Gauge transformations for dynamical systems with first- and second-class constraints

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Gauge theories with second-class constraints are investigated. The relation between primary first-class constraints and gauge degrees of freedom is shown. Next, a method to obtain the generator of the gauge transformation is presented. The generator is expressed in terms of a linear combination of constraints. In the expression, all constraints are employed without distinguishing the first- from the second-class ones. The generator consists of the generator of the pure gauge transformation and that of global symmetry transformations with constant parameters. The method to construct the generator can be applied to a system having only second-class constraints, and extended Noether currents (generators) with constant parameters are obtained. If the first-class constraints and the Hamiltonian are in involution, the generator of pure gauge transformations can be obtained using only the first-class constraints.

I. INTRODUCTION

Recently gauge theories containing second-class constraints have been acquiring importance. In the investigation of such theories, the main problems would be (i) to establish a method of a covariant separation of first-class constraints (FCC's) from second-class constraints (SCC's) in order to perform covariant quantization[1,2] and (ii) to make clear the relation between the FCC's and gauge degrees of freedom, and to give a procedure to obtain the generator of the gauge transformation.

In this paper, the second problem will be considered as an extension of our previous works.[3,4] In Sec. II we will split up constraints into FCC's and SCC's, and obtain a total Hamiltonian, using the stationarity condition of the SCC.

In Sec. III we will present a method to construct the generator of the infinitesimal gauge transformation which makes an action quasi-invariant. In the construction, all constraints are employed without distinguishing the FCC's from SCC's. The generator consists of the generator of the pure gauge transformation and that of the transformation with constant parameters corresponding to global symmetries. This method can be applied to a system with SCC's alone, if it has a nontrivial solution. The generator is an extended Noether charge (or current) with constant parameters. In Sec. IV a few typical examples will be given for illustration. Section V will be devoted to remarks.

II. SEPARATION OF FIRST-CLASS CONSTRAINTS

In order to avoid redundant complexity, we consider the point mechanics described by a Lagrangian $L(q,\dot{q},t)$ with N variables q^i $(i=1,\ldots,N)$. The Hessian matrix

$$A_{ij}\equiv\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j} \tag{2.1}$$

is assumed to be of rank $N-\overline{A}$. Then there exist $\overline{A}$ primary constraints,

$$\phi_\alpha(q,p,t)\approx 0 \quad (\alpha=1,\ldots,\overline{A})\ , \tag{2.2}$$

with $p_i\equiv\partial L/\partial\dot{q}^i$, where the symbol $\approx$ means a weak equality. The total Hamiltonian is given by

$$H_T=H_0(q,p,t)+v^\alpha\phi_\alpha\ , \tag{2.3}$$

where v^α is a multiplier and H_0 is a canonical Hamiltonian.[5,6] The summation convention is used for dummy indices. The stationarity conditions of ϕ_α,

$$\dot{\phi}_\alpha=\frac{\partial\phi_\alpha}{\partial t}+\{\phi_\alpha,H_T\}\approx 0\ , \tag{2.4}$$

yield successively secondary constraints χ_μ $(\mu=1,\ldots,M)$.

Let us first observe the relation between the number of FCC's and gauge degrees of freedom. To do this, we rearrange all the constraints (ϕ_α,χ_μ) by making linear combinations of them, so as to make the number of FCC's maximal.[6] Let us denote the redefined constraints by

FCC: Φ_a $(a=1,\ldots,A)$,

SCC: Θ_s $(s=1,\ldots,S)$

with $A+S=\overline{A}+M$. Here all Φ_a and Θ_s are assumed to be independent. By definition we have

$$\{\Phi_a,\Phi_b\}\approx 0,\quad \{\Phi_a,\Theta_b\}\approx 0\ . \tag{2.5}$$

The number S is given by the rank of the $(\overline{A}+M)\times(\overline{A}+M)$ matrix

$$\begin{vmatrix}\{\phi_\alpha,\phi_\beta\} & \{\phi_\alpha,\chi_\nu\}\\ \{\chi_\mu,\phi_\beta\} & \{\chi_\mu,\chi_\nu\}\end{vmatrix}$$

on the constraint surface $\phi_\alpha=\chi_\mu=0$.

Now, in the following consideration, it is convenient to distinguish independent FCC's and SCC's containing the primary constraints ϕ_α from others among (Φ_a,Θ_s); corresponding to ϕ_α, we express them as [cf. example (i) in Sec. IV]

$$\text{FCC: } \Phi_{a_1},\tilde{\Phi}_{b_1};\quad \text{SCC: } \Theta_{s_1}$$

$$(a_1=1,\ldots,A_1,\quad b_1=1,\ldots,B_1\ ,$$

$$s_1=1,\ldots,S_1,\quad A_1+B_1+S_1=\bar{A})\ ,$$

where

$$\Phi_{a_1}=\kappa^\alpha_{a_1}\phi_\alpha\ , \tag{2.6a}$$

$$\tilde{\Phi}_{b_1}=\tilde{\kappa}^{\,\alpha}_{b_1}\phi_\alpha+\lambda^s_{b_1}\Theta_s\ , \tag{2.6b}$$

$$\Theta_{s_1}=\kappa^\alpha_{s_1}\phi_\alpha \tag{2.6c}$$

(without loss of generality we can choose $\kappa^\alpha_{s_1}=\delta^\alpha_{s_1}$). In (2.6), Φ_{a_1} is a linear combination of ϕ_α alone, while $\tilde{\Phi}_{b_1}$ a linear combination of primary SCC's ϕ_α and secondary SCC's Θ_s. H_T of (2.3) can be rewritten as

$$H_T=H_0+v^{a_1}\Phi_{a_1}+v^{b_1}\tilde{\Phi}_{b_1}+(v^{s_1}\delta^s_{s_1}-v^{b_1}\lambda^s_{b_1})\Theta_s\ . \tag{2.7}$$

The coefficients v^{b_1} and v^{s_1} of Θ_s are determined by the stationarity conditions of Θ_s. We notice that in (2.7) only H_0 and ϕ_α are involved in substance.

Using (2.7), we obtain the stationarity condition for Θ_s:

$$\dot{\Theta}_s\approx\frac{\partial\Theta_s}{\partial t}+\{\Theta_s,H_0\}+(v^{s_1}\delta^r_{s_1}-v^{b_1}\lambda^r_{b_1})\{\Theta_s,\Theta_r\}=0\ . \tag{2.8}$$

Since all Θ_s are independent, the matrix

$$C_{rs}\equiv\{\Theta_r,\Theta_s\}=-C_{sr} \tag{2.9}$$

is regular and hence its inverse exists:

$$C_{rs}C^{st}=\delta^t_r\ . \tag{2.10}$$

Multiplying (2.8) by C^{ts}, we obtain

$$C^{rs}\Theta'_s+v^{s_1}\delta^r_{s_1}-v^{b_1}\lambda^r_{b_1}=0\ , \tag{2.11}$$

where

$$\Theta'_s\equiv\frac{\partial\Theta_s}{\partial t}+\{\Theta_s,H_0\}\ . \tag{2.12}$$

Equation (2.11) turns out to be, for $r\neq s_1$,

$$C^{rs}\Theta'_s=v^{b_1}\lambda^r_{b_1}\ , \tag{2.13}$$

then v^{b_1} is determined to be a definite function of q and p. The number of r $(\neq s_1)$ can be larger than B_1, so that it happens that (2.13) gives overdeterminant conditions for v^{b_1}. In such a case, there is no consistent solution to the dynamical system. We consider here the case where (2.13) has a solution for v^{b_1}. Hence we can put

$$v^{b_1}=\lambda^{b_1}_\sigma C^{\sigma s}\Theta'_s\quad(\sigma\neq s_1)\ . \tag{2.14}$$

Thus from (2.7), (2.11), and (2.14), we obtain[7]

$$H_T=H^*+v^{a_1}\Phi_{a_1} \tag{2.15}$$

with

$$H^*=H_0+\Theta'_r(C^{rs}\Theta_s-C^{r\sigma}\lambda^{b_1}_\sigma\tilde{\Phi}_{b_1})\ . \tag{2.16}$$

H^* is a definite function of q and p.

From the definition of H^*, (2.9), and (2.12), it follows that

$$\{\Theta_s,H^*\}+\frac{\partial\Theta_s}{\partial t}\approx 0\quad(s=1,\ldots,S)\ . \tag{2.17}$$

The stationarity condition of Φ_a using H_T leads to

$$\{\Phi_a,H^*\}+\frac{\partial\Phi_a}{\partial t}\approx 0\quad(a=1,\ldots,A)\ . \tag{2.18}$$

Then v^{a_1} remains to be arbitrary and is associated with a gauge degree of freedom. The similar analysis was made by Mukunda.[8]

Since

$$\{\Phi_a,H_T\}\approx\{\Phi_a,H^*\},\quad\{\Theta_s,H_T\}\approx\{\Theta_s,H^*\}\ , \tag{2.19}$$

the constraint set defined by using H^* is equivalent to the one defined by using H_T of (2.15).

III. GENERATOR OF GAUGE TRANSFORMATION

In this section, we propose a method to construct the generator G of the infinitesimal gauge transformation leaving the action

$$S=\int dt\,L(q,\dot{q},t) \tag{3.1}$$

quasi-invariant. G for special Lagrangians was first obtained by Anderson and Bergmann.[5] For the dynamical system having FCC's alone, G can be given by a linear combination of the FCC's (Refs. 3, 4, 8, and 9). For the system containing FCC's and SCC's, we can also analogously construct G, though the method is rather complicated. In such a case, G turns out in general to be a linear combination of the FCC's and SCC's.

As seen from the derivation of (2.15) and (2.16), we can accomplish the algorithm of the stationarity conditions of ϕ_α and χ_μ using H_T of (2.3), without distinguishing FCC from SCC, and can decide v^α corresponding to v^{b_1} and v^{s_1}. Consequently we will find the function associated with arbitrary v^a to be the FCC.

Now, for systematic considerations in the following, it is convenient to redefine the constraints recurrently, starting from the primary constraint $\phi_\alpha\equiv\phi^1{}_\alpha$ as

$$\phi^{k+1}_\alpha\equiv\frac{\partial\phi^k_\alpha}{\partial t}+\{\phi^k_\alpha,H^*\}\ (\alpha=1,\ldots,\bar{A},k=1,\ldots,K-1) \tag{3.2}$$

and

$$\dot{\phi}^K_\alpha\equiv\frac{\partial\phi^K_\alpha}{\partial t}+\{\phi^K{}_\alpha,H^*\}=C^\beta_{\alpha k}\phi^k_\beta\ , \tag{3.3}$$

where ϕ^k_α $(k\geq 2)$ are secondary constraints χ_μ. (In general, K depends on α, but we omit it for the sake of simplicity. The generalization to an α-dependent case is easy.)

The definition (3.2) of the constraint series gives rise to no problem, provided that all ϕ_α^k are functionally independent, since the constraints are arbitrary within their linear combinations. In the particular case, however, for example,

$$\dot{\phi}_1^{r-1} \equiv \phi_1^r = (\psi_1^r)^2 ,$$

we should notice. In such a case, we usually put $\psi_1^r \approx 0$. It is sufficient with the constraint set $\{\phi_1^1, \ldots, \phi_1^{r-1}, \psi_1^r, \psi_1^{r+1} \equiv \dot{\psi}_1^r, \ldots\}$ to determine the constrained phase space. But we require here $(\psi_1^r)^2 \approx 0$ as the constraint by (3.2). Then there appear dependent constraints. In fact, the matrix

$$(\{\phi_1^k, \phi_1^l\}) = 4(\psi_1^r)^2 \begin{pmatrix} 0 & \{\phi_1^1, \phi_1^2\} & \cdots & \{\phi_1^1, \psi_1^r\} & \cdots & \cdots \\ \cdots & & \cdots & \{\phi_1^2, \psi_1^r\} & \cdots & \cdots \\ \cdots & & & \cdots & \cdots & \cdots \\ \{\psi_1^r, \phi_1^1\} & \{\psi_1^r, \phi_1^2\} & \cdots & 0 & \{\psi_1^r, \phi_1^{r+1}\} & \cdots \\ \cdots & & \cdots & \{\phi_1^{r+1}, \psi_1^r\} & \cdots & \cdots \\ \cdots & & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

vanishes on the surface constrained by $\psi_1^r = 0$. This means that the constraint set $\{\phi_1^k\}$ contains functionally dependent constraints on the basis of the set $\{\phi_1^1, \ldots, \psi_1^r, \psi_1^{r+1}, \ldots\}$. Nevertheless, we define the constraint series by (3.2), since the set $\{\phi_\alpha^k\}$ is necessary to construct the generator for the full transformation related to the constraints that makes the action (3.1) quasi-invariant. In other words, the generator producing the full transformation is not expressed in terms of a linear combination of only the constraints $\{\phi_\alpha^1, \ldots, \psi_\alpha^r, \ldots\}$ [see example (iii) in Sec. IV]. The consideration in Ref. 4 suggests that the generator of the infinitesimal gauge transformation also can be expressed in terms of all constraints ϕ_α^k. Then we put

$$\tilde{G} = \eta_k^\alpha \phi_\alpha^k \quad (\alpha = 1, \ldots, \bar{A},\ k = 1, \ldots, K) , \tag{3.4}$$

where η_k^α is an undetermined multiplier. $\tilde{G}$ must satisfy the stationarity condition[4]

$$\dot{\tilde{G}} = \frac{\partial \tilde{G}}{\partial t} + \{\tilde{G}, H_T\} \equiv 0 \mod(\phi_\alpha^1) . \tag{3.5}$$

This condition splits into the following ones:

$$\frac{\partial \tilde{G}}{\partial t} + \{\tilde{G}, H^*\} \equiv 0 \mod(\phi_\alpha) , \tag{3.6a}$$

$$\{\tilde{G}, \phi_a\} \equiv 0 \mod(\phi_\alpha) \quad (a = 1, \ldots, A_1) , \tag{3.6b}$$

$$v_*^s \{\tilde{G}, \phi_s\} \equiv 0 \mod(\phi_\alpha) \quad (s = 1, \ldots, B_1 + S_1) , \tag{3.6c}$$

and

$$\dot{q}^s \{\tilde{G}, \phi_s\} \equiv \frac{df}{dt} \mod(\phi_\alpha) , \tag{3.6d}$$

where ϕ_a and ϕ_s are the primary FCC and SCC, respectively, and ϕ_α denote all primary constraints. v_*^s is the multiplier fixed by the stationarity condition of the constraint; that is,

$$H^* \equiv H_0 + v_*^s \phi_s .$$

The proof of (3.6) is as follows. By using the infinitesimal transformation

$$\delta q^i = \{q^i, \tilde{G}\}, \quad \delta \dot{q}^i = \frac{d}{dt}\{q^i, \tilde{G}\} , \tag{3.7}$$

we obtain

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial q^i} \delta q^i + p_i \delta \dot{q}^i \\ &= \frac{d}{dt}(p_i \delta q^i - \tilde{G}) + \frac{\partial \tilde{G}}{\partial q^i} \dot{q}^i + \frac{\partial \tilde{G}}{\partial t} + \frac{\partial L}{\partial q^i} \frac{\partial \tilde{G}}{\partial p_i} . \end{aligned}$$

In order to evaluate the right-hand side (RHS) of δL, we introduce the Hamiltonian

$$H = p_i \dot{q}^i - L . \tag{3.8}$$

In the Dirac formalism, all p_i are regarded as independent variables, before taking the Poisson brackets. Then it holds that

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \frac{\partial L}{\partial q^i} = -\frac{\partial H}{\partial q^i} . \tag{3.9}$$

Here it should be noticed that (3.9) does not hold for $H_T = H^* + v^a \phi_a$, because v_*^s has been fixed to a definite function adjusting to the stationarity condition of the constraint. Owing to (3.9), δL turns out to be

$$\delta L = \frac{d}{dt}(p_i \delta q^i - \tilde{G}) + \frac{\partial \tilde{G}}{\partial t} + \{\tilde{G}, H\} . \tag{3.10}$$

H of (3.8) is rewritten as[10]

$$H = H_0 + \dot{q}^\alpha \phi_\alpha \quad (\alpha = 1, \ldots, \bar{A}) \tag{3.11}$$

with the form

$$\phi_\alpha = p_\alpha - \psi_\alpha(\dot{q}^i, p_r) ,$$

where p_r $(r = 1, \ldots, N - \bar{A})$ denotes independent momentum.

On the other hand, we have

$$H_T = H_0 + v^\alpha \phi_\alpha = H^* + v^a \phi_a \; . \qquad (3.12)$$

Then (3.11) becomes

$$H = H^* + (\dot{q}^{\,s} - v_*^s)\phi_s + v^a \phi_a \; . \qquad (3.13)$$

In (3.13) $\dot{q}^{\,a}$ has been absorbed into v^a, as v^a associated with the FCC ϕ_a is arbitrary. Using (3.13), we obtain

$$\frac{\partial \tilde{G}}{\partial t} + \{\tilde{G}, H\} = \frac{\partial \tilde{G}}{\partial t} + \{\tilde{G}, H^*\} + (\dot{q}^{\,s} - v_*^s)\{\tilde{G}, \phi_s\} + v^a\{\tilde{G}, \phi_a\} \mod(\phi_a) \; . \qquad (3.14)$$

If (3.14) reduces to a form df/dt under mod(ϕ_α), Eq. (3.10) turns out to be in velocity phase space (where $\phi_\alpha \equiv 0$)

$$\delta L = \frac{d}{dt}\left[p_i \frac{\partial \tilde{G}}{\partial p_i} - \tilde{G} + f\right] . \qquad (3.15)$$

In (3.14), as all constraints satisfy the stationarity conditions (2.17) and (2.18); hence there exists a solution of $\tilde{G}$ to (3.6a). Since v^a is arbitrary, (3.6b) should hold. No $\dot{q}^{\,s}$ appears in ϕ_s, v_*^s, and $\tilde{G}$, then (3.6c) and (3.6d) are required. We note that (3.6b) and (3.6c) are necessary conditions for the Hamilton vector field for $\tilde{G}$ to be transformed into velocity phase space.[4] Q.E.D.

Equations (3.6a)–(3.6d) decide η_k^α, except for arbitrary gauge functions. From (3.2)–(3.4) and (3.6a), it follows that

$$\eta_{k-1}^\alpha + \dot{\eta}_k^\alpha + \eta_K^\beta c_{\beta k}^\alpha = 0 \quad (k = 2, \ldots, K) \; , \qquad (3.16)$$

where

$$\dot{\eta}_k^\alpha \equiv \frac{\partial \eta_k^\alpha}{\partial t} + \{\eta_k^\alpha, H^*\} \; . \qquad (3.17)$$

The solution $\tilde{G}$ to (3.6) has A_1 arbitrary gauge functions associated with the gauge degrees of freedom and $B_1 + S_1$ $(= \overline{A} - A_1)$ constant parameters (see examples in Sec. IV).

Even for a system containing no FCC, but only SCC's if (3.6) has a nontrivial solution ($\tilde{G} \neq 0$), $\tilde{G}$ gives a generator with constant parameters, corresponding to global symmetries.

Hence, in general we have

$$\tilde{G} = G + N \; ,$$

where N is the extended Noether charge (the generator) with constant parameters.

If H^* and the FCC's Φ_a defined in Sec. II are in involution, that is,

$$\{\Phi_a, \Phi_b\} = C_{ab}^c \Phi_c, \quad \{\Phi_a, H^*\} + \frac{\partial \Phi_a}{\partial t} = \kappa_a^b \Phi_b \; , \qquad (3.18)$$

the generator G of the pure gauge transformation can be expressed in terms of only Φ_a (Refs. 4 and 9). [The Poisson brackets among FCC's are also FC, but there is the possibility of the appearance of quadratic or higher-order product terms of SCC's in the RHS of (3.18).]

In the proof that $\tilde{G}$ is the generator of the transformation leaving the action (3.1) quasi-invariant, it is essential that all primary constraints ϕ_α identically vanishes in velocity phase space $(q, \dot{q})$, and $\dot{q}^{\,i}$ has an additive ambiguity $u^\alpha \partial\phi_\alpha/\partial p_i$ (u^α being arbitrary) in phase space,[4,5] so that H^* reproduces the Lagrangian L same with the one for H_T. It should be noticed that if the last term $v^{b_1}\lambda_{b_1}^s \Theta_s$ in (2.7) is omitted, H_T is not equivalent to the original L, since the secondary constraints are added to H_0 through $\tilde{\Phi}_{b_1}$.

IV. SIMPLE EXAMPLES

For the illustration of our formulation, we present three typical examples.

(i) Let us take the Lagrangian

$$L = \dot{q}^{\,1} q^2 - \dot{q}^{\,2} q^1 - (q^1 - q^2) q^3 \; . \qquad (4.1)$$

The primary constraints are

$$\phi_1 = p_1 - q^2, \quad \phi_2 = p_2 + q^1, \quad \phi_3 = p_3 \qquad (4.2)$$

and the total Hamiltonian is given by

$$H_T = (q^1 - q^2) q^3 + v^\alpha \phi_\alpha \; . \qquad (4.3)$$

The stationarity conditions of constraints yield

$$\dot{\phi}_1 = -q^3 - 2v^2 = 0 \; , \quad \dot{\phi}_2 = q^3 + 2v^1 = 0 \; ,$$
$$\dot{\phi}_3 = -q^1 + q^2 \equiv \chi_1 \; ,$$

and

$$\dot{\chi}_1 = -v^1 + v^2 = 0 \; .$$

Hence

$$v_*^1 = v_*^2 = -\frac{q^3}{2} \; . \qquad (4.4)$$

From (4.3) and (4.4), we obtain

$$H_T = H^* + v\phi_3 \qquad (4.5)$$

with

$$H^* = -\frac{q^3}{2}(p_1 + p_2 - q^1 + q^2) \qquad (4.6)$$

and the rearranged constraint set

$$\begin{aligned} &\Phi_1 = \phi_3 = p_3 \; , \\ &\tilde{\Phi}_1 = \phi_1 + \phi_2 + 2\chi_1 = p_1 + p_2 - q^1 + q^2 \; , \\ &\Theta_1 = \phi_1 = p_1 - q^2 \; , \\ &\Theta_2 = \chi_1 = -q^1 + q^2 \; . \end{aligned} \qquad (4.7)$$

These constraints (4.7) satisfy

$$\begin{aligned} &\{\Phi_1, H^*\} = \tfrac{1}{2}\tilde{\Phi}_1, \quad \{\tilde{\Phi}_1, H^*\} = 0 \; , \\ &\{\Theta_1, H^*\} = 0, \quad \{\Theta_2, H^*\} = 0 \; , \end{aligned} \qquad (4.8)$$

where $H^* = -q^3\tilde{\Phi}_1/2$.

Φ_1, $\tilde{\Phi}_1$, and H^* are in involution; hence the generator of the pure gauge transformation is put as

$$G = \epsilon^1 \Phi_1 + \tfrac{1}{2}\epsilon^2 \tilde{\Phi}_1 \; . \qquad (4.9)$$

The substitutions of (4.9) into (3.6a) and (3.6b) lead to

$$\epsilon^1=-\dot{\epsilon}^2$$

and by putting $\epsilon(t)=\epsilon^2$, which is an arbitrary infinitesimal function of t,

$$G=-\dot{\epsilon}\Phi_1+\frac{\epsilon}{2}\tilde{\Phi}_1 \ . \tag{4.10}$$

The gauge transformation is given by

$$\delta q^1=\frac{\epsilon}{2},\quad \delta q^2=\frac{\epsilon}{2},\quad \delta q^3=-\dot{\epsilon}\ ,$$

by which we obtain

$$\delta L=\frac{d}{dt}\left[\frac{\epsilon}{2}(q^1-q^2)\right] . \tag{4.11}$$

Another method to take into account all constraints $\phi_\alpha\equiv\phi_\alpha^1$ and $\chi_1\equiv\phi_3^2$ is as follows. Here we obtain

$$\begin{aligned} H&=p_i\dot{q}^i-L=(q^1-q^2)q^3+\dot{q}^\alpha\phi_\alpha \\ &=H^*+v^3\phi_3+(\dot{q}^s-v_*^s)\phi_s \quad (s=1,2) \end{aligned} \tag{4.12}$$

and following (3.4) put

$$\tilde{G}=\eta_1^1(p_1-q^2)+\eta_1^2(p_2+q^1)+\eta_1^3 p_3+\eta_2^1(-q^1+q^2)\ . \tag{4.13}$$

From (3.6a), (4.12), and (4.13), it follows that

$$\dot{\eta}_2^1=-\eta_1^3 \tag{4.14a}$$

and, from (3.6c),

$$\eta_1^1=\eta_1^2,\quad \eta_2^1=\epsilon(t)\ . \tag{4.14b}$$

Equation (3.6b) is automatically satisfied. With the help of (3.14b), (3.6d) gives

$$(\dot{q}^1-\dot{q}^2)(2\eta_1^1-\eta_2^1)=\frac{df}{dt}\ . \tag{4.14c}$$

Equation (4.14c) has two solutions;

(a) $2\eta_1^1=\eta_2^1=$ arbitrary function , (4.15a)

(b) η_1^1 and η_2^1: constant parameters , (4.15b)

with

$$f=(2\eta_1^1-\eta_2^1)(q^1-q^2)\ .$$

Combining (4.14a) and (4.14b) and (4.15a) and (4.15b) we find

$$\tilde{G}=G+N\ , \tag{4.16}$$

where G is given by (4.10) with

$$\eta_1^1=\eta_1^2=\tfrac{1}{2}\eta_2^1=\tfrac{1}{2}\epsilon(t),\quad \eta_1^3=-\dot{\epsilon}(t)$$

and

$$N\equiv c_1(\phi_1+\phi_2)+c_2\chi_1\ , \tag{4.17}$$

where $c_1=\eta_1^1=\eta_1^2$ and $c_2=\eta_2^1$ are constant parameters associated with the two primary SCC's.

$\tilde{G}$ generates the transformation

$$\delta q^1=\delta q^2=\frac{\epsilon}{2}+c_1,\quad \delta q^3=-\dot{\epsilon} \tag{4.18}$$

and produces

$$\delta L=\frac{d}{dt}\left[\left[\frac{\epsilon}{2}+c_1\right](q^1-q^2)\right]\equiv\frac{dF}{dt}\ . \tag{4.19}$$

On the other hand, (3.15) gives

$$p_i\frac{\partial\tilde{G}}{\partial p_i}-\tilde{G}+f=F\ . \tag{4.20}$$

Since the term $c_2\chi_1$ has no role, we may put $c_2=0$. Note that without the f term in (4.20), Eq. (4.19) is not consistent with (4.20).

As seen from (4.10), the system has one gauge degree of freedom associated with Φ_1 (but not with $\tilde{\Phi}_1$).

(ii) An example of a system containing SCC's alone is given by

$$L=(\dot{q}^1+\dot{q}^2)q^3+\tfrac{1}{2}(\dot{q}^3)^2-\tfrac{1}{2}(q^2)^2\ . \tag{4.21}$$

From this L, it follows that

$$H_T=\tfrac{1}{2}(p_3)^2+\tfrac{1}{2}(q^2)^2+v^1(p_1-q^3)+v^2(p_2-q^3) \tag{4.22}$$

and SCC's are

$$\begin{aligned} &\phi_1^1\equiv\phi_1=p_1-q^3,\quad \phi_2^1\equiv\phi_2=p_2-q^3\ ,\\ &\phi_1^2\equiv\chi_1=-p_3,\quad \phi_2^2\equiv\chi_2=-p_3-q^2. \end{aligned} \tag{4.23}$$

H^* turns out to be ($v_*^1=v_*^2=0$)

$$H^*=\tfrac{1}{2}(p_3)^2+\tfrac{1}{2}(q^2)^2\ . \tag{4.24}$$

Putting

$$\tilde{G}=\eta_k^\alpha\phi_\alpha^k\quad(\alpha,k=1,2)\ , \tag{4.25}$$

we obtain, from (3.6a),

$$\eta_1^1+\dot{\eta}_2^1=0,\quad \eta_1^2+\dot{\eta}_2^2=0\ , \tag{4.26}$$

where $\dot{\eta}_2^\alpha=\{\eta_2^\alpha,H^*\}+\partial\eta_2^\alpha/\partial t$ and, from (3.6d),

$$\begin{aligned} &\dot{q}^1(\{\eta_2^1,\phi_1^1\}\phi_1^2-\eta_2^1+\{\eta_2^2,\phi_1^1\}\phi_2^2-\eta_2^2)\\ &\quad+\dot{q}^2(\{\eta_2^1,\phi_2^1\}\phi_1^2-\eta_2^1+\{\eta_2^2,\phi_2^1\}\phi_2^2-2\eta_2^2)\\ &\qquad\equiv\frac{df}{dt}\quad \mathrm{mod}(\phi_\alpha^1)\ . \end{aligned} \tag{4.27}$$

A solution to (4.26) and (4.27) is

$$\begin{aligned} &\eta_2^\alpha=a^\alpha\phi_1^1+b^\alpha\phi_2^1+c^\alpha\quad(\alpha=1,2)\ ,\\ &\eta_1^\alpha=-a^\alpha\phi_1^2-b^\alpha\phi_2^2\ , \end{aligned} \tag{4.28}$$

$$f=-(c^1+c^2)q^1-(c^1+2c^2)q^2\ , \tag{4.29}$$

with constants a^α, b^α, and c^α.

Thus we find

$$\tilde{G}=\eta(-\phi_1^1\phi_2^2+\phi_2^1\phi_1^2)+c^1\phi_1^2+c^2\phi_2^2$$
$$=\eta[(p_1-p_2)p_3+(p_1-q^3)q^2]-cp_3-c'q^2 \quad (4.30)$$

and $f=-c(q^1+q^2)-c'q^2$, where $\eta=b^1-a^2$, $c=c^1+c^2$, and $c^2=c'$. The transformation generated by $\tilde{G}$ is

$$\delta q^1=\eta(p_3+q^2)=\eta(\dot{q}^3+q^2) ,$$
$$\delta q^2=-\eta p_3=-\eta\dot{q}^3 , \quad (4.31)$$
$$\delta q^3=\eta(p_1-p_2)-c=-c ,$$

due to $p_1-p_2=\phi_1^1-\phi_2^1\equiv 0$ in velocity phase space, and

$$\delta L=\frac{d}{dt}[\eta q^2q^3-c(q^1+q^2)] , \quad (4.32)$$

which is confirmed by (3.15). In the above, the c' term has no role in velocity phase space, then it may be omitted.

Here we notice that the a^α and b^α terms in η_k^α are linear in the constraints; in other words, the corresponding terms are the quadratic form of the constraints in $\tilde{G}$. Taking into account higher products of the constraints, we can obtain wider transformations. Since such higher-order products strongly vanish on the constraint surface, the essential part of $\tilde{G}$ is $c^\alpha\phi_\alpha^2$, where c^α is associated with the primary SCC ϕ_α. The η term in (4.30) is an apparent one.

(iii) An example containing a quadratic form of SCC:

$$L=a_\alpha\dot{q}^\alpha-q^1(q^2)^2 \quad (4.33)$$

with a_α $(\alpha=1,2)$ being constant. Then we have

$$\phi_\alpha^1=p_\alpha-a_\alpha \quad (4.34)$$

and

$$H_T=q^1(q^2)^2+v^\alpha\phi_\alpha^1 . \quad (4.35)$$

The stationarity condition yields

$$\dot{\phi}_1^1=-(q^2)^2, \quad \dot{\phi}_2^1=-2q^1q^2 . \quad (4.36)$$

In the ordinary method, the secondary constraint is put as

$$\text{case(1)} \quad \psi=q^2 , \quad (4.37)$$

whereas in our method using (3.2), the secondary ones are

$$\text{case(2)} \quad \phi_1^2=-(q^2)^2, \quad \phi_2^2=-2q^1q^2 . \quad (4.38)$$

We will show case (2) to be favorable in construction of $\tilde{G}$.

In case (1), the constraint set is

$$\phi_\alpha^1=p_\alpha-a_\alpha \quad (\alpha=1,2), \quad \psi=q^2 , \quad (4.39)$$

where ϕ_2^1 and ψ are SC and ϕ_1^1 is FC. H^* is given by

$$H^*=q^1(q^2)^2 \quad (4.40)$$

and

$$v^1=v(t), \quad v_*^2=0 . \quad (4.41)$$

Putting

$$\tilde{G}=\eta_1^\alpha\phi_\alpha^1+\eta_2^1\psi ,$$

we obtain, from (3.6a),

$$\eta_1^1q^2+2\eta_1^2q^1-\dot{\eta}_2^1=0 , \quad (4.42a)$$

from (3.6b), $\{\eta_2^1,p_1\}=0$, hence,

$$\frac{\partial\eta_2^1}{\partial q^1}=0 , \quad (4.42b)$$

and, from (3.6d),

$$\dot{q}^2\left[\frac{\partial\eta_2^1}{\partial q^2}q^2+\eta_2^1\right]=\frac{df}{dt} ,$$

hence

$$\eta_2^1=g(q^2), \quad f=q^2g(q^2) , \quad (4.42c)$$

where g is an arbitrary function of q^2. Then, in (4.42a),

$$\dot{\eta}_2^1=\{\eta_2^1,H^*\}=0 .$$

The solution to (4.42a) is

$$\eta_1^1=\epsilon(t)q^1, \quad \eta_1^2=-\tfrac{1}{2}\epsilon(t)q^2 , \quad (4.43)$$

where $\epsilon(t)$ being arbitrary. Thus $\tilde{G}$ is given by

$$\tilde{G}=\epsilon q^1(p_1-a_1)-\tfrac{1}{2}\epsilon q^2(p_2-a_2)+q^2g(q^2) \quad (4.44)$$

and

$$f=q^2g(q^2) .$$

As pointed out in example (ii), though it does not yield substantial change, if we allow a solution of η_1^α comprising the constraint ψ, Eqs. (4.43) and (4.44) turn out to be

$$\eta_1^1=\epsilon^1(t)q^1+\epsilon^2(t)q^2 ,$$
$$\eta_1^2=-\frac{\epsilon^1(t)}{2}q^2-\frac{\epsilon^2(t)}{2}\frac{(q^2)^2}{q^1} , \quad (4.45)$$

and

$$\tilde{G}=(\epsilon^1q^1+\epsilon^2q^2)\phi_1^1-\frac{1}{2}\left[\epsilon^1q^2+\epsilon^2\frac{(q^2)^2}{q^1}\right]\phi_2^1+q^2g(q^2) \quad (4.46)$$

with

$$f=q^2g(q^2) ,$$

where $\epsilon^1(t)$ and $\epsilon^2(t)$ are arbitrary.

In case (1), $\tilde{G}$ of (4.44) is regarded as a reasonable solution, since the FCC is only ϕ_1^1 which gives a gauge degree of freedom. Furthermore, it should be noticed that in both expressions of $\tilde{G}$ (4.44) and (4.46), there is no N term with constant parameter associated with the SCC's ϕ_2^1 and ψ.

On the other hand, in case (2), the constraints are

$$\phi_\alpha^1=p_\alpha-a_\alpha \quad (\alpha=1,2) ,$$
$$\phi_1^2=-(q^2)^2, \quad \phi_2^2=-2q^1q^2 . \quad (4.47)$$

The stationarity condition yields

$$\dot{\phi}_1^2=-2v^2q^2\approx 0,\quad \dot{\phi}_2^2=-2(v^1q^2+v^2q^1)\approx 0\ , \tag{4.48}$$

from which we obtain

$$v^\alpha=u^\alpha(t)q^\alpha \quad (\text{not summed})\ , \tag{4.49}$$

where $u^\alpha(t)$ is arbitrary, and then

$$\dot{\phi}_1^2=2u^2\phi_1^2,\quad \dot{\phi}_2^2=(u^1+u^2)\phi_2^2\ . \tag{4.50}$$

[The simplest choice is $v^\alpha=0$ in (4.48). For this case, we also arrive at the same $\tilde{G}$, as seen below.]

For the choice (4.49), we have

$$H_T=q^1(q^2)^2+u^\alpha q^\alpha(p_\alpha-a_\alpha) \tag{4.51}$$

with $H^*=q^1(q^2)^2$. Hence by putting

$$\Phi_\alpha^1\equiv q^\alpha\phi_\alpha^1 \quad (\text{not summed}) \tag{4.52}$$

all constraints Φ_α^1 and ϕ_α^2 ($\alpha=1,2$) turn out to be FCC's, and we have two gauge degrees of freedom.

Here put

$$\tilde{G}=\eta_1^\alpha\Phi_\alpha^1+\eta_2^\alpha\phi_\alpha^2 \tag{4.53}$$

(or $\tilde{G}=\eta_k^\alpha\phi_\alpha^k$). From (3.6), (4.51), and (4.53), it follows that

$$\eta_1^\alpha q^\alpha+\dot{\eta}_2^\alpha=0 \quad (\text{not summed})\ , \tag{4.54a}$$

$$2\eta_2^2+\frac{\partial\eta_2^1}{\partial q^1}q^2+\frac{\partial\eta_2^2}{\partial q^1}q^1=0\ , \tag{4.54b}$$

$$\left[2\eta_2^1+\frac{\partial\eta_2^1}{\partial q^2}q^2\right]q^2+2\left[\eta_2^2+\frac{\partial\eta_2^2}{\partial q^2}q^2\right]q^1=0\ . \tag{4.54c}$$

Equations (4.54) have the solution

$$\begin{aligned}\eta_2^1&=\epsilon^1(t)q^1+\epsilon^2(t)q^2\ ,\\ \eta_2^2&=-\frac{\epsilon^1(t)}{2}q^2-\frac{\epsilon^2(t)}{2}\frac{(q^2)^2}{q^1}\ ,\end{aligned} \tag{4.55}$$

which leads to

$$\eta_1^1q^1=-\dot{\epsilon}^1q^1-\dot{\epsilon}^2q^2,\quad \eta_1^2q^2=\frac{\dot{\epsilon}^1}{2}q^2+\frac{\dot{\epsilon}^2}{2}(q^2)^2\ . \tag{4.56}$$

Finally we obtain $\tilde{G}$ of (4.46), but with $g=f=0$, which is irrelevant to the transformation (though being relevant to giving δp_i and to the transformation leaving the Hamilton equations of motion invariant). We notice that the constraint is not q^2, but q^1q^2 and $(q^2)^2$ in case (2), in contrast with case (1). Under the gauge transformation produced by $\tilde{G}$, we find

$$\delta L=\frac{d}{dt}(a_\alpha\delta q^\alpha)=\frac{d}{dt}\left[p_\alpha\frac{\partial\tilde{G}}{\partial p_\alpha}-\tilde{G}\right]\ . \tag{4.57}$$

Now choosing the simplest solution $v_*^\alpha=0$ in place of (4.49), we have

$$H_T=H^*\ ,$$

and instead of (4.54b) and (4.54c), due to (3.6d),

$$\begin{aligned}&-\dot{q}^1\left[2\eta_2^2+\frac{\partial\eta_2^1}{\partial q^1}q^2+2\frac{\partial\eta_2^2}{\partial q^1}q^1\right]q^1q^2\\ &-\dot{q}^2\left[\left[2\eta_2^1+\frac{\partial\eta_2^1}{\partial q^2}q^2\right]q^2\right.\\ &\qquad\left.+2\left[\eta_2^2+\frac{\partial\eta_2^2}{\partial q^2}q^2\right]q^1\right]q^2=\frac{df}{dt}\ .\end{aligned} \tag{4.58}$$

The solution to (4.58) is

$$\begin{aligned}\eta_2^1&=\epsilon^1(t)q^1+\epsilon^2(t)\frac{(q^2)^2}{q^1}+\frac{g(q^2)}{q^2}\ ,\\ \eta_2^2&=-\frac{\epsilon^1(t)}{2}q^2-\frac{\epsilon^2(t)}{2}\frac{(q^2)^2}{q^1}\ ,\end{aligned} \tag{4.59}$$

which gives (4.46).

In case (2), there are the two series of FCC's Φ_α^1 and ϕ_α^2 ($\alpha=1,2$), but no SCC. Since $\tilde{G}$ of (4.46) (comprising no N) gives two gauge transformations, case (2) for the choice of the constraints is logically more reasonable than case (1). Thus the definition (3.2) for the constraint series would be preferable

V. REMARKS AND COMMENTS

Owing to the consequence of Sec. III, the inverse of the Noether theorem can be formulated as follows. If the conserved quantity $N(q,p,t)$ satisfies (3.5) or (3.6) in place of $\tilde{G}$, the transformation generated by N,

$$\delta q^i=\{q^i,N\},\quad \delta\dot{q}^i=\frac{d}{dt}\delta q^i\ ,$$

leaves its action invariant, so the inverse theorem is true. For a regular Lagrangian system (with no constraint), any constant of motion N trivially satisfies the condition (3.6), so that the inverse of the Noether theorem holds. For a singular Lagrangian system, however, a constant of motion does not satisfy (3.6), if it is conserved under secondary constraints.

In this paper, we have assumed all Φ_a and Θ_s to be independent except for the special case such as example (iii). If this is not the case, our formulation should be modified according to the method of Dresse *et al.*[2]

Now suppose all FCC's Φ_a and H^* are in involution (3.18). Then as proposed by Fradkin and Vilkovisky,[11] the action

$$\tilde{S}=\int dt(p_i\dot{q}^i-H^*-\lambda^a\Phi_a)\quad (a=1,\ldots,A)\ , \tag{5.1}$$

where the summation of Φ_a runs over all FCC's, is quasi-invariant under the extended gauge transformation

$$\delta q^i=\tau^a(t)\{q^i,\Phi_a\},\quad \delta p_i=\tau^a(t)\{p_i,\Phi_a\} \tag{5.2}$$

with the change of λ^a by

$$\delta\lambda^a=\dot{\tau}^a-C_{bd}^a\lambda^b\tau^d+\kappa_b^a\tau^b \tag{5.3}$$

(τ^a being an independent arbitrary function). Canonical equations also are invariant for this extended transformation.

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