Physique de Théorique, Universite de Genève, CH-1211 Genève 4, Switzerland

G. Pettini
Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Firenze, I-50125 Firenze, Italy
and Dipartimento di Fisica, Università di Pisa, I-56100 Pisa, Italy
(Received 19 October 1989)

We study the diagram for the chiral phases of QCD on the plane of temperature and chemical potential. The calculation is based on the extension of a composite-operator formalism already employed for zero temperature and density. We find that, moving along the critical line for increasing chemical potential, one encounters a point separating second-order and first-order chiral phase transitions. The overall phase diagram is found to qualitatively reproduce the pattern for the Gross-Neveu model. Our study suggests the occurrence of metastable phases.

I. INTRODUCTION

Field theories with spontaneously broken symmetries are generally expected to exhibit symmetry restoration for large temperature and/or chemical potential.[1,2] In particular, special interest has been focused on the study of the phase diagram of QCD in the (λ, T) plane of chemical potential λ and temperature T (Ref. 3), because of its possible relevance to high-energy heavy-ion collisions and to the physics of the early Universe.[4] For the standard model to be consistent with experiment, it is essential that chiral symmetry be spontaneously broken at $T=\lambda=0$. On the other hand, for high temperatures and/or densities, asymptotic freedom suggests that the interaction among quarks and gluons becomes weak, so that chiral symmetry is expected to be restored.[5] Lattice results about the phase transition have been contradictory, especially concerning its nature.[6] Recent results suggest a first-order transition for small current quark masses.

There is therefore a need for alternative studies, by employing a different approach to the problem, in which ones tries to incorporate the main known features of QCD and which allow one to evaluate analytically quantities such as the fermionic condensate as a function of T and λ. Such a dependence might be tested in heavy-ion collisions at high energies. In fact, the existence of critical temperature and chemical potential may already be suggested by certain high-energy cosmic-ray events.[7]

In the present paper we will provide some analytic insight into the mechanism of chiral-symmetry breaking (CSB) for a QCD-like theory at finite temperature and density.[8] The analysis will be performed in the framework of the composite-operator formalism at finite temperature and density, based on a modified effective action[9] within the functional formalism developed by Cornwall, Jackiw, and Tomboulis.[10]

By evaluating the effective potential, we study the fermionic condensate versus temperature and chemical potential. We employ a simple ansatz for the fermionic self-energy, which should contain the essential relevant features. We find a phase diagram which is in qualitative agreement with results occurring in the literature.[3,5] Also, in the whole (λ, T) plane, our phase diagram reproduces the phase pattern of the Gross-Neveu model as found by Wolff.[11]

This paper is organized as follows. To make it self-contained, we give a short review of the composite-operator formalism at $T=\lambda=0$ in Sec. II. In Sec. III we extend this formalism to finite temperature and density. In Sec. IV we discuss the variational ansatz for the fermionic self-energy Σ and we explicitly evaluate the effective potential. In Sec. V we study the phase structure of the theory for varying T and λ. Conclusions are given in Sec. VI. In Appendix A, for convenience of the reader, we list some properties of special functions,

whereas in Appendixes B and C we derive explicitly some of the formulas given in the text.

II. EFFECTIVE ACTION FOR COMPOSITE OPERATORS. REVIEW OF THE ZERO-TEMPERATURE AND ZERO-DENSITY FORMALISM

A successful method to study CSB in QCD-like gauge theories is provided by the effective action for composite operators. In this kind of theory one expects that the breaking of chiral symmetry is dynamical as due to formation of fermion condensates. This phenomenon cannot be easily seen in a perturbative series since it necessarily requires at least an infinite subset of Feynman diagrams. The effective action for composite operators, first introduced by Cornwall, Jackiw, and Tomboulis[10] (CJT), provides for a formalism especially appropriate to the study of dynamical symmetry breaking. In fact this generalized effective action can be expressed as a formal series consisting of a systematic resummation of graphs with a fixed number of loops. This formalism deals with a nonlocal order parameter for which a nonlocal source is introduced in the generating functional. In Ref. 9 a modified version of the CJT effective was given. Since the effective potential corresponds to the vacuum energy only when the source function vanishes, an ambiguity is present of adding an arbitrary polynomial of the source function itself, satisfying some suitable conditions. In particular, the choice of Ref. 9 corresponds to a functional which has the same local extrema of the CJT functional but possesses the convenient property of boundness from below.

The formal expression for this effective action, in the case of a Euclidean fermion gauge theory at zero temperature and density, is[9]

$$\Gamma=-\mathrm{Tr}\ln(S_0^{-1}-\Sigma)-\mathrm{Tr}(\Sigma S)-\Gamma_2(S)\ , \tag{2.1}$$

where S_0 and S are the free and the full fermionic propagators, respectively, $\Gamma_2(S)$ is the sum of all the two-particle-irreducible vacuum diagrams of the theory evaluated with fermionic propagator equal to S, and Σ is the dynamical variable of the theory, defined by the equation

$$\Sigma=-\frac{\delta\Gamma_2}{\delta S}\ . \tag{2.2}$$

At the physical point, Eq. (2.2) is the Schwinger-Dyson equation for the fermion propagator with Σ the fermion self-energy. By taking Γ_2 at the two-loop level (single-gluon exchange) and parametrizing the fermionic propagator as

$$S(p)=iA(p^2)\hat{p}+B(p^2) \tag{2.3}$$

one finds that, in the Landau gauge, no renormalization of the wave function is required at this order [Γ_2 does not depend on $A(p^2)$], and the effective action is completely expressed in terms of Σ:

$$\Gamma(\Sigma)=-\mathrm{Tr}\ln(S_0^{-1}-\Sigma)-\tfrac{1}{2}\mathrm{Tr}(\Sigma B) \tag{2.4}$$

with B related to Σ by

$$\Sigma(q^2)=-3C_2\int\frac{d^4p}{(2\pi)^4}g^2(p,q)\frac{B(p^2)}{(p-q)^2}\ . \tag{2.5}$$

In Eq. (2.5) C_2 is the quadratic Casimir constant of the gauge group and $g(p,q)$ is the running gauge coupling constant. The variational method consists in making use of a parameter-dependent test function for Σ to investigate the stability of the theory. We will here adopt the following ansatz for Σ at zero T and λ:

$$\Sigma(p^2)=\chi f(p^2),\quad f(p^2)=\frac{\mu^3}{\mu^2+p^2}\ . \tag{2.6}$$

This ansatz is different from that used in Ref. 9 but it will be more suitable for the generalization to finite temperature and density. In Eq. (2.6) μ is a momentum scale and χ is a constant field to be taken as variational parameter. Its value at the minimum of the effective potential is related to the fermion condensate renormalized at the scale μ. As in Ref. 9, we assume the momentum dependence of Σ in the ultraviolet region as predicted by the OPE analysis, and a constant behavior for $p\to 0$. To make the calculations simpler, we shall here neglect the logarithmic corrections coming from the renormalization-group analysis. Also, in this simplified study, we will not consider the "running" with the momentum of the gauge coupling constant. Asymptotic freedom will be present in the theory through the assumed dependence of the coupling from T and λ. As it happened for the zero-temperature and zero-density study of the CSB in QCD (Ref. 9), we expect that the inclusion of the renormalization-group corrections in the momentum dependence will not change the qualitative picture of the chiral phase diagram.

In the case of QCD with massless fermions at $T=\lambda=0$, from the analysis of the effective potential V [obtained using the ansatz (2.6)] as a function of χ, one finds that the theory possesses two phases: the chirally symmetric phase and the broken phase to the diagonal flavor subgroup, depending on the value of the gauge coupling constant. In particular one finds that CSB occurs for

$$c=\frac{2\pi^2}{g^2}<\tfrac{3}{2}\ , \tag{2.7}$$

where g is the gauge coupling constant renormalized at the scale μ.

In order to fix the values of the parameters c and μ we will follow the same procedure of Ref. 12. We analyze the more realistic situation in which both spontaneous and explicit breakdown of the chiral symmetry takes place, by including current quark masses for the three lightest flavors. In this massive case the effective potential is UV divergent and from the renormalization condition in the small-mass limit one is able to fix the value $c=0.554$. By inserting this value in the effective potential for massless quarks, one finds that chiral symmetry is spontaneously broken via a minimum of V located at $\chi_{\min}=4.06$, and the point $\chi=0$ is a local maximum. In order to determine the mass scale μ from the experimental data, one has to derive the explicit expressions for the masses and for the decay constants of the pseudoscalar

octet mesons which are the pseudo-Goldstone bosons of chiral-symmetry breaking. These expressions constitute a system of coupled equations which we have solved by an approximation method. The experimental inputs are the decay constant and the mass of the charged pion, the charged-kaon mass, and the electromagnetic mass difference between the neutral and the charged kaon. The outputs of the numerical fit for the octet meson masses (agreement within 3%) are the masses of the u, d, and s quarks and the mass parameter μ which turns out to be

$$\mu = 282 \text{ MeV} . \tag{2.8}$$

From the relation between $\chi_{\min}$ and the fermionic condensate at μ (see Ref. 9), for massless quarks one gets the numerical value

$$\langle 0|\bar{\psi}\psi|0\rangle_{\mu} = \frac{3\mu^3}{2\pi^2} c\chi_{\min} \approx (197 \text{ MeV})^3 . \tag{2.9}$$

III. EXTENSION TO FINITE TEMPERATURE AND DENSITY

We now extend the preceding model, for vanishing current quark masses, to finite temperature and density. In this case, as suggested by asymptotic freedom and renormalization-group considerations, we expect the strong force to become weak at high temperatures and/or densities. We shall assume that in the UV region the coupling constant g depends logarithmically on the temperature T and the chemical potential λ in the same way as it usually depends on the momentum.[13] We take into account this assumption by writing

$$g^2(T,\lambda) = \frac{g^2}{1+\ln(1+\xi T^2/\mu^2+\zeta\lambda^2/\mu^2)g^2/2b} , \tag{3.1}$$

where $b = 24\pi^2/(11N-2n)$, with N the dimension of the fundamental representation of the gauge group SU(N) and n the flavor number. This expression gives the desired UV and $T=\lambda=0$ limits, with two parameters ξ and ζ. As an example which is consistent with this assumption, we recall the expression of the baryon density as computed from the thermodynamics of a free gas of quarks and gluons (see, for instance, Cleymans, Gavai, and Suhonen[5]). The expression for massless quarks is

$$n_B = \frac{Nn}{9}(\lambda T^2 + \lambda^3/\pi^2) . \tag{3.2}$$

Let us add a further comment on the parameters ξ and ζ: as stressed in Ref. 14, in the intermediate region of temperatures and densities, we have no hint for the dependence from the coupling, so that we have to let them be arbitrary unless we guess some interpolation formula. Since in this model we are more interested in a qualitative rather than in a quantitative picture of the CSB pattern, we shall prefer to leave them arbitrary, studying in the following how possible choices of their values affect the dynamics.

Let us now generalize the expressions (2.1)–(2.5) to the present case. The calculations will be performed in the imaginary-time formalism[2] and, for our purpose, it is useful to use Poisson's formula[15] (the introduction of the chemical potential implies the shift $p_0 \to p_0 + i\lambda$)

$$\sum_l (-)^l e^{il\beta p_0 + l\beta\lambda} = \frac{2\pi}{\beta} \sum_l \delta\left[p_0 - \frac{(2l+1)\pi}{\beta} - i\lambda\right] \tag{3.3}$$

(l integer and $\beta = 1/T$) which allows for an equivalent version of the Dolan-Jackiw finite-temperature Feynman rule:

$$\int \frac{d^4p}{(2\pi)^4} f(p_0,\mathbf{p}) \to \frac{1}{\beta} \sum_l \int \frac{d^3p}{(2\pi)^3} f\left[\frac{\pi(2l+1)}{\beta} + i\lambda, \mathbf{p}\right] = \sum_l (-)^l \int \frac{d^4p}{(2\pi)^4} f(p_0,\mathbf{p}) e^{il\beta p_0 + l\beta\lambda} . \tag{3.4}$$

Using Eq. (3.4) and the same procedure as for $T=\lambda=0$, one gets the extension of Eq. (2.5):

$$\Sigma(q) = -3g^2(T,\lambda)C_2 \sum_l (-)^l \int \frac{d^4p}{(2\pi)^4} \frac{B(p)}{(p-q)^2} e^{il\beta p_0 + l\beta\lambda} . \tag{3.5}$$

Inverting Eq. (3.5) one gets the formal expression

$$B(p) \sum_l (-)^l e^{il\beta p_0 + l\beta\lambda} = \frac{4\pi^2}{3g^2(T,\lambda)C_2} \Box_q \Sigma(q) \tag{3.6}$$

and thus the generalization of the effective action is

$$\Gamma(\Sigma) = -\frac{8\Omega Nn\pi^2}{3g^2(T,\lambda)C_2} \int \frac{d^4p}{(2\pi)^4} \Sigma(p)\Box_p \Sigma(p) - 2\Omega Nn \sum_l (-)^l \int \frac{d^4p}{(2\pi)^4} \ln\left[1 + \frac{\Sigma^2(p)}{p^2}\right] e^{il\beta p_0 + l\beta\lambda} \tag{3.7}$$

(where Ω is the four-dimensional volume). Equation (3.7) shows that Γ_2 has the same functional dependence on Σ as for zero temperature and zero density (see Ref. 9) and that the $l=0$ term in the sum reproduces the one-loop zero-temperature functional. Thus the effective action can be finally expressed as

$$\Gamma(\Sigma) = \Gamma^{(0)}(\Sigma) + \Gamma^{(1)}_{\ln}(\Sigma) , \tag{3.8}$$

where $\Gamma^{(0)}$ depends on the temperature and on the chemical potential only through $g^2(T,\lambda)$, since it is the same functional of Σ as in Sec. II, whereas $\Gamma^{(1)}_{\ln}$ is the new term to compute, and the subscript reminds us that it comes

from the one-loop term

$$\Gamma_{\ln}^{(1)}(\Sigma)=-2\Omega Nn\sum_{l\neq 0}(-)^l\int\frac{d^4p}{(2\pi)^4}\ln\left[1+\frac{\Sigma^2(p)}{p^2}\right]\times e^{il\beta p_0+l\beta\lambda}. \tag{3.9}$$

IV. EFFECTIVE POTENTIAL

The explicit computation of the effective potential can be performed once we have chosen a test function for Σ, which is the variational input in our approach (see Ref. 9). Unfortunately the finite-temperature Schwinger-Dyson equations for the self-energy (in principle both the temperature and the density may require including a vector component in the decomposition of the self-energy) are very hard to treat and we must limit ourselves to attempt the generalization of Eq. (2.6):

$$\Sigma(p)=\chi(T,\lambda)f(p^2). \tag{4.1}$$

With these assumptions, recalling that the effective potential is defined by $V=\Gamma/\Omega$, putting $\bar{V}=4\pi^2V/Nn\mu^4$, $C_2=\frac{4}{3}$ for $N=3$, and $n=3$, and using Eqs. (2.6) and (3.7), we are left with ($r=T/\mu, s=\lambda/\mu$)

$$\bar{V}(\chi,r,s)=\frac{c(r,s)}{3}\chi^2-\frac{1}{2}\int_0^\infty dx\,x\ln\left[1+\frac{\chi^2}{x^3+2x^2+x}\right]+\bar{V}_{\ln}^{(1)}(\chi,r,s), \tag{4.2}$$

where [see Eqs. (2.7) and (3.1)]

$$c(r,s)=\frac{2\pi^2}{g^2(r,s)}=c+\tfrac{9}{8}\ln(1+\xi r^2+\zeta s^2). \tag{4.3}$$

Computing the last term in Eq. (4.2) one gets (see Appendix B)

$$\bar{V}_{\ln}^{(1)}(\chi,r,s)=8r^2\sum_{k=1}^{3}z_k\sum_{l=1}^{\infty}\frac{(-)^l}{l^2}\cosh\left[\frac{ls}{r}\right]\times K_2\left[\frac{l\sqrt{z_k}}{r}\right],\quad |\arg\sqrt{z_k}|<\frac{\pi}{2}, \tag{4.4}$$

where K_2 is a modified Bessel function and $z_k=z_k(\chi)$ are the roots (reversed in sign) of the cubic equation

$$x^3+2x^2+x+\chi^2=0. \tag{4.5}$$

Let us notice that $\bar{V}_{\ln}^{(1)}(\chi,r,s)$ can be equivalently expressed as (see Ref. 2)

$$\bar{V}_{\ln}^{(1)}(\chi,r,s)=-4r\sum_{k=1}^{3}\int_0^\infty dx\,x^2\left\{\ln\left[1+\exp\left[-\frac{1}{r}[s+(x^2+z_k)^{1/2}]\right]\right]+\ln\left[1+\exp\left[-\frac{1}{r}[-s+(x^2+z_k)^{1/2}]\right]\right]\right\} \tag{4.6}$$

which can be obtained by making use of the Dolan-Jackiw rule in the fermionic case. It is possible to obtain one formula from the other by using integral representations for the modified Bessel functions. This shows the close relationship among the two formulations. The expressions involving sums of modified Bessel functions are mostly used in the low-temperature regime, by inserting the asymptotic expansions of $K_\nu(z)$ and of the polylogarithm functions, whereas the integral representation is more convenient in the high-temperature region.[2,16]

To get an insight on the large-χ behavior of the effective potential (as well as on other properties useful in the following), let us solve explicitly Eq. (4.5):

$$z_1=\tfrac{2}{3}-f^+-f^-,$$
$$z_2=\tfrac{2}{3}+\tfrac{1}{2}(f^++f^-)+i\frac{\sqrt{3}}{2}(f^+-f^-), \tag{4.7}$$
$$z_3=z_2^*,$$

where

$$f^+\left[\frac{1}{27}-\frac{\chi^2}{2}+\chi\left[\frac{\chi^2}{4}-\frac{1}{27}\right]^{1/2}\right]^{1/3},$$
$$f^-=\left[\frac{1}{27}-\frac{\chi^2}{2}-\chi\left[\frac{\chi^2}{4}-\frac{1}{27}\right]^{1/2}\right]^{1/3}. \tag{4.8}$$

Thus, in the $\chi\to\infty$ limit, we get [using the prescription (4.4) for the phase of z_k]

$$z_1(\chi)\sim\chi^{2/3},\quad z_2(\chi)\sim\chi^{2/3}e^{i2\pi/3},$$
$$z_3(\chi)=z_2^*(\chi). \tag{4.9}$$

This implies that $\mathrm{Re}(x^2+z_k)^{1/2}$ [see (4.6)] is always positive. In this way we find that there always exists a value $\bar{\chi}=\bar{\chi}(\bar{r},\bar{s})$, for any pair of fixed $\bar{r}$ and $\bar{s}$, such that V increases as χ^2 for $\chi>\bar{\chi}$, as in the $T=\lambda=0$ case, since the finite-temperature and -density corrections vanish in this limit.

V. PHASE TRANSITION AT FINITE TEMPERATURE AND CHEMICAL POTENTIAL

Let us now study the stability properties of the vacuum beginning with the two limiting cases $r\neq 0$ and $s=0$, $s\neq 0$ and $r=0$, and finally going to the general case.

It turns out from the numerical analysis that, for growing temperatures but zero chemical potential, the minimum of the effective potential, $\chi_{\min}(r)$, goes to zero continuously and then it remains zero. From the relation (2.9) between the fermion condensate $\langle\bar{\psi}\psi\rangle_r$ and $\chi_{\min}(r)$, it follows that chiral symmetry is restored via a second-order phase transition. (See Fig. 1.) This turns out to be true for any choice of the parameter ξ (see Fig. 2).

To find the critical temperature r_c it is sufficient to look at the second derivative of the effective potential at the origin. That is we have to solve the following equation in r:

$$\left.\frac{\partial^2\bar{V}}{\partial\chi^2}\right|_{\chi=0}=\frac{2c}{3}-1+\tfrac{3}{4}\ln(1+\xi r^2)+\tfrac{2}{3}\pi^2r^2 -4\int_0^\infty dx\frac{2x^2+1}{\sqrt{x^2+1}\left[1+\exp\left[\frac{1}{r}\sqrt{x^2+1}\right]\right]} =0\ . \tag{5.1}$$

Solutions to this equation for some values of ξ are given in Table I. Since the critical temperature is less than one, it is also useful to adopt the equivalent expression

$$\left.\frac{\partial^2\bar{V}}{\partial\chi^2}\right|_{\chi=0}=\frac{2c}{3}-1+\tfrac{3}{4}\ln(1+\xi r^2)+\tfrac{2}{3}\pi^2r^2 +4\sum_{l=1}^{\infty}(-)^lK_0\left[\frac{l}{r}\right]+8r\sum_{l=1}^{\infty}\frac{(-)^l}{l}K_1\left[\frac{l}{r}\right], \tag{5.2}$$

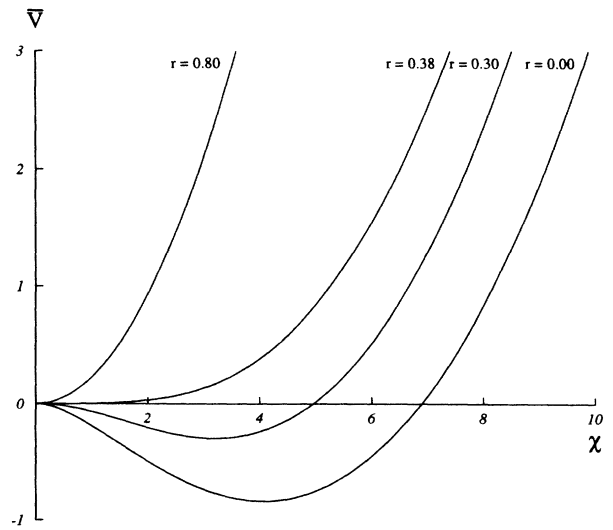

FIG. 1. Typical evolution of the effective potential for increasing temperature at zero chemical potential. The case shown is for $\xi=0.6$. [In this figure $\bar{V}$ is defined in such a way that $\bar{V}(0)=0$.]

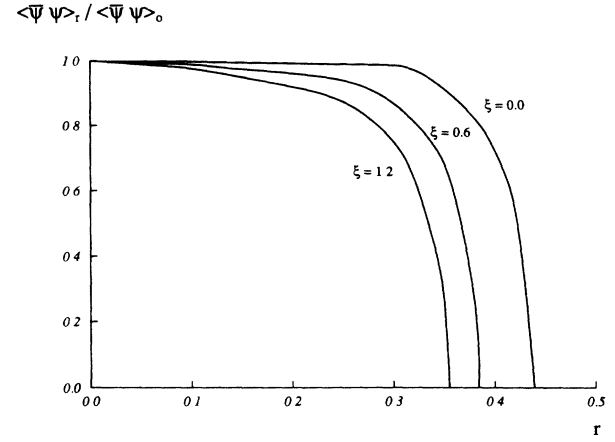

FIG. 2. Behavior of the normalized condensate at zero chemical potential vs temperature for some values of the parameter ξ. The condensate approaches zero with continuity.

which reduces to a finite sum using the expansion (A10) for large l/r. This formula can be derived from Eq. (4.4) using Eq. (A9).

After the symmetry has been restored, no other transition occurs, and in particular, the effective potential comes out to be a monotone (increasing) function of χ for any $r>r_c$. Furthermore, the curvature in the origin, for high r, can be worked out in an analytic way. In fact expanding the second derivative of $\bar{V}$ for high $r/\sqrt{z_k}$ (small $\beta\mu\sqrt{z_k}$), one gets (see Appendix C)

$$\frac{\partial^2\bar{V}}{\partial\chi^2}=\frac{2c}{3}+\tfrac{3}{4}\ln(1+\xi r^2) +2\sum_{k=1}^{3}\sum_{l=1}^{\infty}\frac{(-1)^l}{2^l}\frac{(2l-1)!!}{l!}\frac{z_k^l}{(\pi r)^{2l}} \times\left[\frac{z_k''z_k}{(l+1)}+(z_k')^2\right] \times\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2l+1}} \tag{5.3}$$

which gives, at the origin,[17]

$$\left.\frac{\partial^2\bar{V}}{\partial\chi^2}\right|_{\chi=0}=\frac{2c}{3}+\tfrac{3}{4}\ln(1+\xi r^2)-\frac{7}{8\pi^2r^2}\zeta(3)+O((\beta\mu)^4) \tag{5.4}$$

(here ζ is the Riemann function), which is positive for

TABLE I. The critical point r_c vs ξ.

ξ	r_c
0	0.44
0.3	0.41
0.6	0.38
0.9	0.36
1.2	0.35

high r. Combining this with the asymptotic behavior of the potential, one can conclude that the absolute minimum remains at the origin also for very high temperature (see Fig. 1).

The critical temperature does not depend strongly from the value of ξ (see Table I). A possible indication for the parameter ξ can be obtained by considering the region of very low temperatures. In fact, there, the leading contribution in T to the effective potential comes from the two-loop term containing $g^2(T)$ which goes as T^2, whereas the $\bar{V}_{\rm ln}^{(1)}$ term goes as $T^{3/2}e^{-1/T}$. Evaluating the minimum of $\bar{V}$ for $T\to 0$ one finds that the behavior of the condensate is of the form $\langle\bar{\psi}\psi\rangle_T \approx \langle\bar{\psi}\psi\rangle_0 + A(\xi)T^2$. Comparing this behavior with that given in Ref. 18, we find $\xi\simeq 1$, implying a critical temperature $T_c \simeq 0.36\mu$.

Let us now consider the second limiting situation, zero temperature and finite chemical potential. In this case we have to distinguish between the case in which Eq. (4.5) has real roots and that in which it has complex roots. It turns out that for $\chi^2 \le \frac{4}{27}$ the roots are real and positive, whereas for $\chi^2 > \frac{4}{27}$ one root is real and positive and two are complex conjugate. In the first case it is possible to get the analytic expression for the density correction to the effective potential by taking Eq. (4.6) for $r\to 0$:

$$\begin{aligned}\bar{V}_{\rm ln}^{(1)} &= 4\sum_{k=1}^{3}\theta(s^2-z_k)\int_0^{(s^2-z_k)^{1/2}} x^2[(x^2+z_k)^{1/2}-s]dx \\ &= \sum_{k=1}^{3}\theta(s^2-z_k)\left[\frac{s}{3}(s^2-z_k)^{1/2}\left(\tfrac{5}{2}z_k-s^2\right)+\frac{z_k^2}{2}\ln\left[\frac{\sqrt{z_k}}{s+(s^2-z_k)^{1/2}}\right]\right] \\ &\equiv \sum_{k=1}^{3}\bar{V}_{\rm ln}^{z_k}\ . \end{aligned} \tag{5.5}$$

On the other hand, for $\chi^2 > \frac{4}{27}$, one obtains, after some algebra,

$$\begin{aligned}\bar{V}_{\rm ln}^{(1)} = \bar{V}_{\rm ln}^{z_1} &+ 8\theta\left[s^2-\frac{z_{2r}}{2}-\frac{|z_2|}{2}\right] \\ &\times\left[-\tfrac{1}{3}s\left[s^2-z_{2r}-\frac{z_{2i}^2}{4s^2}\right]^{3/2}+\frac{1}{\sqrt{2}}\int_0^{(s^2-z_{2r}-z_{2i}^2/4s^2)^{1/2}} dx\, x^2\{[(x^2+z_{2r})^2+z_{2i}^2]^{1/2}+(x^2+z_{2r})\}^{1/2}\right]\end{aligned} \tag{5.6}$$

(z_{2r}, z_{2i} are the real and imaginary parts of z_2). In particular we easily get information about the stability of the point $\chi=0$, occurring in the region where the roots are real. In fact, by taking the second derivative of $\bar{V}$ with respect to χ [see (4.2) and (5.5)],

$$\frac{\partial^2\bar{V}}{\partial\chi^2} = \tfrac{2}{3}c+\tfrac{3}{4}\ln(1+\zeta s^2)-1+\sum_{k=1}^{3}\theta(s^2-z_k)\left[z_k''s(s^2-z_k)^{1/2}+\ln\left[\frac{\sqrt{z_k}}{s+(s^2-z_k)^{1/2}}\right](z_kz_k''+z_k'^2)\right] \tag{5.7}$$

and using the expression of the z_k's and of their derivatives in $\chi=0$ one finds two cases:

$$\begin{aligned}\left.\frac{\partial^2\bar{V}}{\partial\chi^2}\right|_{\chi=0} &= \tfrac{2}{3}c+\tfrac{3}{4}\ln(1+\zeta s^2)-1+2s^2 \quad \text{for } s\le 1\ , \\ \left.\frac{\partial^2\bar{V}}{\partial\chi^2}\right|_{\chi=0} &= \tfrac{2}{3}c+\tfrac{3}{4}\ln(1+\zeta s^2)-1+2s^2 \\ &\quad -2s\sqrt{s^2-1} \quad \text{for } s>1\ .\end{aligned} \tag{5.8}$$

Thus, starting from $s=0$, the maximum in $\chi=0$ becomes a minimum as the chemical potential increases. Precisely, from Eq. (5.8), it is easy to see that the transition point is always located below $s\simeq 0.56$, which corresponds to $\zeta=0$. This point will be denoted as s_1. In addition to this, Eq. (5.8) shows that for $s>1$ the origin is always a stable minimum, which becomes narrower as s increases. The numerical analysis shows that, even when the point $\chi=0$ has become a minimum, it is not the absolute one. Thus the chiral symmetry is still broken and it gets restored via a first-order phase transition when the absolute minimum become as deep as the one at $\chi=0$. This point will be denoted as s_2. As an example, for $\zeta=0.6$, the chiral symmetry is restored at $s_2\simeq 0.75$. Decreasing the value of ζ, s_2 becomes sensibly larger. Finally for $s>s_2$ another metastable region (corresponding now to the minimum outside the origin not being the absolute one) is present. This region disappears at higher values of s. The corresponding point will be denoted as s_3. For instance, for $\zeta=0.6$, we find $s_3\simeq 3.3$. Finally, as ζ decreases, s_2, s_3 approach ∞ and chiral symmetry is never restored in this limit. A list of values of s_1, s_2, and s_3 is given in Table II. In Fig. 3 we show the typical evolution of $\bar{V}$ with the chemical potential, whereas Fig. 4 reproduces the behavior of the (normalized) condensate for some values of ζ, showing that it always vanishes with a jump, and that for $s>s_2$ it remains equal to zero.

Let us now look at the general case of finite temperature and chemical potential, $r\neq 0$, $s\neq 0$. The previous analysis can be easily generalized by studying the phase diagram in the (s,r) plane. We start by looking at the

TABLE II. The points s_1, s_2, s_3 vs ζ.

ζ	s_1	s_2	s_3
0	0.56	∞	∞
0.3	0.53	2.22	5.0
0.6	0.51	0.75	3.3
0.9	0.49	0.68	2.7
1.2	0.47	0.63	2.3

line l_1 (represented by a dashed line in Fig. 5), defined as the set of points at which the second derivative of the potential at the origin vanishes. This line signals the transition from a minimum at the origin of the potential to a maximum. The end points of this line will be denoted by $(0,r_c)$ and $(s_1,0)$. For small values of s, by continuity from the case of the vanishing chemical potential, l_1 is the critical line corresponding to the phase transition from the unbroken to the broken phase. Notice that the transition is second order, corresponding to the minimum outside the origin going smoothly into the minimum at the origin (see Fig. 1). However there must exist a point $t \equiv (s_t, r_t)$ on l_1 at which the situation changes. In fact, we know that at $r=0$ the phase transition occurs at $s = s_2 \neq s_1$. That is, if we cross l_1 at the right of t (see Fig. 5), starting from the broken phase, the minimum at the origin is no longer the absolute one: more than one minimum is present. We will call the phase in which more than one minimum is present a metastable one. Increasing again the temperature, at fixed chemical potential, the minimum at the origin will become the absolute one, but at least one of the minima outside the origin will still be present. That is we encounter a second metastable phase. The separation line l_2 (continuous line in Fig. 5) of the two metastable phases is, of course, the true critical line. In fact it separates the broken phase (absolute minimum outside the origin) from the unbroken one (absolute minimum at the origin). Furthermore, l_2 joins $(0,r_c)$ to $(s_2,0)$, and it separates from l_1 at the point t. If we continue to increase the temperature, all the minima outside the origin disappear, and we end up into a pure unbroken phase. The line dividing this phase from the metastable one will be denoted by l_3 (dashed line in Fig. 5). Clearly l_3 coincides with l_1 and l_2 up to the point t. After this point the three lines separate and l_3 ends up at $(s_3,0)$. The existence of metastable phases has the impli-

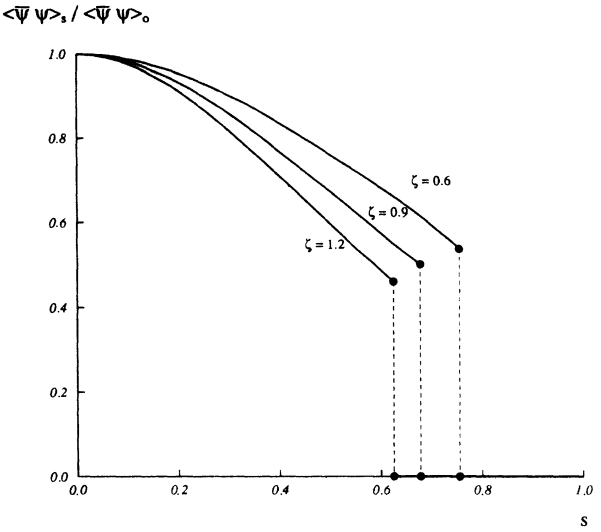

FIG. 4. Behavior of the normalized condensate at zero temperature vs chemical potential for some values of the parameter ζ. The condensate always vanishes with a jump.

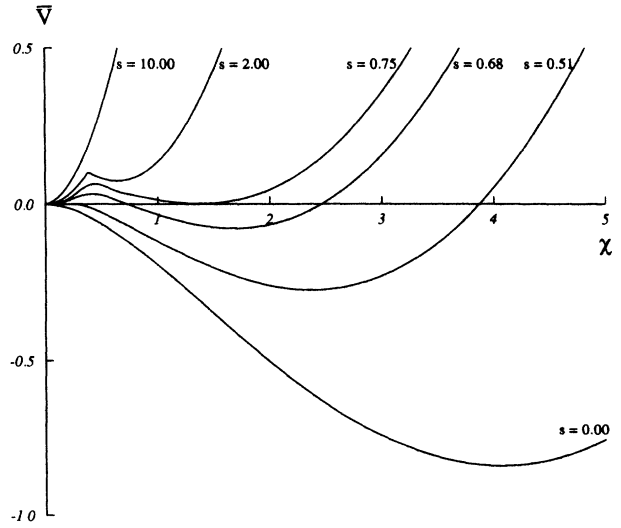

FIG. 3. Typical evolution of the effective potential for increasing chemical potential at zero temperature. The case shown is for $\zeta=0.6$. [In the figure $\bar{V}$ is defined in such a way that $\bar{V}(0)=0$.]

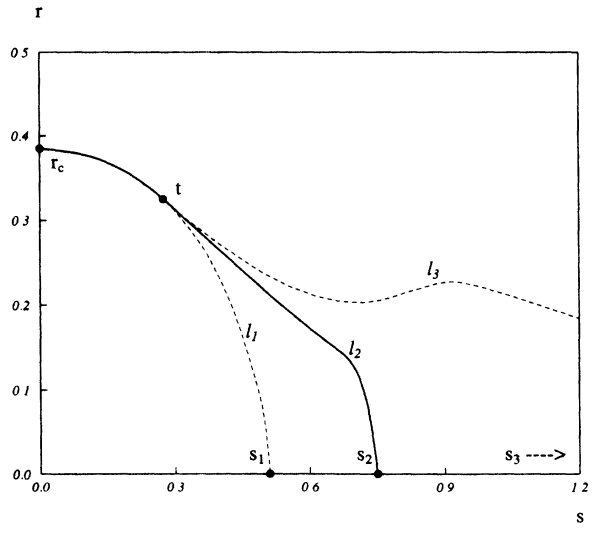

FIG. 5. The phase diagram in the plane (s,r). The curves l_1 and l_3 (see text) are denoted by dashed lines, the critical curve, l_2, by a continuous one. The coordinates of the point t are $s_t=0.27$ and $r_t=0.33$. In the present case $\xi=0.6$ and $\zeta=0.6$. An analogous structure is found for the other values listed in the tables.

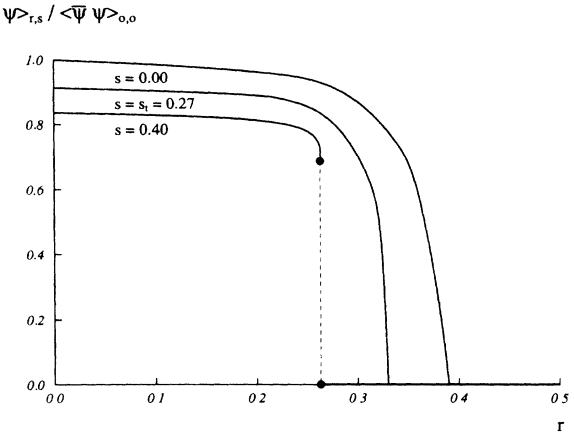

FIG. 6. Behavior of the normalized condensate at finite r and s for $\xi=0.6$ and $\zeta=0.6$. The curves are found for increasing temperatures at different chemical potentials, i.e. (see Fig. 5), following vertical lines into the phase diagram. The curve in the middle is found along the vertical line crossing the point t.

cation that the transition is first order, that is the condensate has a discontinuity at the critical point. This corresponds to the absolute minimum jumping from a minimum outside the origin to a minimum at the origin. Therefore, the phase transition is second order along l_2 at the left of t, whereas it is first order at its right (see Fig. 6). Similar behavior occurs in the Gross-Neveu model.[11]

VI. CONCLUSIONS

We have studied the diagram of the chiral phases of QCD for finite temperatures and densities. The phase diagram we find has strong similarity to that of the Gross-Neveu model. The existence of metastable phases is a new result, suggested by our analysis. The kind of phase diagram we have described in the main text is found for any choice of the variables ξ and ζ which parametrize the behavior of the gauge coupling constant as a function of T and λ. Therefore, although our analysis depends quantitatively from the particular ansatz we have done for the coupling constant, we feel that the qualitative results are quite general, and they should not change by a different choice of the ansatz itself. Our study is entirely analytic and should be complemented by lattice calculations.

ACKNOWLEDGMENTS

We would like to thank D. Dominici for fruitful discussions.

APPENDIX A: SPECIAL FUNCTIONS

For the convenience of the reader we briefly list some properties of special functions used in this study.

The Gegenbauer polynomials in four dimensions are[19,20]

$$C_n^1(\hat{p}\cdot\hat{q})=\frac{2\pi^2}{n+1}\sum_{lm} Y^*_{nlm}(\hat{p})Y_{nlm}(\hat{q}) , \tag{A1}$$

where Y_{nlm} are the spherical harmonics and $\hat{p}$ and $\hat{q}$ are two unit four-vectors. They satisfy the parity relation

$$C_n^1(-x)=(-)^n C_n^1(x) \tag{A2}$$

and orthogonality condition

$$\int d\Omega_{\hat{q}} C_{n_1}^1(\hat{p}\cdot\hat{q})C_{n_2}^1(\hat{k}\cdot\hat{q})=2\pi^2\delta_{n_1,n_2}\frac{1}{n_1+1}C_{n_1}^1(\hat{p}\cdot\hat{k}) . \tag{A3}$$

A special case is

$$C_0^1(x)=1 . \tag{A4}$$

The following expansion holds:

$$e^{ip\cdot x}=\sum_{n=0}^{\infty} i^n(n+1)C_n^1(\hat{p}\cdot\hat{x})\left[\frac{p^2x^2}{4}\right]^{n/2} \times j_{n+1}\left[\frac{p^2x^2}{4}\right] , \tag{A5}$$

where

$$j_\nu(z)=\sum_{n=0}^{\infty}\frac{(-)^n z^n}{n!\Gamma(n+\nu+1)} \tag{A6}$$

and

$$J_\nu(z)=\left[\frac{z}{2}\right]^\nu j_\nu\left[\frac{z^2}{4}\right] \tag{A7}$$

are the usual Bessel functions. A useful integral representation for the modified Bessel functions[21,22] is

$$K_\nu(z)=\int_0^\infty e^{-z\cosh(t)}\cosh(\nu t)dt \quad |\arg z|<\frac{\pi}{2} . \tag{A8}$$

The following relations hold

$$\left[\frac{1}{z}\frac{d}{dz}\right]^l[z^\nu K_\nu(z)]=(-)^l z^{\nu-l}K_{\nu-l}(z) . \tag{A9}$$

Useful expansions for small arguments are

$$K_0(z)\simeq -\ln z ,$$
$$K_\nu(z)\simeq\tfrac{1}{2}\Gamma(\nu)\left[\frac{z}{2}\right]^{-\nu} \quad (\mathrm{Re}\,\nu>0) \tag{A10}$$

while for large arguments

$$K_\nu(z)=\left[\frac{\pi}{2z}\right]^{1/2}e^{-z}\left[\sum_{m=0}^{M-1}(\nu,m)(2z)^{-m} +O(|z|^{-M})\right], \quad |\arg z|<\frac{3}{2}\pi , \tag{A11}$$

where (ν,m) are the Hankel symbols[22]

$$(\nu,m)=\frac{2^{-2m}}{m!}\{(4\nu^2-1) \times(4\nu^2-3^2)\cdots[4\nu^2-(2m-1)^2]\} . \tag{A12}$$

APPENDIX B: DERIVATION OF EQ. (4.4)

From Eqs. (4.2) and (3.9) one finds, neglecting an infinite χ-independent constant,

$$\bar{V}^{(1)}_{\ln}(\chi,r,s)=\frac{4\pi^2}{9\mu^4\Omega}\Gamma^{(1)}_{\ln}(\chi)=-\frac{1}{2\pi^2}\sum_{l=1}^{\infty}(-)^l e^{ls/r}\int d^4y\,\ln(y^6+2y^4+y^2+\chi^2)e^{ily_0/r}+(l\to -l)\ , \tag{B1}$$

where the sum has been divided according to positive and negative integers. Introducing the four-vectors $\tilde{\beta}^{\gamma}_{l}=(l/r,0)$ and using the expansion (A5) one has

$$\bar{V}^{(1)}_{\ln}(\chi,r,s)=-\frac{1}{2\pi^2}\sum_{l=1}^{\infty}(-)^l e^{ls/r}\int_0^{\infty}dy\,y^3\ln(y^6+2y^4+y^2+\chi^2)$$
$$\times\int d\Omega_{\hat{y}}\sum_{n=0}^{\infty}i^n(n+1)\left[\frac{y^2l^2}{4r^2}\right]^{n/2}j_{n+1}\left[\frac{y^2l^2}{4r^2}\right]C^1_n(\cos\phi)+(l\to -l) \tag{B2}$$

(where ϕ is the angle between the directions of $\tilde{\beta}^{\gamma}_{l}$ and y^{ν}). Performing the angular integration, using (A3) and (A4), and recombining the sums over positive and negative l, one obtains

$$\bar{V}^{(1)}_{\ln}(\chi,r,s)=-4r\sum_{k=1}^{3}\sum_{l=1}^{\infty}\frac{(-)^l}{l}\cosh\left[l\frac{s}{r}\right]\int_0^{\infty}dx\,x^2\ln(x^2+z_k)J_1\left[\frac{lx}{r}\right]$$
$$\equiv F(z_k(\chi),r,s)\ . \tag{B3}$$

Performing the derivative of $F(z_k(\chi),r,s)$ with respect to z_k, and the integration over x (Ref. 20), one gets

$$\frac{\partial F}{\partial z_k}=-4r\sum_{k=1}^{3}\sum_{l=1}^{\infty}\frac{(-)^l}{l}\cosh(ls/r)\int_0^{\infty}dx\frac{x^2}{x^2+z_k}J_1\left[\frac{lx}{r}\right]$$
$$=-4r\sum_{k=1}^{3}\sqrt{z_k}\sum_{l=1}^{\infty}\frac{(-)^l}{l}\cosh(ls/r)K_1\left[\frac{l\sqrt{z_k}}{r}\right],\quad |\arg\sqrt{z_k}|<\frac{\pi}{2}\ . \tag{B4}$$

Finally, integration with respect to z_k and use of Eq. (A9), lead to the expression (4.4) (again a χ-independent term is neglected).

APPENDIX C: DERIVATION OF EQ. (5.3)

A possible short way to derive Eq. (5.3) is to start from Eq. (4.4) and to evaluate the derivative of $\bar{V}$ with respect to χ [with the aid of Eq. (A9)]:

$$\frac{\partial\bar{V}^{(1)}_{\ln}}{\partial\chi}=-4\sum_{l=1}^{\infty}\frac{(-)^l}{l}r\sum_{k=1}^{3}z_k'\sqrt{z_k}K_1\left[\frac{l}{r}\sqrt{z_k}\right]\ . \tag{C1}$$

Using the representation (A8) and calculating the subsequent derivative gives rise to two terms where there appear expressions (their high $r/\sqrt{z_k}$ expansion has been studied originally in Ref. 2) such as

$$I\left[\frac{\sqrt{z_k}}{r}\right]=\int_0^{\infty}\frac{dx}{(x^2+z_k/r^2)^{1/2}[1+\exp(x^2+z_k/r^2)^{1/2}]}=-\tfrac{1}{2}\ln\left[\frac{\sqrt{z_k}}{r\pi}\right]-\frac{\gamma}{2}+\tilde{I}\left[\frac{\sqrt{z_k}}{r}\right], \tag{C2}$$

$$\tilde{I}\left[\frac{\sqrt{z_k}}{r}\right]=-\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[\left[1+\frac{z_k}{2l+1^2\pi^2r^2}\right]^{-1/2}-1\right] \tag{C3}$$

(γ is the Euler's constant) and their integrals with respect to z_k. Then, using simple relations among the z_k's and their derivatives, one obtains

$$\frac{\partial^2\bar{V}}{\partial\chi^2}=\frac{2c(r,0)}{3}+2\sum_{k=1}^{3}z_k'^2\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{1}{[1+z_k/(2n+1)^2\pi^2r^2]^{1/2}}$$
$$+4\sum_{k=1}^{3}z_k''\sum_{n=0}^{\infty}(2n+1)\pi^2r^2[1+z_k/(2n+1)^2\pi^2r^2]^{1/2}\ . \tag{C4}$$

Expanding the square roots and using again relations among the derivatives of the z_k's, after some algebra, Eq. (5.3) is recovered.

[1]D. A. Kirzhnits and A. D. Linde, Phys. Lett. **42B**, 471 (1972).

[2]L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974); C. W. Bernard, *ibid.* **9**, 3312 (1974); S. Weinberg, *ibid.* **9**, 3357 (1974).

[3]D. Bailin, J. Cleymans, and M. D. Scadron, Phys. Rev. D **31**, 164 (1985); P. Castorina, Nuovo Cimento **93A**, 125 (1986); J. Cleymans, A. Kocic, and M. D. Scadron, Phys. Rev. D **39**, 323 (1989); T. S. Evans and R. J. Rivers, Z. Phys. C **40**, 293 (1988); P. Gasser and H. Leutwyler, Nucl. Phys. **B321**, 387 (1989); A. Cabo, O. K. Kalashnikov, and E. Kh. Veliev, *ibid.* **B299**, 367 (1988).

[4]A. D. Linde, Rep. Prog. Phys. **42**, 389 (1979).

[5]For some reviews, see E. V. Shuryak, Phys. Rep. **61**, 72 (1980); **115**, 151 (1984); D. Gross, R. Pisarski, and L. Yaffe, Rev. Mod. Phys. **53**, 43 (1981); O. K. Kalashnikov, Fortsch. Phys. **32**, 525 (1984); J. Cleymans, R. V. Gavai, and E. Suhonen, Phys. Rep. **130**, 217 (1986); L. Van Hove, Report No. CERN TH 5069/88 (unpublished).

[6]For a review of recent results on lattice field theory, see *Lattice '88,* proceedings of the Symposium, Batavia, Illinois, 1988, edited by A. S. Kronfeld and P. B. Mackenzie [Nucl. Phys. B (Proc. Suppl.) **9** (1989)].

[7]J. Cleymans, M. Dechantsreiter, and F. Halzen, Z. Phys. C **17**, 341 (1983).

[8]For a preliminary study, see A. Barducci, R. Casalbuoni, S. De Curtis, R. Gatto, and G. Pettini, Phys. Lett. B **231**, 463 (1989).

[9]R. Casalbuoni, S. De Curtis, D. Dominici, and R. Gatto, Phys. Lett. **150B**, 295 (1985); A. Barducci, R. Casalbuoni, S. De Curtis, D. Dominici, and R. Gatto, Phys. Rev. D **38**, 238 (1988).

[10]J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974); A. Barducci, R. Casalbuoni, D. Dominici, R. Gatto, and G. Pettini, Phys. Lett. B **179**, 275 (1986).

[11]U. Wolff, Phys. Lett. **157B**, 303 (1985).

[12]A. Barducci, R. Casalbuoni, S. De Curtis, D. Dominici, and R. Gatto, Phys. Lett. **147B**, 460 (1984); R. Casalbuoni, in *Proceedings of the International Symposium on Composite Models of Quarks and Leptons,* Tokyo, Japan, 1985, edited by H. Terazawa and M. Yasué (INS, University of Tokyo, Tokyo, 1985).

[13]J. C. Collins and M. J. Perry, Phys. Rev. Lett. **34**, 1353 (1975); P. D. Morley and M. B. Kislinger, Phys. Rep. **51**, 63 (1979); H. Matsumoto, Y. Nakano, and H. Umezawa, Phys. Rev. D **38**, 3211 (1984); H. Nakkagawa, Physica **158A**, 235 (1989).

[14]O. K. Kalashnikov, Z. Phys. C **39**, 427 (1988).

[15]I. Stakgold, *Green's Functions and Boundary Value Problems* (Wiley, New York, 1979).

[16]A. Actor, Nucl. Phys. **B256**, 689 (1986); H. A. Weldon, *ibid.* **B270**, 79 (1986); T. F. Treml, Phys. Rev. D **39**, 679 (1989).

[17]E. R. Hansen, *A Table of Series and Products* (Prentice-Hall, Englewood Cliffs, N.J., 1975).

[18]J. Gasser and H. Leutwyler, Phys. Lett. B **184**, 83 (1987).

[19]P. Pascual and R. Tarrach, *QCD: Renormalization for the Practitioner* (Springer, Berlin, 1984).

[20]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).

[21]M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[22]W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, Berlin/Heidelberg, 1966).