Bifurcation to a chiral-symmetry-breaking state in continuum quantum electrodynamics

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Dyson-Schwinger equations for a fermion propagator in the Landau gauge are studied in the approximation of a small-momentum-transfer vertex function. There exists a critical value of the coupling constant above which the ordinary solution bifurcates to another, chiral-symmetry-breaking solution. The new solution does not require either infrared or ultraviolet momentum cutoffs.

Recent results obtained within the framework of lattice field theory[1] provide convincing evidence that quantum electrodynamics may possess a second, chiral-symmetry-breaking phase. The existence of such a phase would greatly increase the family of theories that exhibit physically interesting features such as dynamical-symmetry breaking, anomalous scaling, etc.

The problem of dynamical chiral-symmetry breaking was also studied in the past with use of analytical methods. References 2–8 provide a small (and by no means exhaustive) sample of various attempts that are relevant to this contribution.

An interesting approach to the study of the Dyson-Schwinger (DS) equation for the fermion propagator in chirally symmetric electrodynamics was lately undertaken by Atkinson and Johnson.[5] They used various regularization cutoff procedures, with and without a momentum-dependent running coupling constant mimicking the asymptotic behavior of the vertex function in quantum chromodynamics. In such an approach, symmetry breakdown is indicated by the existence of a positive critical value of the coupling constant, above which the trivial, chirally invariant solution bifurcates away to a nontrivial solution that violates chiral symmetry and creates a fermion mass. Such critical values were indeed found; however, in all cases some kind of momentum cutoff was needed to be introduced in order to prove their existence.` Unfortunately, momentum cutoffs necessarily bring a mass scale parameter into the theory. This is conceptually troubling, mainly because one cannot know whether the fermion mass is an artifact of a hidden transmutation of the cutoff scale parameter or, indeed, a product of truly dynamical process of mass generation.

Before we proceed further, let us first briefly analyze the source of divergences and formal reasons for the introduction of cutoffs in the DS equation.

The equation for the fermion propagator has the form

$$\begin{aligned} S^{-1}(p) &= A(p)\not{p} - B(p) \\ &= \not{p} + ie^2 \int đq\, \Gamma_\mu(p,q) S(q) \Gamma^0{}_\nu D^{\mu\nu}(p-q)\ , \end{aligned} \tag{1}$$

where $đq = d^4q/(2\pi)^4$.

In the above expression, $\Gamma^0{}_\nu$ represents the fundamental vertex of the theory (as determined by the bare Lagrangian), while $\Gamma_\mu(k,q)$ represents the full, dressed vertex function. Both $\Gamma_\mu(k,q)$ and the boson propagator $D^{\mu\nu}(p-q)$ are given by their own DS equations. These equations involve Green's functions of higher order in the number of external legs, and their presence initiates an infinite hierarchy of similar equations. For the purpose of any practical calculations, such hierarchy must be broken, e.g., by postulating a particular form of certain Green's functions, or by performing partial summations over subclasses of Feynman diagrams. In the widely used truncation procedure of Johnson, Baker, and Willey,[9] one substitutes the bare values of Γ_μ and $D^{\mu\nu}$. This method, although simple, has serious disadvantages: most notably, it breaks the Ward-Takahashi identity by disregarding an important class of nonplanar Feynman diagrams.[10] In addition, what makes it unsuitable for our purpose is that the use of a constant vertex and a free photon propagator means that standard rules of traditional power counting are in effect and the approximation produces ultraviolet- (UV-) divergent integrals. Regularization of UV divergences introduces an undesirable dimensional parameter into the theory.

To circumvent this difficulty, we shall leave the gluon propagator undressed, and for $\Gamma_\mu(k,q)$ we shall adopt an ansatz we used in a similar context in Ref. 8 and postulate that the vertex function is dominated by the contribution from the vicinity of the photon pole, i.e., the zero-momentum-transfer sector:

$$\Gamma^\mu(k,p) \approx \Gamma^\mu(p,p)\ . \tag{2}$$

The longitudinal part of $\Gamma^\mu(p,p)$ is fully determined by the Ward identity

$$\Gamma^\mu(k,p) = \Gamma^\mu(p,p) = \partial S^{-1}(p)/\partial p_\mu\ . \tag{3}$$

It is known that, in the Landau gauge and with massive fermions, the above approximation produces the DS equations that are free of UV divergences.[11]

Substituting (3) into (1), rotating to the Euclidean momentum space and integrating over angles we obtain the following equations for the functions $A(x)$ and $B(x)$:

$$\begin{aligned} x^2 A(y) = x^2 &+ (g/2)\int_0^x dy\, y^2[(G-3)\tau + 2G\xi] \\ &+ (g/2)x^2 \int_x^\infty dy [(G-3)\tau + 2G\xi]\ , \end{aligned} \tag{4}$$

and

$$xB(x)=\tfrac{1}{2}(3+G)(e/4\pi)^2\int_0^x dy\, y(y\eta+2\zeta)$$
$$+(e/4\pi)^2 x\int_x^\infty dy[2G\eta+(3+G)\zeta]$$
$$+\tfrac{1}{2}(3-3G)x^2\int_x^\infty \eta dy\ , \tag{5}$$

where x and y represent Euclidean momenta squared and

$$\tau(x)=[x(dA/dx)A+(dB/dx)B]/(xA^2+B^2)\ , \tag{6a}$$

$$\eta(x)=[(dA/dx)B-A(dB/dx)]/(xA^2-B^2)\ , \tag{6b}$$

$$\zeta(x)=AB/(xA^2+B^2)\ , \tag{6c}$$

$$\xi(x)=A^2/(xA^2+B^2)\ . \tag{6d}$$

The infrared (IR) divergence is inherent in the massless truncated theory which has the structure of integrands similar to the ordinary perturbation expansion. In our approach there is no need to introduce IR cutoffs in (4) from the very beginning, as a precondition assuring consistency at zero momentum. This is because in the massless ($B\equiv 0$) case the equation may still possess a solution for $A(x)$ which tends to a finite value at $x=0$. Then the numerator in (6a) vanishes at $x=0$ and the integrand is regular. We shall see that such a solution exists only for the values of the coupling constant above a critical value. Below this value, there is only an infrared-divergent solution which near $x=0$ behaves as x^{-2}. The model lacks a dimensional parameter to scale this divergence and such a scale must be introduced, by hand, in the form of an IR cutoff in the integrals in (4). We shall see that for the latter solution the boundary conditions for the equivalent differential equation can be imposed in the vicinity of zero, but not at $x=0$.

The disappearing of the UV divergence is caused by the presence of the derivative of $A(x)$ in the numerator. The subsequent analysis will show that the solution tends to a finite limit at infinity. Then $dA/dx=0$, the integrand tends to zero at $x=\infty$, and thus the UV divergence is also tamed.

The chiral-invariant ($B\equiv 0$) solution bifurcates to a chiral-symmetry-breaking solution at the lowest value of the coupling constant for which the linearized (in B) version of Eqs. (4) and (5) has a solution. In the Landau gauge ($G=0$) the bifurcation equations have the form

$$A(x)=1-(g/2x^2)\int_0^x dy(G-3)y^2(dA/dy)/A(y)$$
$$-3(g/2)\int_x^\infty dy(dA/dy)/A(y)\ , \tag{7}$$

and

$$x\beta(x)=(g/2)\int_0^x dy\{y[(dA/dy)\beta(y)-A(y)(d\beta/dy)]/A^2(y)+2\beta/A\}$$
$$+gx\int_x^\infty dy\,\beta/(yA)+(g/2)x^2\int_x^\infty dy[(dA/dy)\beta-A(d\beta/dy)]/(yA^2)\ , \tag{8}$$

where

$$\beta\equiv\delta B\ ,$$

and

$$g=3(e/4\pi)^2\ .$$

The nonlinear equation (7) for $A(x)$ must be solved first, and then its solution will be substituted in the kernel of the integro-differential equation (8) for $\beta(x)$.

Equation (7) is equivalent to the nonlinear differential equation

$$d^2A(t)/dt^2+2\,dA(t)/dt+g(dA/dt)/A(t)=0\ , \tag{9}$$

where $t=\ln(x/\mu)$.

The only role of the arbitrary dimensional parameter μ introduced above is to convert the differential equation to an autonomous form. Autonomous equations do not explicitly involve the independent variable. Techniques for such equations are readily available,[12] and this greatly simplifies our presentation. Although the dependence of physical quantities on μ may be important from the point of view of the renormalization-group analysis, the results presented in this paper are μ independent and all steps of the forthcoming analysis could be repeated with t eliminated in terms of x. Let us arbitrarily substitute $\mu=1$.

The boundary conditions for (9) are

$$\lim_{t\to\infty} e^{2t}(dA/dt)=0 \tag{10}$$

and

$$\lim_{t\to+\infty}[A(t)+\tfrac{1}{2}(dA/dt)]=1\ . \tag{11}$$

Following the standard procedures for autonomous equations, let us define

$$X(t)=A(t),\quad Y(t)=dA/dt\ . \tag{12}$$

Then

$$dY/dt\equiv P(X,Y)=-(2+g/X)Y\ , \tag{13}$$

$$dX/dt\equiv Q(X,Y)=Y\ , \tag{14}$$

$$dY(X)/dX\equiv F(X,Y)$$
$$=Q(X,Y)/P(X,Y)=-2-gX-1\ . \tag{15}$$

The last equation is exactly solvable in an implicit form. The solution is

$$t=-\int_{A(0)}^{A(t)} d\zeta[2(\zeta-\zeta_0)+g\ln(|\zeta/\zeta_0|)]^{-1}\ . \tag{16}$$

There are several distinct branches of this solution. For the first one, the IR point $t=-\infty$ is reached at $\zeta=\infty$ and the UV point $t=+\infty$ at $\zeta=\zeta_0$.

Near the IR point, this solution behaves asymptotically as $t\sim\ln(A)^{-1/2}$, and $A(t)\sim\exp(-2t)$ increases too fast to satisfy the IR boundary condition without an additional cutoff. Therefore condition (10) must be applied at a finite point.

The second solution corresponds to another interval of integration in (16), with the $t=+\infty$ point at ζ_0 and the $t=-\infty$ at a finite point ζ_1, such that

$$2(\zeta_1-\zeta_0)=g\ln(-\zeta_1/\zeta_0)\ . \tag{17}$$

If such a point does not exist, the integration extends to $\zeta=-\infty$. Then the asymptotic behavior is again $A(t)\sim\exp(-2t)$ and the boundary condition cannot be met without an IR cutoff. If (17) has a solution, condition (10) is trivially satisfied because the expression in the square brackets in (16) is nothing else but dA/dt.

It is easy to find that condition (17) is satisfied independently of the value of the constant ζ_0, provided $g[\ln(g/2)+1]>2$, i.e., for

$$g>g_{\text{crit}}=7.18\ . \tag{18}$$

The constant ζ_0 is equal to the value of $A(x)$ at $x=\infty$ and is determined by condition (11). For $\zeta=\zeta_0$ the denominator in the integrand in (16) vanishes; hence, $\lim_{t\to\infty}dA/dt=0$ and $A(\infty)=\zeta_0=1$.

For the values of the coupling constant exceeding the critical value (18), lies a third solution which corresponds to the integration interval extending from ζ_1 to $\zeta=-\infty$. This solution not only requires an IR cutoff, but also violates the UV condition. At $A=\zeta_1$, $dA/dt=0$, but also $A<0$, and therefore condition (11) cannot be met (even with cutoffs).

We can now proceed to the study of the bifurcation equation for δB. The integral equation (8) can be converted to the differential form

$$d^3(x\beta)/dx^3=g\beta/(x^2A)+g\{d[(d\beta/dx)/A-(dA/dx)\beta/A^2]/dx\}\ . \tag{19}$$

The boundary conditions for (19) are

$$\lim_{x\to 0}(x\beta)=0\ , \tag{20}$$

$$\lim_{x\to\infty}d(x\beta)/dx=0\ , \tag{21}$$

$$\lim_{x\to\infty}\{[d^2(x\beta)/dx^2]+g[\beta/xA+(dA/dx)\beta/A^2-(d\beta/dx)/A]\}=0\ . \tag{22}$$

As $x\to 0$, $dA/dx\to 0$ and $A(x)\to A_0=\zeta_1$, with ζ_1 given by (17). The asymptotic solution of (19) has the form

$$\beta(x)\sim c_1x^{n_1}+c_2x^{n_2}+c_3x^{n_3}\ , \tag{23}$$

where the exponents n_i satisfy

$$n^3+(g/A_0-1)n^2-(g/A_0)n-g/A_0=0\ . \tag{24}$$

At the critical point $n_1\approx-0.596$, $n_2\approx 1.69$, $n_3\approx 11.2$, and, as g increases, n_2 and n_3 remain positive, while the value of n_1 does not fall below -1. The IR condition (20) is therefore satisfied by all three particular solutions.

For large x, the asymptotic equation admits two types of solutions: one exhibiting ordinary power behavior x^n and another oscillating very rapidly:

$$\beta\sim x^{\alpha}\{c_2\sin[\omega\ln(x)]+c_2\cos[\omega\ln(x)]\}\ . \tag{25}$$

The x^n solution must be rejected since at the critical point the exponent $n=6.39$, and for larger values of g it even further increases. At the critical point, the exponent α and the "frequency" ω in (25) have the values 0.396 and 0.984, respectively. They change relatively slowly with the coupling, e.g., for $g=10^6$ their respective values are slightly less than 0.50 and 10.0. Because of this infinitely quick oscillatory behavior around zero at infinity, the average values of (21) and (22) (which appear only in the integrands) are zero, and the boundary conditions are satisfied in a generalized sense. A similar oscillatory behavior was found in the solutions for B obtained with use of the constant vertex approximation.[2]

Below the critical value of the coupling constant, cutoffs must be introduced, and the boundary conditions (20)–(22) are imposed at finite points. In the UV region the asymptotic behavior of Eq. (19) has the form of (23) with n determined by the solutions of

$$n^3-gn^2+(g-3)n+(2-g)=0\ . \tag{26}$$

In the IR region the solution behaves as

$$\beta\sim c_1x^{-1}+c_2x+c_3\ . \tag{27}$$

The conditions (21) and (22) fix two out of the three arbitrary constants in the general solution, leaving only an overall normalization factor. Condition (20) then implies that $\beta\equiv 0$, unless lucky cancellations occur for a few (cutoff-dependent) values of the coupling constant g. We are not yet able to rule out a possiblity that, for certain positions of the cutoff points, such additional critical points show up below the critical value (8). Even if they do, the principal conclusion remains unaltered. For small values of the coupling constant chiral symmetry is not broken. Because of the inherent infrared divergence, chiral-symmetric solutions of the Dyson-Schwinger equation for the propagator require a momentum cutoff. Above the critical value of the coupling constant, $g_{\text{crit}}=3(e_{\text{crit}}/4\pi)^2>7.18$, another solution appears. It breaks chiral symmetry and introduces the fermion mass term. Naturally enough, in the latter case the IR cutoff is no longer needed. The lack of an UV cutoff is due to the choice of Landau gauge in which the (logarithmic) divergence generating $\xi(x)$ term is eliminated from the integral (4).

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[12]See, e.g., H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations,* 2nd ed. (Dover, New York, 1962).