Numerical solution for cosmological evolution of Newton's gravitational constant in superstring theories

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By using a computer to solve the ten-dimensional Einstein equations in superstring theories, we find that the present value for the time variation of Newton's gravitational constant is in the range of -1×10^{-11} to -6×10^{-12} yr^{-1} in the case of the flat internal potential, which confirms the previous perturbative estimate.

In recent years there has been much interest in exploring the time variation of fundamental constants in Kaluza-Klein theories and ten-dimensional superstring theories.[1-4] Generally in higher-dimensional theories the extra spatial dimensions form a very small compact manifold K (10^{-32} cm) in order to make these theories realistic. Since the coupling constants in the four-dimensional world are related to those in higher dimensions by a factor of the inverse volume of K, a cosmological evolution of the size of K would be reflected in a time variation of the coupling constants in four dimensions. Furthermore, in ten-dimensional superstring theories, the metric and other bosonic backgrounds in K are constrained by compactification and particle-phenomenology considerations.[5] Thus the cosmological evolution of the size of K can be determined dynamically and hence the time variation of the coupling constants in four dimensions may be calculable.[2] Generally in a field theory[6] with extra spatial dimensions, quantum effects in K (Ref. 7) gives rise to an effective potential which may fix the size of the internal space R in a vacuum and influence its cosmological evolution. In superstring theories, however, due to the nonrenormalization theorem,[8] such a potential for R_6 (the size of the six-dimensional internal space) is flat up to all orders in σ-model perturbation theory. So far, the exploration of nonperturbative supersymmetry-breaking effects,[9,10] such as world-sheet instantons, also has failed to produce a potential with a minimum at finite R_6, whose existence is only expected by conventional wisdom. In Ref. 2 it has been shown that the time variation of coupling constants critically depends on the shape of this potential. If the potential is flat, the present value for the time variation of Newton's gravitational constant, $\dot{G}/G$, is calculable, for example, for an open universe (the Robertson-Walker parameter $k=-1$):[2]

$$\left[\frac{\dot{G}}{G}\right]_0=(q_0-13\Omega_0H_0^2t_0^2/8)/t_0\ , \tag{1}$$

where H_0 is the Hubble constant, t_0 the age of the Universe, q_0 the deceleration parameter, and $\Omega\equiv8\pi G_0\rho_0/3H_0^2$. Here ρ_0 is the density in ordinary three-space and the subscript 0 denotes the present value of the quantity. By using the observational values[11] for these cosmological parameters, Eq. (1) gives us an estimate for $(\dot{G}/G)_0$ as[2]

$$\left[\frac{\dot{G}}{G}\right]_0\approx-1\times10^{-11\pm1}\ \mathrm{yr}^{-1}\ , \tag{2}$$

which overlaps the present observational upper bound[12]

$$\left|\frac{\dot{G}}{G}\right|\le1\times10^{-11}\ \mathrm{yr}^{-1}\ . \tag{3}$$

However, if the potential really has a minimum at finite R_6, $(\dot{G}/G)_0$ will be suppressed and become unobservably small.[2] Thus an improvement on the measurements of $\dot{G}/G$ will give us important information about the shape of the potential.

In Ref. 2 the ten-dimensional matrix is assumed to be of the generalized Robertson-Walker form

$$g_{MN}=\begin{bmatrix}-1 & & \\ & R_3^2(t)\tilde{g}_{ij}(x) & \\ & & R_6^2(t)\tilde{g}_{mn}(y)\end{bmatrix}\ , \tag{4}$$

where $i,j=1,2,3$; $m,n=4,\dots,9$; and $R_3(t)$ and $R_6(t)$ are the scale factors. $\tilde{g}_{ij}(x)$ is assumed to be maximally symmetric in three-space and $\tilde{g}_{mn}(y)$ is a Ricci-flat metric. In Eq. (4) factoring out a time-dependent scale for the internal space is due to an assumption of the existence of a "breathing mode" for the internal space.

From the theoretical point of view, in superstring theories it is possible to choose definitions of the four-metric, $g_{\mu\nu}$, $\mu,\nu=0,1,2,3$, that differ by conformal rescalings from each other. This is equivalent to choosing definitions of the four-dimensional gravitational constant G which have different functional dependences on the fields of the theory. It has been found[13] that the quantity GM_{GUT}^2 is invariant under any conformal rescaling of the metric. Another feature of superstring theories is that there is a dilaton field ϕ in addition to the size of the internal space, R_6, which can be seen as a scalar field in four dimensions. In Ref. 2 we have used the metric given by Eq. (4) and assumed ϕ to be constant since all particle-physics constants in ten dimensions do not vary with time. From the ten-dimensional point of view this seems to be natural. Experimentally what can be derived from observations should be independent of the choice of the metric. Since we have argued[2] that the time variation of the electromagnetic coupling constant $\dot{\alpha}/\alpha$ is 2 orders of magnitude lower than $\dot{G}/G$, we would like to point out that the estimate of $(\dot{G}/G)_0$ given by Eq. (2), in fact, also gives the relative deviation of the rate of the gravitational clock from that of the atomic clock, which could be tested directly.

In Ref. 2, Eq. (1) is obtained by solving ten-dimensional Einstein equations:

$$\frac{\ddot{R}_3}{R_3}+2\frac{\ddot{R}_6}{R_6}=-\frac{7}{24}\kappa_{10}^2\rho_0\left[\frac{R_3(t_0)}{R_3(t)}\right]^3, \tag{5a}$$

$$\frac{2k}{R_3^2}+\frac{\ddot{R}_3}{R_3}+\frac{2\dot{R}_3^2}{R_3^2}+6\frac{\dot{R}_3\dot{R}_6}{R_3R_6}=\frac{1}{8}\frac{\kappa_{10}^2\rho_0}{R_6^6}\left[\frac{R_3(t_0)}{R_3(t)}\right]^3, \tag{5b}$$

$$\frac{\ddot{R}_6}{R_6}+5\frac{\dot{R}_6^2}{R_6^2}+3\frac{\dot{R}_3\dot{R}_6}{R_3R_6}=\frac{1}{8}\frac{\kappa_{10}^2\rho_0}{R_6^6}\left[\frac{R_3(t_0)}{R_3(t)}\right]^3 \tag{5c}$$

for a matter-dominated universe in the perturbation theory context; we have shown[2] that for an open universe $(k=-1)$, in the large-t limit, the asymptotic solutions of Eqs. (5) are critically stable against time-dependent perturbations in $R_3(t)$ and $R_6(t)$.

Now we are going to find the numerical solutions of Eqs. (5) in order to further study the stability of the solutions. Equations (5) can be rewritten as

$$\frac{\ddot{R}_3}{R_3}=\frac{2}{R_3^2}-2\frac{\dot{R}_3^2}{R_3^2}-6\frac{\dot{R}_3\dot{R}_6}{R_3R_6}+\frac{1}{8}\rho(t), \tag{6a}$$

$$\frac{\ddot{R}_6}{R_6}=-5\frac{\dot{R}_6^2}{R_6^2}-3\frac{\dot{R}_3\dot{R}_6}{R_3R_6}+\frac{1}{8}\rho(t), \tag{6b}$$

$$\rho(t)=\frac{-3}{R_3^2}+3\frac{\dot{R}_3^2}{R_3^2}-18\frac{\dot{R}_3\dot{R}_6}{R_3R_6}+10\frac{\dot{R}_6^2}{R_6^2}, \tag{6c}$$

where

$$\rho(t)=\frac{\kappa_{10}^2\rho_0}{R_6^6}\left[\frac{R_3(t_0)}{R_3(t)}\right]^3 =8\pi G(t)\rho_0\left[\frac{R_3(t_0)}{R_3(t)}\right]^3. \tag{7}$$

For the present time, i.e., $t=t_0$, Eq. (7) reduces to

$$\rho(t_0)=8\pi G_0\rho_0=8\pi G_0\Omega_0\rho_c, \tag{8}$$

where $\rho_c=3H_0^2/8\pi G_0=1.1\times10^{-29}$ $[H_0/75\ \mathrm{km\,sec^{-1}\,Mpc^{-1}}]^2\mathrm{g\,cm^{-3}}$, being the critical density, and $G_0=6.6732\times10^{-8}$ dyn cm^{2}g^{-2}. We use the most "satisfactory" set of cosmological parameters:[11]

$$(\Omega_0,H_0)=(0.05,67\ \mathrm{km\,sec^{-1}Mpc^{-1}}) \tag{9a}$$

and the extreme sets[11]

$$(\Omega_0,H_0)=\begin{cases}(0.05,100\ \mathrm{km\,sec^{-1}\,Mpc^{-1}}), & (9b)\\ (1,40\ \mathrm{km\,sec^{-1}Mpc^{-1}}). & (9c)\end{cases}$$

We obtain the corresponding values for $\rho(t_0)$ as follows:

$$\text{(i)}\ \rho(t_0)=7.36\times10^{-37}\ \mathrm{dyn\,g^{-1}cm^{-1}}, \tag{10a}$$

$$\text{(ii)}\ \rho(t_0)=1.64\times10^{-36}\ \mathrm{dyn\,g^{-1}cm^{-1}}, \tag{10b}$$

$$\text{(iii)}\ \rho(t_0)=5.24\times10^{-36}\ \mathrm{dyn\,g^{-1}cm^{-1}}. \tag{10c}$$

From the numerical-solution point of view, solving Eqs. (5) or (6) is an initial boundary-value problem of a system of second-order ordinary differential equations. The initial boundary conditions are the asymptotic solutions[2] of $R_3(t)$ and $R_6(t)$ in the infinitely dilute limit of matter:

$$\rho(t)\xrightarrow[t\to\infty]{}0, \tag{11}$$

namely,

$$R_3(t)\xrightarrow[t\to\infty]{}\infty,\qquad \dot{R}_3(t)\xrightarrow[t\to\infty]{}\mathrm{const}, \tag{12a}$$

and

$$R_6(t)\xrightarrow[t\to\infty]{}\mathrm{const},\qquad \dot{R}_6(t)\xrightarrow[t\to\infty]{}0. \tag{12b}$$

Since in Eqs. (5), and hence in Eqs. (6), only two of them are independent due to the Bianchi identities, we need four boundary conditions to determine four integration constants in $R_3(t)$ and $R_6(5)$. In what follows, for the convenience of numerical computation, we choose $R_3(t_0)$, $\dot{R}_3(t_0)$, $R_6(t_0)$, and $\dot{R}_6(t_0)$ as the four initial boundary values for Eqs. (6), and seek its solutions which satisfy the asymptotic conditions (11) and (12).

From dimensional analysis, we can see that each term on both sides of Eqs. (6) is proportional to $1/t^2$, because t is the only relevant cosmological time scale. This fact restricts the order of magnitude of the ratio of $\dot{R}_6(t_0)$ to $R_6(t_0)$ to be

$$\frac{\dot{R}_6(t_0)}{R_6(t_0)}\sim\frac{1}{t_0}\approx1.98\times10^{-18}\ \mathrm{sec}^{-1}, \tag{13}$$

although we do not know the exact value for $R_6(t_0)$ at present. Similarly, the possible initial values for $R_3(t_0)$ and $\dot{R}_3(t_0)$ can be chosen as

$$R_3(t_0)\sim act_0,\quad \dot{R}_3(t_0)\sim ac, \tag{14}$$

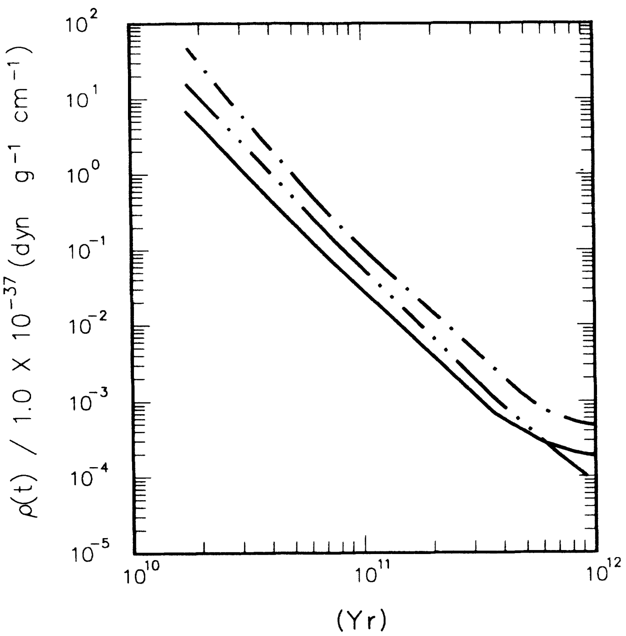

FIG. 1. The scale factor of the ordinary three-space $R_3(t)$; 1 light year $=9.4605\times10^{17}$ cm.

where the coefficient a is the order of unity and $c=2.9979\times10^{10}$ cm sec^{-1}, being the speed of light. We use the LSODA (Ref. 14) subroutine to solve Eqs. (6). The possible initial values for $R_3(t)$, $\dot{R}_3(t)$, $R_6(t)$, and $\dot{R}_6(t)$ at $t\sim1\times10^{10}$ yr are input by using the trial-and-error method. Corresponding to the satisfactory and the extreme values for $\rho(t_0)$, i.e., Eqs. (10a)–(10c), the numerical solutions of Eqs. (6), which satisfy the asymptotic

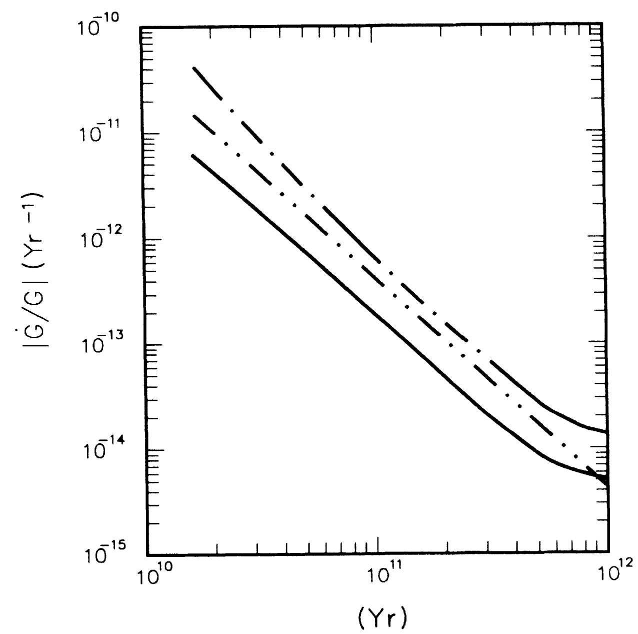

FIG. 2. The scale factor of the internal six-dimensional space $R_6(t)$, assuming its present value to be taken as $R_6(t_0)\sim10^{-32}$ cm.

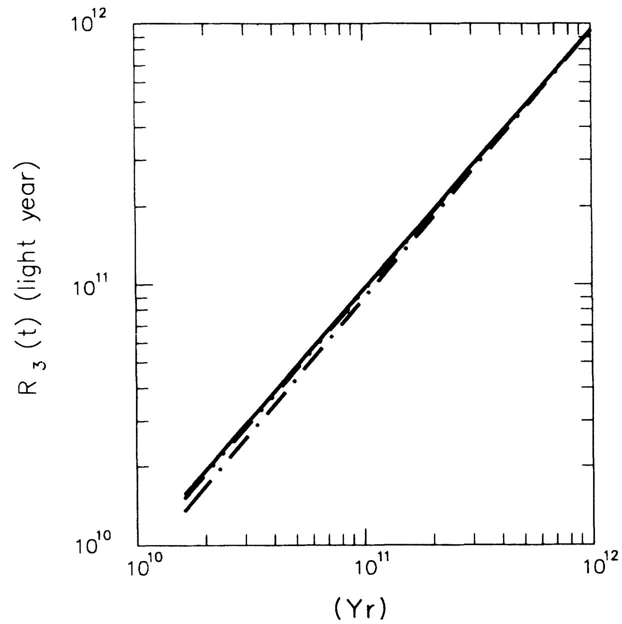

FIG. 3. The ten-dimensional density, $\rho(t)=(\kappa_{10}^2\rho_0/R_6^6(t))(R_3(t_0)/R_3(t))^3$, where κ_{10} is the ten-dimensional gravitational constant, ρ_0 the present value of the density in ordinary three-space, t_0 the age of the Universe, and $R_3(t)$ and $R_6(t)$ the scale factor of the ordinary three-space and the internal six-space, respectively.

conditions (11) and (12), and of $\dot{G}/G$ are given in Figs. 1–4. In these figures, the solid lines, the one-point-dashed lines, and the triplet-points-dashed lines correspond to cases (i)–(iii) above, respectively.

The present values for $(\dot{G}/G)_0$ and $\rho(t_0)$, which are obtained from these numerical solutions, are listed as

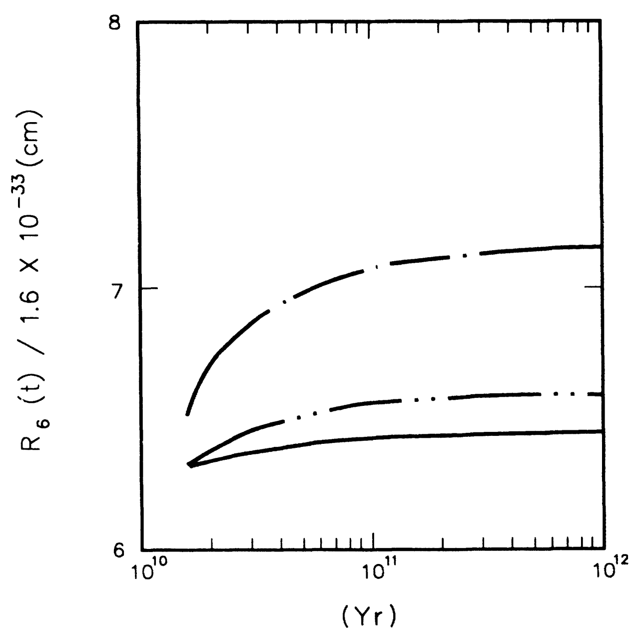

FIG. 4. The time variation of Newton's gravitational constant in superstring theories $|\dot{G}/G|$ means the absolute values of $\dot{G}/G$.

$$\text{(i) } t_0=1.6\times10^{10}\text{ yr: } (\dot G/G)_0=-6.47\times10^{-12}\text{ yr}^{-1}, \quad \rho(t_0)=7.36\times10^{-37}\text{ dyn g}^{-1}\text{ cm}^{-1}; \tag{15a}$$

$$\text{(ii) } t_0=1.6\times10^{10}\text{ yr: } (\dot G/G)_0=-1.47\times10^{-11}\text{ yr}^{-1}, \quad \rho(t_0)=1.65\times10^{-36}\text{ dyn g}^{-1}\text{ cm}^{-1}; \tag{15b}$$

$$\text{(iii) } t_0=1.6\times10^{10}\text{ yr: } (\dot G/G)_0=-4.33\times10^{-11}\text{ yr}^{-1}, \quad \rho(t_0)=5.25\times10^{-36}\text{ dyn g}^{-1}\text{ cm}^{-1}. \tag{15c}$$

Thus, from the numerical solutions, the range of $(\dot G/G)_0$ can be expressed as

$$\left[\frac{\dot G}{G}\right]_0 \approx -1\times10^{-11}\text{ yr}^{-1}\text{ to }-6\times10^{-12}\text{ yr}^{-1}, \tag{16}$$

which confirms the perturbative estimate Eq. (2).

Some remarks about the numerical solutions of Eqs. (6) are in order. These solutions, within the accuracy of the computer, only depend on the ratio of $\dot R_6(t_0)$ to $R_6(t_0)$, because $R_6(t)$ only appears in combination with $\dot R_6(t)$ and $\ddot R_6(t)$ in Eqs. (6). In order to obtain a smooth behavior of $R_3(t)$, the coefficient a in $R_3(t)$ has to be in the range $0.8 < a < 1$. Within this range we have not seen any rapid instability[13] of solutions arise.

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