Z-boson decays to heavy quarkonium

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We evaluate the decays $Z \to Q+\bar{Q}+S(Q\bar{Q})$ of Z bosons to ${}^3S_1(Q\bar{Q})$ and ${}^1S_0(Q\bar{Q})$ quarkonium states of charm and bottom quarks. We find partial widths of $\Gamma(Z\to c\bar{c}\psi/J)=47$ keV, $\Gamma(Z\to c\bar{c}\eta_c)=145$ keV, $\Gamma(Z\to b\bar{b}\Upsilon_b)=6.4$ keV, and $\Gamma(Z\to b\bar{b}\eta_b)=6.5$ keV, that might be observable at the CERN e^+e^- collider LEP. We also calculate the partial widths for $W^+\to c\bar{s}\psi/J$, $t\to W^+ b\Upsilon_b$, $\eta_Q\to Zgg$, and $Z\to\eta_{c,b}gg$ decays.

The study of rare Z decays should soon be possible at the e^+e^- colliders CERN's LEP and the SLAC Linear Collider (SLC) (Ref. 1). A data sample of 10^7 Z events could be obtained in a year of running at LEP. Thus branching fractions down to 10^{-6}, corresponding to partial widths of a few keV, might be measured. An interesting possibility is the decays of the Z to a quarkonium state, which we study in this paper. Previously the partial widths for the two-body decays, $Z\to\gamma,H+{}^{2S+1}L_J(Q\bar{Q})$, to a photon or a Higgs boson and quarkonium states were calculated[2] and found to be too small to be observable at LEP (e.g., a partial width for $Z\to\gamma+\psi/J$ of 0.2 keV is predicted). However we find that the inclusive decays $Z\to Q+\bar{Q}+S(Q\bar{Q})$ to ${}^3S_1(Q\bar{Q})$ and ${}^1S_0(Q\bar{Q})$ quarkonium states are substantially larger (e.g., a partial width for $Z\to c\bar{c}+\psi/J$ of 47 keV is predicted) and may be observable at LEP. We also address the backgrounds to these decay modes.

$Z\to Q\bar{Q}+$quarkonium

First we calculate Z decay into a heavy ${}^3S_1$ quarkonium state $\psi(Q\bar{Q})$ together with its associated quark pair $Q\bar{Q}$. The amplitude for $Z\to Q\bar{Q}\psi$ involves four Feynman diagrams with different attachments of the exchanged gluon, as shown in Fig. 1. In these graphs the color factor of the exchanged gluon is $T^a_{ij}T^a_{jk}=\frac{4}{3}1_{ik}$. The $\psi Q_i\bar{Q}_k$ vertex is

$$\frac{\Psi(0)}{2\sqrt{2m}}\frac{1_{ik}}{\sqrt{3}}\not{\epsilon}(\not{\psi}+2m) \quad (1)$$

in the nonrelativistic approximation. The complete amplitude for the decay is

$$\mathcal{M}^Z=\frac{4}{3}\frac{g_s^2 g}{(1-x_W)^{1/2}}\frac{\Psi(0)}{2\sqrt{2m}}\frac{1_{ik}}{\sqrt{3}}\bar{u}(Q)\times(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4)v(\bar{Q}) , \quad (2)$$

with

$$\begin{aligned}\mathcal{A}_1&=\gamma^\alpha\frac{\not{\epsilon}(\psi)(\not{\psi}+2m)}{(\frac{1}{2}\psi+Q)^2}\gamma_\alpha\frac{\not{\psi}+\not{Q}+m}{(\psi+Q)^2-m^2}\not{\epsilon}(Z)P_Z ,\\ \mathcal{A}_2&=\gamma^\alpha\frac{\not{\epsilon}(\psi)(\not{\psi}+2m)}{(\frac{1}{2}\psi+Q)^2}\not{\epsilon}(Z)P_Z\frac{\frac{1}{2}\not{\psi}-\not{Z}+m}{(\frac{1}{2}\psi-Z)^2-m^2}\gamma_\alpha ,\\ \mathcal{A}_3&=\gamma^\alpha\frac{\not{Z}-\frac{1}{2}\not{\psi}+m}{(Z-\frac{1}{2}\psi)^2-m^2}\not{\epsilon}(Z)P_Z\frac{\not{\epsilon}(\psi)(\not{\psi}+2m)}{(\frac{1}{2}\psi+\bar{Q})^2}\gamma_\alpha ,\\ \mathcal{A}_4&=\not{\epsilon}(Z)P_Z\frac{-\not{\psi}-\not{\bar{Q}}+m}{(\psi+\bar{Q})^2-m^2}\gamma^\alpha\frac{\not{\epsilon}(\psi)(\not{\psi}+2m)}{(\frac{1}{2}\psi+\bar{Q})^2}\gamma_\alpha ,\end{aligned} \quad (3)$$

where the projection P_Z is defined as

$$P_Z=g_L\tfrac{1}{2}(1-\gamma_5)+g_R\tfrac{1}{2}(1+\gamma_5) \quad (4)$$

and

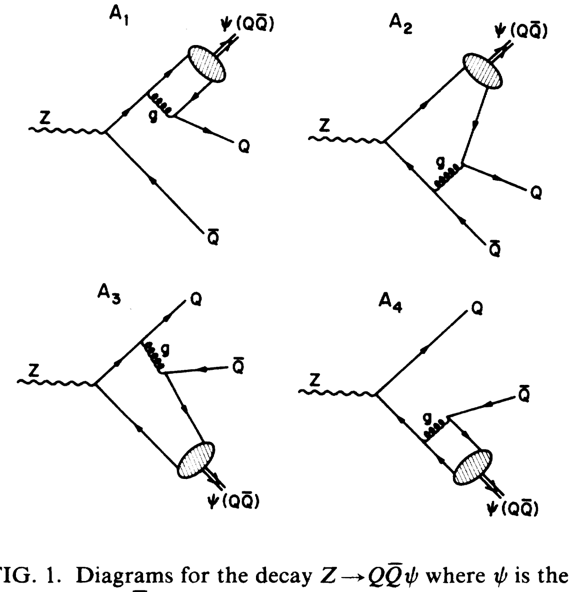

FIG. 1. Diagrams for the decay $Z\to Q\bar{Q}\psi$ where ψ is the ${}^3S_1$ bound state of $Q\bar{Q}$.

$$g_L = t_L^3(Q) - e_q x_W, \quad g_R = -e_q x_W \ . \tag{5}$$

The Feynman gauge for the gluon field is used in the above expressions.

The square of the full amplitude obtained using the REDUCE program[3] is given in Appendix A. If we take $g_L = g_R = 1$, then these calculations also apply to the process $\gamma^* \to c\bar{c}\psi/J$ that was studied in Ref. 4 in the approximation that charm mass terms are neglected in the numerator; for $r=0$ the expressions in Appendix A reproduce their analytic result for the γ^* process.

We also used helicity amplitude methods to directly calculate the $Z \to c\bar{c}\psi/J$ amplitudes numerically in arbitrary gauges, following the techniques of Ref. 5. These calculations confirmed the validity of the results obtained from the analytic expressions in Appendix A.

We find that there is little interference between $\mathcal{A}_1 + \mathcal{A}_2$ and $\mathcal{A}_3 + \mathcal{A}_4$. For example, the contributions of these amplitude combinations to the $Z \to c\bar{c}\psi$ partial width are

$$\Gamma(|\mathcal{A}_1 + \mathcal{A}_2|^2) = 25 \text{ keV} \ ,$$
$$\Gamma(|\mathcal{A}_3 + \mathcal{A}_4|^2) = 23 \text{ keV} \ , \tag{6}$$
$$\Gamma(2\,\mathrm{Re}[(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{A}_3 + \mathcal{A}_4)^*]) = -1.4 \text{ keV} \ .$$

The amplitudes for the $^1S_0$ pseudoscalar quarkonium decays are obtained from Eqs. (2)–(4) by the replacement $\not{\epsilon} \to \gamma_5$. We calculated the $^1S_0$ decay rates numerically with the helicity amplitude method but did not derive analytic formulas for this case.

The wave functions at the origin for ψ/J and Υ are determined through their leptonic decay widths Γ_{ee},

$$|\psi(0)|^2 = M_S^2 \Gamma_{ee} / (16\pi\alpha^2 e_Q^2) \ , \tag{7}$$

which gives

$$|\psi_{c\bar{c}}(0)|^2 = 0.063 \ (\text{GeV})^3 \ ,$$
$$|\psi_{b\bar{b}}(0)|^2 = 0.44 \ (\text{GeV})^3 \ .$$

The results of our helicity-amplitude calculations are tabulated below:

Process $Z \to Q\bar{Q}S$	$\Gamma(Z \to Q\bar{Q}s)$ (keV)	$\Gamma(Z \to Q\bar{Q}S)/\Gamma(Z \to Q\bar{Q})$
$Z \to c\bar{c}\psi/J$	47	1.6×10^{-4}
$Z \to c\bar{c}\eta_c$	145	4.8×10^{-4}
$Z \to b\bar{b}\Upsilon$	6.4	1.7×10^{-5}
$Z \to b\bar{b}\eta_b$	6.5	1.7×10^{-5}

The partial widths for these channels range from 6 to 145 keV, from which 60 to 1450 events could result in a year of running at LEP.

The most readily identifiable quarkonium decay modes are ψ/J, $\Upsilon \to l^+ l^-$ (with $l = e, \mu$) for which the branching fractions are $B(\psi/J \to l^+ l^-) = 0.14$ and $B(\Upsilon \to l^+ l^-) = 0.05$. Hence, the partial width for $Z \to c\bar{c}\psi$ decay with $\psi \to l^+ l^-$ is 6.5 keV, potentially giving 65 events/year at LEP. However, unless heavy-quark flavors are efficiently tagged, there is a large background to $Z \to c\bar{c}\psi$ resulting from $Z \to b\bar{b}$ with $b \to \psi s$ (or $\bar{b} \to \psi\bar{s}$) giving $Z \to b\bar{s}\psi$. The partial width for this background,

$$\Gamma(Z \to b\bar{b}) \times 2B(b \to \psi s) \simeq 4.6 \times 10^3 \text{ keV} \ ,$$

is two orders of magnitude larger than $\Gamma(Z \to c\bar{c}\psi)$. Nevertheless, the use of microvertex detectors, invariant-mass constraints, particle identification, and the difference in signal and background distributions may give sufficient background suppression to extract the signal. Detailed detector simulations are necessary to see if this is indeed possible.

The $Z \to b\bar{b}\Upsilon$ decay does not suffer from such backgrounds but unfortunately its rate is only marginally detectable. The partial width for $Z \to b\bar{b}\Upsilon$ with $\Upsilon \to l^+ l^-$ is 0.3 keV.

$W \to Q\bar{Q}' +$ quarkonium

The process $W \to Q\bar{Q}'\psi(Q\bar{Q})$ can be studied with the same techniques. For the case $\bar{Q}' = \bar{s}$, $Q = c$, the process is $W^+ \to c\bar{s}\psi/J$. There are only two Feynman diagrams here. The amplitude is

$$\mathcal{M}^W = \frac{4}{3} \frac{g_s^2 g}{\sqrt{2}} \frac{\Psi(0)}{2\sqrt{2m}} \frac{1_{ik}}{\sqrt{3}} \bar{u}(Q) \left\{ \frac{A_1 + A_2}{(\frac{1}{2}\psi + Q)^2} \right\} v(\bar{Q}') V_{QQ'} \ , \tag{8}$$

with

$$A_1 = \gamma^a \not{\epsilon}(\psi)(\not{\psi} + 2m)\gamma_a \frac{\not{\psi} + \not{Q} + m}{(\psi + Q)^2 - m^2} \not{\epsilon}(W) P_L \ ,$$
$$A_2 = \gamma^a \not{\epsilon}(\psi)(\not{\psi} + 2m)\not{\epsilon}(W) P_L \frac{\frac{1}{2}\not{\psi} - \not{W} + m'}{(\frac{1}{2}\psi - W)^2 - (m')^2} \gamma_a \ , \tag{9}$$

and $P_L = \frac{1}{2}(1 - \gamma_5)$. The matrix squared can be evaluated by the REDUCE program. The result is given in Appendix B. The numerical result for the decay rate is

$$\Gamma(W^+ \to c\bar{s}\psi/J) = 57 \text{ keV} \tag{10}$$

giving a branching fraction

$$B(W^+ \to c\bar{s}\psi/J) = 2.3 \times 10^{-5} \ . \tag{11}$$

$t \to W^+ b\Upsilon$

The partial width for the decay $t \to W^+ b\Upsilon$ of a heavy top quark to a W boson and the upsilon resonance can be obtained from the preceeding calculation by crossing, but with additional factors of $\frac{1}{3}$ due to averaging of the initial color, 3 due to summing over polarizations of W, and $\frac{1}{2}$ due to the initial spin average. The result for $m_t = 100$ GeV is

$$\Gamma(t \to W^+ b\Upsilon) = 0.04 \text{ keV} \ , \tag{12}$$

giving

$$\frac{\Gamma(t \to W^+ b\Upsilon)}{\Gamma(t \to W^+ b)} = 4.2 \times 10^{-7} \ . \tag{13}$$

$Z\to gg$ quarkonium and quarkonium $\to Zgg$

The process $Z\to gg\,^3S_1(Q\bar{Q})$ which proceeds through $Z\to Q\bar{Q}$ with the bremsstrahlung of two gluons has been studied previously.[6] Taking the strong coupling $\alpha_s=0.15$, the partial widths are

$$\Gamma(Z\to\psi/Jgg)=0.4\ \text{keV}\ , \tag{14}$$

and

$$\Gamma(Z\to\Upsilon gg)=1.0\ \text{keV}\ . \tag{15}$$

By crossing the amplitude for $Z\to gg\,^3S_1(Q\bar{Q})$ in Ref. 6, we can also calculate the decay $^3S_1(Q\bar{Q})\to Zgg$ of a heavy vector quarkonium state. We use the Coulomb-type binding potential to estimate the wave function at the origin

$$|\psi(0)|^2=\alpha_s^3M_S^3/(27\pi)\ .$$

The results for $\alpha_s=0.15$ are

$M(^3S_1)$ (GeV)	$\Gamma(^3S_1\to ggZ)$ (eV)	$\Gamma(^3S_1\to ggZ)/\Gamma(^3S_1\to gg\gamma)$
100	9.2	3×10^{-3}
120	32	7×10^{-3}
140	57	11×10^{-3}
160	83	14×10^{-3}
180	110	17×10^{-3}
200	140	20×10^{-3}

where the third column is based on a quark charge $e_Q=\frac{2}{3}$. The increase of Γ with M is a phase-space effect. Note that the $gg\gamma$ width is related to the ggg width by

$$\Gamma(^3S_1\to gg\gamma)/\Gamma(^3S_1\to ggg)=\tfrac{36}{5}e_Q^2(\alpha/\alpha_s)\simeq0.156\ . \tag{16}$$

Finally, we make similar calculations of pseudoscalar heavy quarkonium $\eta_Q\equiv{}^1S_0(Q\bar{Q})$ decays. For $\eta_Q\to ggZ$ decay, there are six Feynman graphs corresponding to the six permutations of the final-state particles. These graphs can be grouped into two sets, which are charge conjugates of each other. Hence only the axial-vector coupling of Z to $Q\bar{Q}$ contributes. The $\eta Q_i\bar{Q}_k$ vertex is

$$\frac{\Psi(0)}{\sqrt{2m}}\frac{1_{ik}}{\sqrt{3}}\gamma_5(\not{Q}-m) \tag{17}$$

in the nonrelativistic approximation; i.e., the constituent momentum satisfy $Q=\frac{1}{2}\eta_Q$. The axial-vector coupling is $g_Z^A(Q)=\pm\frac{1}{4}$ for $e_q=-\frac{1}{3},\frac{2}{3}$, respectively, in the standard model. (We note that exotic isosinglet quarks in E_6 models have zero axial-vector coupling to the Z boson but nonzero axial-vector coupling to the extra Z boson.) The $\eta_Q\to ggZ$ decay amplitude is

$$\mathcal{M}=\frac{g_s^2gg_Z^A(Q)}{(1-x_W)^{1/2}}\frac{\Psi(0)}{\sqrt{2m}}\sqrt{3}\times2\,\mathrm{Tr}\mathcal{A}\ , \tag{18}$$

with

$$\mathcal{A}=\frac{\gamma^\zeta(\not{Q}-\not{g}_1-\not{g}_2+m)\gamma^\beta(\not{Q}-\not{g}_1+m)\gamma^\alpha(\not{Q}+m)}{[(Q-g_1-g_2)^2-m^2][(Q-g_1)^2-m^2]}+\frac{\gamma^\zeta(\not{Q}-\not{g}_1-\not{g}_2+m)\gamma^\alpha(\not{Q}-\not{g}_2+m)\gamma^\beta(\not{Q}+m)}{[(Q-g_1-g_2)^2-m^2][(Q-g_2)^2-m^2]}-\frac{\gamma^\beta(\not{Q}-\not{g}_2+m)\gamma^\zeta(\not{Q}-\not{g}_1+m)\gamma^\alpha(\not{Q}+m)}{[(Q-g_1)^2-m^2][(Q-g_2)^2-m^2]}\ . \tag{19}$$

Here ζ, α, and β are the polarization indices of Z, g_1, and g_2, respectively. The negative sign of the last term in the above expression occurs from elimination of the γ_5 matrix. The gamma matrix algebra was performed using REDUCE (Ref. 3) and SCHOONSCHIP (Ref. 7) programs with the same result. We obtain the following concise expression for the partial width:

$$\frac{\Gamma(\eta_Q\to ggZ)}{\Gamma(\eta_Q\to gg)}=\frac{\sqrt{2}G_FM_\eta^2}{2!2\pi^2}\frac{(xy-\zeta)^2+r^2\zeta^2+r\delta}{x^2y^2(z-2r)^2}dx\,dy\ . \tag{20}$$

Here x, y, and z are the scaling energy variables,

$$x=2\eta\cdot g_1/M_\eta^2,\quad y=2\eta\cdot g_2/M_\eta^2,\quad z=2\eta\cdot Z/M_\eta^2\ , \tag{21}$$

and

$$\begin{aligned}&\zeta=1+r-z,\quad r=M_Z^2/M_\eta^2\ ,\\&\delta=xy(x+y)(2\zeta-xy)\\&\quad+2\zeta(xy-\zeta)[(x+y-\zeta)^2-2xy]\\&\quad-2\zeta(2+3xy)+xy\zeta(2+5xy)-x^2y^2\ .\end{aligned} \tag{22}$$

The δ term is the correction to the result of the equivalent Goldstone-boson calculation. If δ is set to zero, the formula becomes identical to that[8] for the decay $\eta_Q\to gga$ to a pseudoscalar boson a.

The calculated $\eta\to Zgg$ decay rates for $\alpha_s=0.15$ are

$M_{\eta Q}$ (GeV)	$\Gamma(\eta_Q\to ggZ)$ (GeV)	$\Gamma(\eta_Q\to gg)$ (GeV)	$\Gamma(\eta_Q\to ggZ)/\Gamma(\eta_Q\to gg)$
100	5.24×10^{-9}	3.0×10^{-3}	1.7×10^{-6}
150	1.14×10^{-6}	4.5×10^{-3}	2.5×10^{-4}
200	5.36×10^{-6}	6.0×10^{-3}	8.9×10^{-4}
1000	1.24×10^{-3}	3.0×10^{-2}	0.041

Here we find the interesting result that the decay of a superheavy quarkonium into ggZ may be observable, provided that its single-quark decay is suppressed by small mixing (e.g., for a fourth generation $e_Q=-\frac{1}{3}$ v quark with mass less than the t quark).

Our formula can be generalized to the production of any gauge boson which has significant axial-vector coupling to the heavy quark. An example is the axial-gluon production[9] from the heavy-quarkonium decay.[10] With appropriate change in couplings Eq. (20) agrees with the result in Ref. 10 for $\eta\to ag\gamma$ decay, where a is an axial gluon.

The crossed channels $Z\to gg\eta_{c,b}$ were also studied. The decay rates for $\alpha_s=0.15$ are

$$\Gamma(Z\to gg\eta_c)=5.6\ \text{keV}\ ,\qquad \Gamma(Z\to gg\eta_b)=3.6\ \text{keV}\ , \tag{23}$$

for which the ratios to the $Z\to\mu^+\mu^-$ keV partial width are

$$\Gamma(Z\to gg\eta_c)/\Gamma(Z\to\mu^+\mu^-)=6.6\times10^{-5}\ ,\qquad \Gamma(Z\to gg\eta_b)/\Gamma(Z\to e^+e^-)=4.3\times10^{-5}\ . \tag{24}$$

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APPENDIX A

In this appendix we give the squared amplitudes for the decay process $Z\to Q\bar{Q}\psi$. We introduce the notation

$$x=2Z\cdot\psi/M_Z^2,\quad y=2Z\cdot Q/m_Z^2,\quad z=2Z\cdot\bar{Q}/M_Z^2\ ,$$

and define the mass-squared ratio $r=m_Q^2/M_Z^2$. Then the square of A_1+A_2 in Eq. (3) summed over spins and Z,ψ polarizations is given by the expression

$$
\begin{aligned}
\sum_{\text{spin}}|A_1+A_2|^2=&\left[\frac{2}{(1-z)M_Z^2}\right]^2\\
&\times\left[\frac{4M_Z^2}{(1-z)^2}(96r^3(g_L^2+g_R^2-6g_Lg_R)\right.\\
&\quad+16r^2\{(g_L^2+g_R^2)[3(1-z)(x+y)+2(z-4)]+6g_Lg_R(z-1)\}\\
&\quad+2r[(g_L+g_R)^2(z^2-6z+5)+4(z^2-1)g_Lg_R+4(g_L^2+g_R^2)(xz-y)(1-z)]\\
&\quad+(g_L^2+g_R^2)(z-1)^2(1+yz-x))\\
&+\frac{M_Z^2}{\left[1-\frac{x}{2}\right]^2}\{384r^3(g_L^2-6g_Lg_R+g_R^2)+8r^2(g_L^2+g_R^2)(5x^2-2xy-2xz-68x-4yz-4y-20z+32)\\
&\quad+16r^2g_Lg_R(x^2+2xy+2xz+68x+4yz-44y+36z-72)\\
&\quad+4r(g_L^2+g_R^2)(-2x^3+32x^2-xy^2-5xy-xz^2+23xz-10x+6y^2+12yz-6y+6z^2\\
&\qquad-14z-16)\\
&\quad+8rg_Lg_R[x(x^2+y^2+z^2+xy+xz)-2(z-1)(z-16)\\
&\qquad-25x^2-3xy-15xz+38x-18y^2-8yz+26y]\\
&\quad+(g_L^2+g_R^2)[-16x^3+x(4+xy)(z-1)+x(-xz+17x-8y^2-4yz+12y)\\
&\qquad+4(z-1)(1-y)]\}\\
&+\frac{2M_Z^2}{(1-z)\left[1-\frac{x}{2}\right]}(384r^3(g_L^2+g_R^2-6g_Lg_R)\\
&\quad+16r^2(g_L^2+g_R^2)(2x^2+xy-5xz-20x-4yz-7y-7x+4)\\
&\quad+32r^2g_Lg_R(x^2+2xy+2xz+14x+4yz-17y+3z-12)\\
&\quad+r(g_L^2+g_R^2)[-8x^2(x+y)+2x^2z+70x^2+8xy^2+32xy-8xz^2\\
&\qquad+28xz-52x+8y^3-8y^2+32yz-96y-16z^2-24z+40]\\
&\quad+4rg_Lg_R(4x^2y+x^2z-x^2+2xyz-6xy+2xz^2-12xz+10x-4y^3\\
&\qquad+4y^2+12yz-12y+4z^2-8z+4)\\
&\quad+(g_L^2+g_R^2)[(1-z)(x^3-3x^2-xy^2+xy-xz^2-6y^2-4yz+10y)\\
&\qquad\left.-2(1-z)^2(z+2)+x(2-z-z^2)])\right]\ .
\end{aligned}
$$

The corresponding result for the square of A_3+A_4 in Eq. (3) summed over spins and polarizations can be obtained by the interchange $y\leftrightarrow z$ (i.e., $Q\leftrightarrow\bar{Q}$) in the expression above.

The remaining interference contributions are given by expressions

$$\sum 2\,\mathrm{Re}(A_1A_4^*)=\frac{32}{(1-z)^2(1-y)^2}\frac{1}{M_Z^2}$$
$$\times\{32r^3(5g_L^2+2g_Lg_R+5g_R^2)+8r^2(g_L^2+g_R^2)[(x-1)(y+z)+2y(y-z)-8x+2z^2+10]$$
$$+16r^2g_Lg_R(-x^2-6x-3y-3z+12)$$
$$+r(g_L^2+g_R^2)[(y+z)(-4x^2+12x)+8x(x-yz)-32x+4y^2(y-z)$$
$$+14yz-6y+4z^2(z-1)-4y(y+z^2)-6z+22]$$
$$+12rg_Lg_R(1-z)(y-1)+(g_L^2+g_R^2)(1-z)(y-1)(yz-x+1)\}\ ,$$

$$\sum 2\,\mathrm{Re}(A_2A_3^*)=\frac{8}{\left[1-\frac{x}{2}\right]^2(1-y)(1-z)}\frac{1}{M_Z^2}$$
$$\times(128r^3(5g_L^2+2g_Lg_R+5g_R^2)+8r^2(g_L^2+g_R^2)[-3x^2+20x-4yz-32+(36-2x)(y+z)]$$
$$+16r^2g_Lg_R[-3x^2-20x+4yz+40+(2x-20)(y+z)]$$
$$+4r(g_L^2+g_R^2)\{2x^3-16x^2+2x+4+(2-x)[y^2+z^2+3(y+z)]\}$$
$$+4rg_Lg_R[-2x^3+46x^2-28x+8yz-40+(y+z)(-2x^2+30x-4)+(y^2+z^2)(2x+4)]$$
$$+(g_L^2+g_R^2)(x-2)^2(y+z-yz-1)+32g_Lg_Rx^2(1-x))\ ,$$

$$\sum 2\,\mathrm{Re}(A_2A_4^*)=\frac{8}{(1-y)^2(1-z)\left[1-\frac{x}{2}\right]}\frac{1}{M_Z^2}$$
$$\times(128r^3(5g_L^2+2g_Lg_R+5g_R^2)+16r^2(g_L^2+g_R^2)(-x^2+4x-2yz+21y+2z^2+z-8)$$
$$+16r^2g_Lg_R[-4x^2+(6x+4z)(y-z)-12x-18y-2z+32]$$
$$+r(g_L^2+g_R^2)[2x^2(4x-y-11)+8(2-x)(y^2+z^2)+4x(y+2z+5)$$
$$-8y(yz-5z+1)-8(6z+1)]$$
$$+4rg_Lg_R[x^2(9-y)+6x(y^2+1)-2xy(z+2)-14xz$$
$$+4(y^2+5)(z-1)-8(z^2-3y+2yz)]$$
$$+(g_L^2+g_R^2)\{(y-1)[x^2(x-3)+2z(z+1)-4yz]+x(y+z^2-z-2)$$
$$-xy(y^2-2y+z^2-z)-6y^3+16y^2-14y+4\})\ .$$

The result for $\sum 2\,\mathrm{Re}(A_1A_3^*)$ is obtained from that of $\sum 2\,\mathrm{Re}(A_2A_4^*)$ by the replacement $y\leftrightarrow z$.

To calculate the partial widths, the above amplitude squares must be divided by 3 (the Z-polarization average) and integrated over the final-state phase space.

APPENDIX B

In this appendix we give the squared amplitudes for the decay process $W\to Q\bar{Q}'\psi$. In terms of the scaling variables defined as

$$x=2W\cdot\psi/M_W^2,\quad y=2W\cdot Q/M_W^2,\quad z=2W\cdot Q'/M_W^2\ ,$$

and the mass-squared ratios

$$r=m^2/M_W^2,\quad r'=m'^2/M_W^2\ ,$$

the squares of the amplitudes in Eq. (9) summed over spins are

$$\sum_{\text{spin}} |A_1|^2 = 4M_W^2(1-z+r'-r)^{-2}$$
$$\times\{(z-1-r')^2(yz-x+1-r')$$
$$+r[3(1-z^2)+z(8x-2y)(r'-z)-3r'^2+(2x-8y+6)(1+r')]$$
$$+r^2(73r'+27r-56xz-47yz+47x+56y+30z-133)\}\ ,$$

$$\sum_{\text{spin}} |A_2|^2 = M_W^2(1+r-r'-x/2)^{-2}$$
$$\times[264r^3+168r^2r'-40rr'^2-8r'^3-x^3y-x^2y^2-15x^3+6x^2y-4xy^2+11x^2+4xy+4y^2-8y+4$$
$$+r^2(51x^2+64xy+16y^2-388x+64y-188)$$
$$+r(-11x^3-10x^2y+126x^2-56xy+8y^2+44x+8y-64)$$
$$+rr'(6x^2-24xy+16y^2+16x+16y+104)$$
$$+r'(-x^3+2x^2y-8xy^3-50x^2-40xy-8y^2+44x+24y-16)$$
$$+r'^2(-x^2-8xy-44x-16y+20)]\ ,$$

$$\sum_{\text{spin}} 2\,\mathrm{Re}(A_2A_1^*) = -4M_W^2(1-z+r'-r)^{-1}(1+r-r'-x/2)^{-1}$$
$$\times[-102r^3-98r^2r'+6rr'^2+2r'^3+x^3y+2x^2y^2$$
$$+xy^3-x^3-8x^2y-7xy^2+6x^2+15xy+4y^2-9x-8y+4$$
$$+r^2(-57x^2-44y^2-101xy+181x+64y+128)$$
$$+r(9x^3+13x^2y-4y^3-31x^2+26y^2+16xy-36x-32y+42)$$
$$+rr'(x^2+12y^2+22xy-4y-48)$$
$$+r'(-2xy+2y^2+6x+4y-6)+r'^2(-xy+3x+4y)]\ .$$

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[1]For recent summaries of rare Z-boson decay model calculations within the standard model, see W. Bernreuther, M. J. Duncan, E. W. N. Glover, R. Kleiss, J. J. van der Bij, J. J. Gomez-Cadenas, and C. A. Heusch, CERN Report No. TH.5484/89 (unpublished); T. J. Weiler, in *Proceedings of the Second International Symposium on the Fourth Family of Quarks and Leptons,* Santa Monica, California, 1989, edited by D. B. Cline and A. Soni (New York Academy of Sciences, New York, in press).

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