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Partial vorticity in classical Yang-Mills solutions

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The classical SU(2) Yang-Mills equation is solved numerically in $2+1$ dimensions with a static central point source. We study the static potentials whose rotational symmetry in two-space is manifest, i.e., does not require a gauge transformation for its realization. In the radial gauge, such a potential typically consists of two functions of the radial coordinate: a highly oscillatory tangential component A_ϕ and a timelike component A_0 which is essentially monotonic. These two components point in two perpendicular SU(2) directions. The interest of these solutions lies in the following features. First, the asymptotic behavior of A_0 at large distances is linear and hence more confining than the logarithmic, Coulomb-type behavior resulting from $A_\phi \equiv 0$. Second, in a suitably defined strong-coupling limit, the vorticity oscillates so tightly as to disappear, while A_0 remains linear instead of approaching the Coulomb form. Therefore the strong-coupling behavior could be relevant to $3+1$ dimensions with spherical symmetry. A third feature is that, in the strong-coupling limit, the source becomes renormalized to an infinitesimal bare charge.

I. INTRODUCTION

The purpose of this paper is to call attention to the results of a straightforward exercise in solving the Yang-Mills equation. The interest in these solutions arises from their somewhat unexpected features, which to this author's knowledge have not been brought to light before, and which conceivably might be a classical reflection of a confinementlike mechanism.

One reason for studying the classical solutions of the Yang-Mills equation with a static central source has been the hope that its large-distance behavior would offer a clue to the confinement mechanism of QCD. One might have expected that the flux-tube mechanism of Kogut and Susskind[1] and the lattice analysis of Wilson[2] would find their confirmation in a linearly rising quark-antiquark potential, as would be displayed on the most elementary level by the potential of a *single* central source. However, several studies[3-7] of the static centrally symmetric classical Yang-Mills solutions with a central source have consistently shown that, outside the source volume, the solutions are of the Coulomb form $V \sim r^{-1}$ in three space dimensions; for a sufficiently small value of the coupling constant, that form is stable under small perturbations, while for larger values no stable, static, spherically symmetric solution exists. Some static, spherically nonsymmetric solutions[8] with a central source have also been found, but their message, if any, concerning the $q\bar{q}$ system is not clear.

More recently, an alternative approach to confinement has postulated random fields in the Yang-Mills vacuum.[9] Specifically, a result of Olesen[10] states that the hypothesized existence of random SU(n) fluxes in the Yang-Mills vacuum is equivalent to that of the confinement property. The confining effect of the random flux has been described by that author and others as a "dimensional reduction." (The earliest discovery of this effect occurred in solid-state theory; see Ref. 10 for additional literature.)

In the fully quantized theory, the randomness would have to be temporal as well as spatial. It is therefore of considerable interest that in a classical context Matinyan, Savvidi, and Ter-Arutyunyan-Savvidi have discovered that the Yang-Mills equation (without external source) can have solutions with a stochastic time dependence.[11-15] In these references, the potentials are chosen to have a simple spatial behavior, such as constancy or spherical symmetry.

The present paper examines numerically the static classical SU(2) Yang-Mills solutions in $2+1$ dimensions, with a central point source. We specialize to those solutions which exhibit rotational symmetry in space without the need for a gauge transformation to accompany each rotation ("manifest rotational symmetry"). The $(2+1)$-dimensional case has the unique feature that the rotational symmetry allows a vortex solution, i.e., a potential with a nonzero tangential component. Thus, on the one hand, novel features such as a non-Coulomb behavior are observed; but, on the other hand, these features might seem irrelevant to the $(3+1)$-dimensional world. There exists a strong-coupling limit, however, which restores a possible relevance to $3+1$ dimensions due to an effective disappearance of the vorticity, while the non-Coulomb features are retained.

In view of the stochastic behavior in time discussed in Refs. 11–15, it is natural to examine the static model for any indications of spatial chaos. No such chaos is visible here. However, the "next best" phenomenon does occur: while, in the radial gauge, the time component A_0 of the potential appears to be smooth and, beyond a certain value of r, monotonic and even linear, the tangential component A_ϕ exhibits rapid oscillations whose wavelength and amplitude both tend to zero at large r. In the appropriate strong-coupling limit the vorticity oscillates so tightly as a function of r as to become invisible on a finite scale; A_0 remains smooth and linear. (The simultaneous nonzero value of A_0 and A_ϕ in the radial gauge may be referred to as "partial velocity.")

We note that the lack of spatial chaos in the present calculations does not preclude its existence in less symmetric situations.

II. SU(2) YANG-MILLS EQUATIONS IN 2+1 DIMENSIONS

Let A^μ be the matrix representation

$$A^\mu(x)=\tfrac{1}{2}\sigma_a A_a^\mu(x) \quad (\mu=0,1,2;a=1,2,3) , \tag{2.1}$$

where the A_a^μ are real functions and the σ_a are the Pauli matrices. We consider the Yang-Mills equation with a static external point source:

$$D_\mu F^{\mu\nu}=-(2\pi e')(\tfrac{1}{2}\sigma_3)g^{0\nu}\delta(\mathbf{r}) , \tag{2.2}$$

where

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu-ie[A^\mu,A^\nu] , \tag{2.3}$$

$$D_\mu=\partial_\mu-ie[A_\mu,\ ] ; \tag{2.4}$$

e and e' are adjustable constants; the external source is oriented without loss of generality in the third SU(2) direction; the factor -2π is for later convenience.

In polar coordinates

$$x^1=r\cos\phi,\quad x^2=r\sin\phi , \tag{2.5}$$

without time dependence, and with

$$\begin{aligned}F^{12}&=-B ,\\ F^{01}&=-E_r\cos\phi+E_\phi\sin\phi ,\\ F^{02}&=-E_r\sin\phi-E_\phi\cos\phi ,\end{aligned} \tag{2.6}$$

and similarly for A^1,A^2, Eq. (2.2) reads

$$\partial_r E_r+\frac{1}{r}E_r+\frac{1}{r}\partial_\phi E_\phi+ie[A_r,E_r]+ie[A_\phi,E_\phi]=-(2\pi e')(\tfrac{1}{2}\sigma_3)\delta(\mathbf{r}) , \tag{2.7a}$$

$$\frac{1}{r}\partial_\phi B+ie[A_0,E_r]+ie[A_\phi,B]=0 , \tag{2.7b}$$

$$\partial_r B-ie[A_0,E_\phi]+ie[A_r,B]=0 , \tag{2.7c}$$

where, from (2.3),

$$\begin{aligned}E_r&=-\partial_r A_0+ie[A_0,A_r] ,\\ E_\phi&=-\frac{1}{r}\partial_\phi A_0+ie[A_0,A_\phi] ,\\ B&=-\frac{1}{r}\partial_\phi A_r+\partial_r A_\phi+\frac{1}{r}A_\phi+ie[A_r,A_\phi] .\end{aligned} \tag{2.8}$$

Going over to the radial gauge

$$A_r=0 \tag{2.9}$$

and, assuming manifest rotational symmetry,

$$\partial_\phi A_0=\partial_\phi A_\phi=0,\quad \partial_\phi E_r=\partial_\phi E_\phi=\partial_\phi B=0 , \tag{2.10}$$

the above simplifies to

$$\begin{aligned}&\partial_r E_r+\frac{1}{r}E_r+ie[A_\phi,E_\phi]=-(2\pi e')(\tfrac{1}{2}\sigma_3)\delta(\mathbf{r}) ,\\ &[A_0,E_r]+[A_\phi,B]=0 ,\\ &\partial_r B-ie[A_0,E_\phi]=0 ,\end{aligned} \tag{2.11}$$

where

$$E_r=-\partial_r A_0,\quad E_\phi=ie[A_0,A_\phi],\quad B=\partial_r A_\phi+\frac{1}{r}A_\phi . \tag{2.12}$$

If we renounce vorticity, $A_\phi=0$, the solution reduces to the Coulomb form $B=E_\phi=0$, $E_r\propto r^{-1}$. Instead, the present paper focuses on that particular class of solutions with $A_\phi\neq0$ whose SU(2) form is

$$A_0=\tfrac{1}{2}\sigma_3 a_0(r),\quad A_\phi=\tfrac{1}{2}\sigma_1 a_\phi(r) \tag{2.13}$$

for some real functions a_0,a_ϕ. Denoting the radial derivative by $\partial_r=\partial$, we find, from (2.12),

$$\begin{aligned}E_r&=-\tfrac{1}{2}\sigma_3\partial a_0 ,\\ E_\phi&=-\tfrac{1}{2}e\sigma_2 a_0 a_\phi ,\\ B&=\tfrac{1}{2}\sigma_1\left[\partial a_\phi+\frac{1}{r}a_\phi\right] ,\end{aligned} \tag{2.14}$$

while (2.11) reduces to

$$\begin{aligned}&\partial^2 a_0+\frac{1}{r}\partial a_0-e^2a_0a_\phi^2=2\pi e'\delta(\mathbf{r}) ,\\ &\partial^2 a_\phi+\frac{1}{r}\partial a_\phi-\frac{1}{r^2}a_\phi+e^2a_0^2a_\phi=0 .\end{aligned} \tag{2.15}$$

III. SOLVING FOR THE POTENTIALS

Let

$$a_0(r)=e'u(r),\quad a_\phi(r)=e'v(r),\quad (e'e)^2=\alpha . \tag{3.1}$$

Then Eqs. (2.15) read

$$\partial^2u+\frac{1}{r}\partial u-\alpha v^2u=2\pi\delta(\mathbf{r}) , \tag{3.2}$$

$$\partial^2v+\frac{1}{r}\partial v-\frac{1}{r^2}v+\alpha u^2v=0 . \tag{3.3}$$

Although with $v\neq0$ these equations could not be solved in closed form, some features of the solution for $r\to0$ and $r\to\infty$ can be stated analytically.

Equations (3.2) and (3.3) involve $\alpha^{1/2}r$ rather than α and r separately, and thus can be standardized, e.g., to $\alpha\to1$, by a scale transformation. Later on in this paper, however, we perform a scaling transformation that we shall prefer to view as a strong-coupling limit, in order to facilitate comparison with other models. Therefore we choose to keep an adjustable α. We also note that the individual solutions of (3.2) and (3.3) do, of course, in general depend on α and r separately; the scaling symmetry only implies an invariance group acting on the space of solutions.

A. Small-distance behavior

Let us further restrict the solutions by assuming that when r becomes sufficiently small, and for any given value of α, the functions u,v approach some multiple of the $\alpha=0$ solutions. (In a classical model, this is as close as we can come to the asymptotic freedom of quantum chromodynamics.)

We introduce an iteration solution near $r=0$,

$$u=\sum_{n=0}^{\infty} u_n, \quad v=\sum_{n=0}^{\infty} v_n , \tag{3.4}$$

where v is assumed nonsingular at $r=0$. The zeroth-order term is

$$u_0=\ln\frac{r}{a}, \quad v_0=\frac{r}{b} \tag{3.5}$$

for some constants a,b; we take u_0 and v_0 as defining the $r\to 0$ behavior of the solution; in this way, the higher iterations include no arbitrary multiples of u_0 and v_0. By reinserting u_0,v_0 into the interaction term we obtain the next order:

$$\begin{aligned} u_1&=\frac{\alpha r^4}{32b^2}\left[2\ln\frac{r}{a}-1\right] , \\ v_1&=\frac{\alpha r^3}{64b}\left[-8\left[\ln\frac{r}{a}\right]^2+12\ln\frac{r}{a}-7\right] . \end{aligned} \tag{3.6}$$

For r small enough and any given α, this is an arbitrarily small relative correction to (3.5). (As is to be expected, we *formally* obtain a perturbation series in α.)

B. Large-distance behavior

The numerical illustrations shown further on indicate that, for α,a,b in certain ranges, and as $r\to\infty$, the function v has ever-tightening oscillations, and u has a leading term linear in r. In the following, we show that this information is consistent with Eqs. (3.2) and (3.3); we also derive some exact statements on the $r\to\infty$ behavior of this class of solutions; the approach is patterned after the WKB method.

Let

$$v=w(r)\sin\left[\int^r k(r)dr\right] , \tag{3.7}$$

where we assume $k>0$ and w nonoscillatory. Equation (3.3) now reads

$$\left[\partial^2 w+\frac{1}{r}\partial w-\frac{1}{r^2}w+(\alpha u^2-k^2)w\right]\sin\left[\int^r k\,dr\right] +\frac{1}{rw}\partial(rkw^2)\cos\left[\int^r k\,dr\right]=0 . \tag{3.8}$$

Requiring the cosine term to vanish identically, we have

$$k=\frac{c}{rw^2} , \tag{3.9}$$

where $c>0$ is a constant; the sine term then gives

$$\partial^2 w+\frac{1}{r}\partial w+\left[\alpha u^2-\frac{c^2}{r^2w^4}-\frac{1}{r^2}\right]w=0 . \tag{3.10}$$

In (3.2) we make use of the tight asymptotic oscillations of v by using its local root mean square; according to (3.7) we should set $v^2\to\frac{1}{2}w^2$. Equation (3.2) now reads

$$\partial^2 u+\frac{1}{r}\partial u-\frac{\alpha}{2}w^2u=0 . \tag{3.11}$$

We can find an exact solution to the coupled equations (3.10) and (3.11). Trying a linear form

$$u=\mu r \tag{3.12}$$

for some constant $\mu\neq 0$, we have, from (3.11),

$$w=\left[\frac{2}{\alpha}\right]^{1/2}\frac{1}{r} . \tag{3.13}$$

Insertion in (3.10) gives

$$c=\frac{2|\mu|}{\sqrt{\alpha}} ; \tag{3.14}$$

from (3.9),

$$k=\frac{\alpha c r}{2}=\sqrt{\alpha}|\mu|r . \tag{3.15}$$

In conclusion, we find a class of asymptotic behaviors of the form

$$u\approx\mu r , \tag{3.16}$$

$$v\approx\left[\frac{2}{\alpha}\right]^{1/2}\frac{1}{r}\sin(\tfrac{1}{2}\sqrt{\alpha}\mu r^2+\delta) . \tag{3.17}$$

(Since the constant phase term δ is unknown, there is no loss of generality in the replacement $|\mu|\to\mu$.) We note that (3.12) is only a leading-term estimate, and that therefore δ in (3.17) may be found to vary slowly with r in an improved approximation. Such refinements will not affect the strong-coupling considerations discussed further on.

The numerical results do indeed yield solutions which behave nearly as predicted by (3.16) and (3.17).

IV. SOME NUMERICAL EXAMPLES

The solutions are determined by the parameters a,b in (3.5) and by α. Actually, we need to consider only two parameters κ,λ, defined by the scaling transformation (in terms of a new coordinate s)

$$r=bs, \quad \alpha=\lambda/b^2, \quad a=\kappa b . \tag{4.1}$$

Equations (3.2) and (3.3) now read

$$\left[\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}\right]u-\lambda v^2 u=2\pi\delta(s) , \tag{4.2}$$

$$\left[\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}-\frac{1}{s^2}\right]v+\lambda u^2 v=0\ , \tag{4.3}$$

unchanged except for notation. However, the $r \to 0$ conditions are now

$$u_0=\ln\frac{s}{\kappa}\ ,\quad v_0=s \tag{4.4}$$

instead of (3.5).

In Figs. 1–3, the functions u and v are plotted for three illustrative sets of values of κ and λ. For reference, the short-range potential $u \propto A^0$ will be called "attractive," because we take it as the result of a perturbative quark-antiquark interaction in an attractive state. By comparison, we see that the (nonperturbative) long-range potential can be linearly attractive (Figs. 1 and 2) or linearly repulsive (Fig. 3). There are also special loci in the $\kappa\lambda$ plane where the intermediate feature—a potential damped to zero—prevails; this case is not studied here.

V. A STRONG-COUPLING LIMIT

The limiting case $\alpha \to \infty$ is not necessarily unique. Indeed, it is up to us to specify the behavior of $a(\alpha)$ and $b(\alpha)$ during the limiting process.

One case is of special interest because of its simplicity and because it may bear on the extension of the model to three space dimensions: the limit $\alpha \to \infty$ can be made into a scaling process, as follows.

First we need a "physical" normalization criterion; for that purpose we take the potential $a_0=e'u$, see Eq. (3.1), at some fixed distance r from the source. Thus we have

$$e'u=\text{fixed}\ , \tag{5.1}$$

$$r=\text{fixed}\ . \tag{5.2}$$

Next, noting that a and b have dimensions of length while α has dimensions of $(\text{length})^{-2}$, we choose the scaling behavior

$$a=\frac{\xi}{\sqrt{\alpha}}\ ,\quad b=\frac{\eta}{\sqrt{\alpha}} \tag{5.3}$$

(ξ,η fixed as $\alpha \to \infty$). Since the complete solution u,v is entirely determined by (3.5) and by α, the result of $\alpha \to \infty$ is to compress the horizontal scale of u and v when plotted as functions of r. Hence *the asymptotic behavior* (3.16) *and* (3.17) *becomes the exact behavior* everywhere except in an ever-shrinking region around the source. We see that

$$\mu=\sqrt{\alpha}\,\xi \tag{5.4}$$

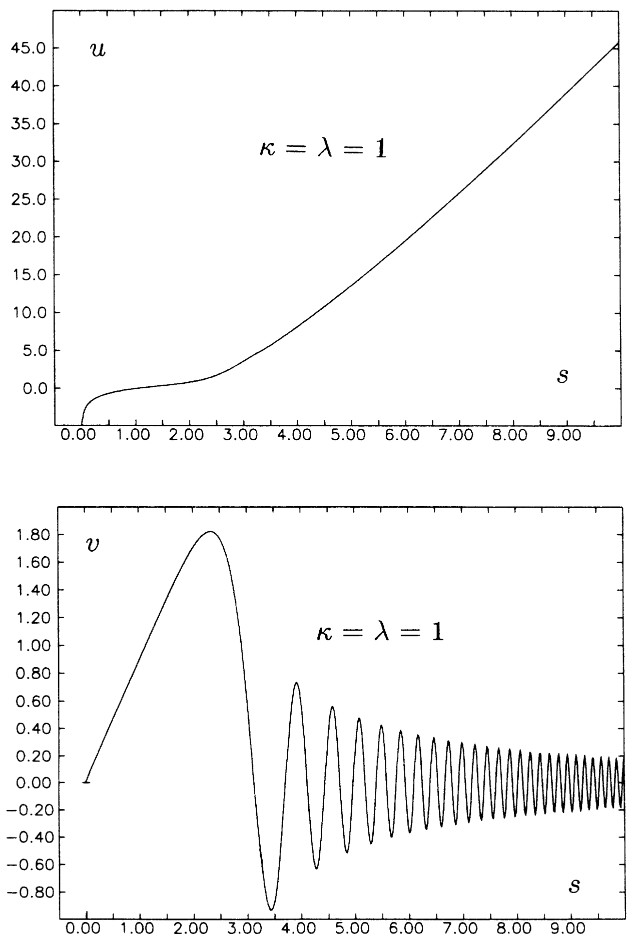

FIG. 1. Computer plot of u and v vs s for $\kappa=\lambda=1$.

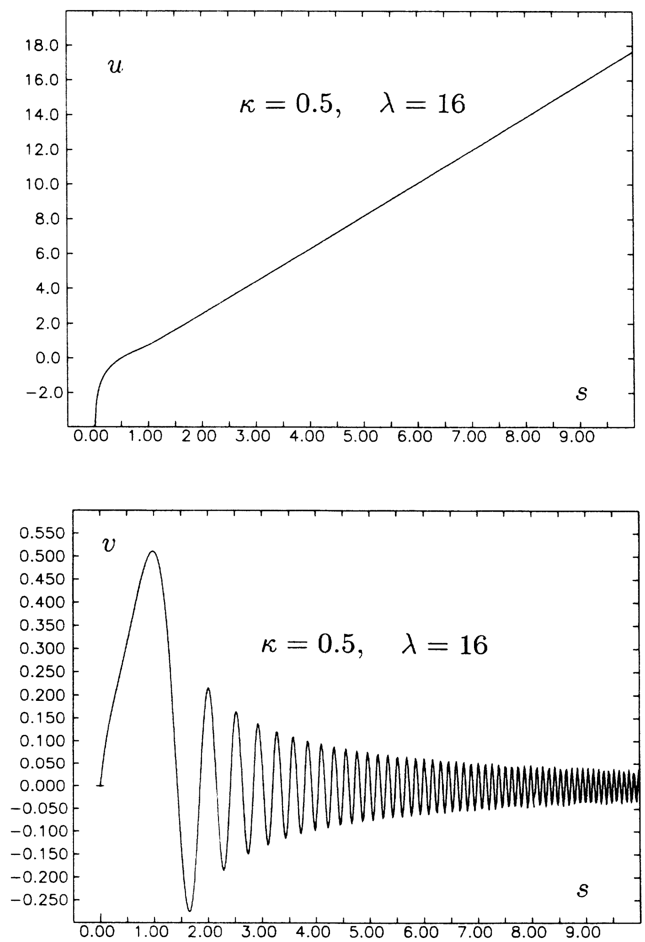

FIG. 2. u and v vs s for $\kappa=0.5$, $\lambda=16$.

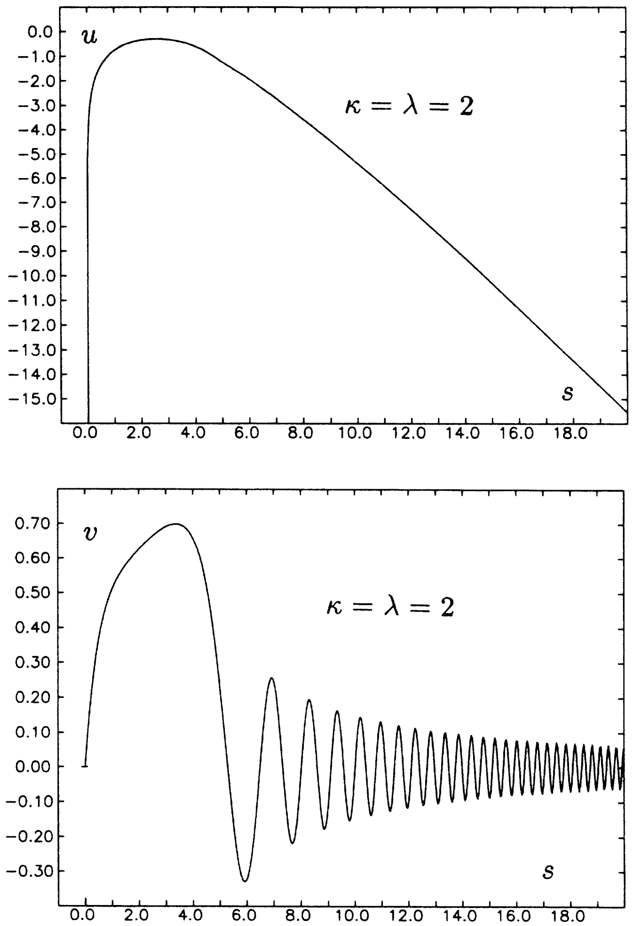

FIG. 3. u and v vs s for $\kappa=\lambda=2$.

(ζ fixed as $\alpha\to\infty$). From (3.16) and (3.17) we find (everywhere except near the origin)

$$u=\sqrt{\alpha}\zeta r\ , \tag{5.5}$$

$$v=\left[\frac{2}{\alpha}\right]^{1/2}\frac{1}{r}\sin(\tfrac{1}{2}\alpha\zeta r^2+\delta)\ ; \tag{5.6}$$

the $\alpha^{1/2}r$ dependence of δ is negligible compared to that of the preceding term.

Now, according to (5.1) and (5.2), we have

$$e'\sqrt{\alpha}=\text{fixed} \tag{5.7}$$

as $\alpha\to\infty$, or

$$e'=\epsilon\alpha^{-1/2} \tag{5.8}$$

(ϵ fixed). Thus, the external source is "renormalized" to an infinitesimal value; in the limit, it is completely "antishielded" by the Yang-Mills field.

Turning to the tangential component v, Eq. (5.6), we see that its amplitude approaches zero like $\alpha^{-1/2}$; perhaps more physically, we have

$$e'v\sim\alpha^{-1}\ . \tag{5.9}$$

In addition, the wavelength of v approaches zero like α^{-1} as well. We reach a paradoxical situation where only u survives (as a linearly rising potential) and v disappears, whereas we recall that the conventional calculation with $v=0$ yields a Coulomb-type (logarithmic) result for u.

Finally, we examine the fields. In the $\alpha\to\infty$ limit we have, from Eqs. (2.14),

$$\begin{aligned} E_r&=-\tfrac{1}{2}\epsilon\zeta\sigma_3\ ,\\ E_\phi&=-\frac{1}{\sqrt{2}}\epsilon\zeta\sigma_2\sin(\tfrac{1}{2}\alpha\zeta r^2+\delta)\ ,\\ B&=\frac{1}{\sqrt{2}}\epsilon\zeta\sigma_1\cos(\tfrac{1}{2}\alpha\zeta r^2+\delta)\ . \end{aligned} \tag{5.10}$$

Thus, E_r is finite and constant. On the other hand, E_ϕ and B oscillate everywhere with infinitesimal wavelength and have a finite constant amplitude. The force exerted by E_ϕ and B on a test quark of fixed charge vanishes for $\alpha\to\infty$ if that quark is of nonzero size, no matter how small. Their force on a point quark is "bumpy," but, at least in classical mechanics, it becomes unobservable as $\alpha\to\infty$.

VI. CONCLUSION

The (2+1)-dimensional static Yang-Mills equation with partial vorticity offers some nontrivial nonlinear mathematics whose possible relevance to the (3+1)-dimensional case arises from a strong-coupling limit. In that limit, the tangential features disappear, but leave a permanent dimensional-reduction effect on the radial features. The potential rises or falls linearly instead of logarithmically. Thus, in the rising case, we are led to expect a good (rather than marginal) quark-antiquark confinement mechanism. The disappearance of tangential features is essential if a manifestly spherically symmetric confining classical gluon field is ever to be contemplated in three space dimensions, since only radial vectors would be allowed.

An additional characteristic of the present study is the renormalization of the source to an infinitesimal value in the strong-coupling limit.

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