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Wick rotation and infinities in the superstring chiral-anomaly graph

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Sufficient conditions are found which, if satisfied, would justify the use of Wick rotation in superstring loop amplitudes. Unfortunately, infinities in the superstring chiral-anomaly amplitude block this justification. The point amplitude is first presented in terms of both string parameters and Feynman parameters. An interchange of loop-momentum and string-parameter integrals is found to violate Fubini's theorem by equating a Minkowski integral to a Euclidean integral. In the full superstring anomaly amplitude, an interchange of loop-momentum and string-parameter integrals also requires simultaneous Wick rotation if the infinite sums involved are uniformly convergent in a particular sense. It is observed that the amplitude actually contains infinities in canceled-propagator terms which violate this condition. It must therefore be regulated. Clavelli, Cox, and Harms [Phys. Rev. D **35**, 1908 (1987)] only proved the finiteness of an amplitude with such terms subtracted out, corresponding to a point-splitting regularization. In fact, if Wick rotation is valid for dimension-changing regularizations, the result of Mann [Nucl. Phys. **B303**, 99 (1988)] implies that the Green-Schwarz anomaly is not regularizable and anomaly cancellation does not hold.

I. INTRODUCTION

Superstring theory has encountered both nonperturbative difficulties[1,2] and a breakdown of uniqueness when the perturbation theory is compactified to four dimensions.[3] Proton decay and cosmology are likely the only areas of observational physics where relevant constraints may be found.[4] This paper comments on a more formal question: is the Green-Schwarz anomaly cancellation mathematically consistent?

Wick rotation is an essential procedure required for the regularization of open superstring anomalies.[5] It is of interest, therefore, to determine general conditions under which it is justified. For the first time a convergence criterion is found for this. Unfortunately, it requires the open-superstring chiral-anomaly amplitude to be finite which, as pointed out in Sec. IV, is not true. Clavelli, Cox, and Harms[6] did *not* prove its finiteness as they had claimed since they used the canceled-propagator argument in an unregulated formula.[7] Natural ultraviolet finiteness, which is desirable for a realistic extended-particle model,[8] does not occur for superstrings.[9] In fact, recent work shows that the full amplitude cannot even be unambiguously regulated since the corresponding current divergences were found to be infinite when conventional dimensional regularization (CDR) and regularization by dimensional reduction (RDR) were used.[10] The validity of Wick rotation must be checked for each regularization technique separately.

In the next section, the usual point quantum-field-theory anomaly is briefly reviewed and, for the first time, Wick rotation is justified by its necessity for the interchange of string-parameter integrals[11] and loop-momentum integrals. The regularization independence of the alternating sum of current divergences is also commented on. In Sec. III the resultant anti-Fubini theorem is generalized to the full superstring. Finally, in Sec. IV, the superstring is examined to find where in the amplitude the infinities reside which prevent the use of this theorem to justify an *ad hoc* Euclidean approach. A discussion of this result is also presented there.

II. STRING PARAMETRIZATION OF THE POINT AMPLITUDE

A typical chiral-anomaly graph with polarizations ξ_i may be written as[12,13]

$$S(\xi_0,\ldots,\xi_n;k_1,\ldots,k_n)=\int \frac{d^{2n}p'}{(2\pi)^{2n}}\mathrm{Tr}\left[\gamma_{2n+1}\prod_{j=0}^{n}\not{\xi}_j\not{p}_j^{-1}\right]$$

$$=\epsilon_{\alpha_0\alpha_1\rho_1\cdots\alpha_{n-1}\rho_{n-1}\alpha_n}\xi_0^{\alpha_0}\cdots\xi_n^{\alpha_n}\sum_{r=1}^{n}A_r k_1^{\rho_1}\cdots[k_r]\cdots k_n^{\rho_{n-1}} \tag{2.1}$$

$$+\sum_{p=1}^{n}\epsilon_{\alpha_0\alpha_1\rho_1\cdots[\alpha_p]\cdots\alpha_n\rho_n}\xi_0^{\alpha_0}\cdots[\xi_p]\cdots\xi_n^{\alpha_n}\sum_{l=1}^{n}{}^{p}B_l\,\xi_p\cdot k_l k_1^{\rho_1}\cdots k_n^{\rho_n}\ . \tag{2.2}$$

where notation and conventions from Appendix A is used. Expression (2.2) is a generalized Adler-Rosenberg expansion with coefficients A_r and ${}^pB_l$ yet to be determined. When the resultant current divergences are compared with the generic regulated form

$$S(\xi_1,\ldots,k_r,\ldots,\xi_n;k_1,\ldots,k_n)=\frac{2i^n}{(2\pi)^n n!}(-1)^r X_r \epsilon_r'(\xi,k)\ , \qquad (2.3)$$

the X_r being some real parameters, the relationship

$$A_r=\frac{2X_r}{(2\pi i)^n n!}-(-1)^{n-r}\sum_{l=1}^{n}{}^rB_l k_l\cdot k_r \qquad (2.4)$$

is found. Thus if the B's are calculated to be finite, the A's are also finite but ambiguous. Such an ambiguity does not affect the full amplitude

$$\sum_{r=0}^{n}(-1)^r S(\xi_0,\ldots,k_r,\ldots,\xi_n;k)=-\sum_{r,l=1}^{n}{}^rB_l k_l\cdot k_r \epsilon(\xi,k)\ . \qquad (2.5)$$

Note that although the current divergences are determined by the X_r's, the full graph is independent of them. This shows that the anomaly is an intrinsic result—it is fundamental to the system rather than to the description of the system.

Strictly speaking, the formal amplitude (2.1) has poles for any $p_j^2=0$. In order to perform the analytic integration required for Wick rotation, the anomaly amplitude is rigorously defined to be the $\epsilon\to 0$ limit of

$$S_\epsilon=\int\frac{d^{2n}p'}{(2\pi)^{2n}}\frac{\mathcal{T}_{\rm PT}(\xi,k;p')}{\prod_{j=0}^{n}(p_j^2+i\epsilon)}\ , \qquad (2.6)$$

where

$$\mathcal{T}_{\rm PT}(\xi,k;p')={\rm Tr}\left[\gamma_{2n+1}\prod_{l=0}^{n}\xi\!\!\!/_l p\!\!\!/_l\right]\ . \qquad (2.7)$$

The phrase "finiteness of the anomaly" refers to the finiteness of this limit. We shall discuss this further at the end of this section.

The amplitude (2.6) was evaluated using Feynman parameters in Ref. 13. Let us compare this analysis with the corresponding analysis using string parameters.[11] First of all, let the Feynman parameters be y_i and define, using the metric g in (A1),

$$x_i=\sum_{j=i}^{n}y_j,\quad i=0,\ldots,n\ , \qquad (2.8)$$

$$D(k,x)=2\sum_{1\le r<s\le n}k_r\cdot k_s x_s(1-x_r)+\sum_{r=1}^{n}k_r^2 x_r(1-x_r)\ . \qquad (2.9)$$

Since $\sum_{i=0}^{n}y_i=1$ and $y_i\ge 0$, then $x_i\in[0,1]$ and $x_{i+1}\ge x_i$. Also, since

$$k_r\cdot k_s=\sqrt{(m^2+\mathbf{k}_r^2)(m^2+\mathbf{k}_s^2)}-|\mathbf{k}_r||\mathbf{k}_s|\cos\theta\ge 0 \qquad (2.10)$$

for $k^2=m^2$, therefore $D(k,x)\ge 0$. By momentum conservation, all the k_r cannot point in the same direction and hence $D>0$ for nonzero x's, even if $k_r^2=0$.

Next, the string parameters are also denoted by x_i and

$$\rho_i=\prod_{j=0}^{i}x_j,\quad w=\rho_n\ , \qquad (2.11)$$

$$\nu_i=\frac{\ln\rho_i}{\ln w},\quad \nu_n=1,\quad \nu=-\ln w\ . \qquad (2.12)$$

Here $x_j\in(0,1]$ and so $\nu_i\in[0,1]$, $\nu\in[0,\infty)$ and $\nu_{i+1}\ge\nu_i$. Let us now revise (8.1.16) of Ref. 5 for these parameters. In this case,

$$\sum_{r=0}^{n}p_r^2\ln x_r=\ln w\left[p'-\sum_{s=1}^{n+1}k_s\frac{\ln\rho_{s-1}}{\ln w}\right]-\sum_{r<s}k_r\cdot k_s\left[\ln c_{sr}-\frac{(\ln c_{sr})^2}{\ln w}\right]\ , \qquad (2.13)$$

where

$$c_{sr}=\rho_{s-1}/\rho_{r-1}\ . \qquad (2.14)$$

Then,

$$\prod_{j=0}^{n}x_j^{p_j^2/2}=w^{\left[p'-\sum_{s=1}^{n+1}k_s\nu_{s-1}\right]^2/2}\times\exp\left[\sum_{r<s}\left[-\frac{1}{2}\ln c_{sr}+\frac{1}{2}\frac{(\ln c_{sr})^2}{\ln w}\right]k_r\cdot k_s\right]\ . \qquad (2.15)$$

The expression in the large square brackets may be rewritten as

$$\frac{\nu}{2}\sum_{0\le r\le s\le n+1}[\nu_{r-1}-\nu_{s-1}+(\nu_{s-1}-\nu_{r-1})^2](-k_r\cdot k_s)=-\frac{\nu}{2}\bar{D}(k,\nu_r)\ . \qquad (2.16)$$

where

$$\bar{D}(k,\nu_r)=2\sum_{0\le r\le s\le n}k_r\cdot k_s\nu_{r-1}(1-\nu_{s-1})+\sum_{r=1}^{n}k_r^2\nu_{r-1}(1-\nu_{r-1})\ . \qquad (2.17)$$

Now the sum in the first factor of (2.15) may be written as

$$\sum_{s=1}^{n+1}k_s\nu_{s-1}=\sum_{s=1}^{n}k_s\nu_{s-1}-\sum_{r=1}^{n}k_r=-\sum_{r=1}^{n}k_r(1-\nu_{r-1})=-\sum_{r=1}^{n}k_r\tilde{\nu}_r\ , \qquad (2.18)$$

where

$$\tilde{\nu}_r=1-\nu_{r-1}\ . \qquad (2.19)$$

Rewriting (2.17) in terms of $\tilde{v}_r$ gives

$$\bar{D}(k,v_r)=D(k,\tilde{v}_r)\ , \qquad (2.20)$$

and (2.15) becomes

$$\prod_{j=0}^{n} x_j^{p_j^2/2}=w^{\left[p'-\sum_{r=1}^{n}k_r\tilde{v}_r\right]^2/2}e^{-vD(k,\tilde{v}_r)/2}\ . \qquad (2.21)$$

Let us now expand (2.6),

$$S_\epsilon=\int\frac{d^{2n}p'}{(2\pi)^{2n}}n!\int_0^\infty\frac{\prod_{j=0}^{n}dy_j\delta\left[\sum_{l=0}^{n}y_l-1\right]}{\left[\sum_{i=0}^{n}y_i(p_i^2+i\epsilon)\right]^{n+1}}\mathcal{T}_{PT}(p') \qquad (2.22)$$

$$=n!\int\frac{d^{2n}p'}{(2\pi)^{2n}}\int_0^\infty\frac{\prod_{j=0}^{n}dy_j\delta\left[\sum_{l=0}^{n}y_l-1\right]\mathcal{T}_{PT}(p')}{\left[\left[p'+\sum_{j=1}^{n}k_jx_j\right]^2+D(k,x)+i\epsilon\right]^{n+1}} \qquad (2.23)$$

and, similarly,

$$S_\epsilon=\int\frac{d^{2n}p'}{(2\pi)^{2n}}\frac{1}{2^n}\int_0^1\prod_{i=0}^{n}\frac{dx_i}{x_i}x_i^{(p_i^2+i\epsilon)/2}\mathcal{T}_{PT}(p') \qquad (2.24)$$

$$=\frac{1}{2^n}\int\frac{d^{2n}p'}{(2\pi)^{2n}}\mathcal{T}_{PT}(p')\int_0^1\prod_{i=0}^{n-1}\theta(v_{i+1}-v_i)dv_i\int_0^\infty dv\,v^n\exp\left\{-\frac{v}{2}\left[\left[p'+\sum_{r=1}^{n}k_r\tilde{v}_r\right]^2+D(k,\tilde{v}_r)+i\epsilon\right]\right\}. \qquad (2.25)$$

Here the changes of variable (2.11) and (2.12) have been made to give

$$\prod_{i=0}^{n}dx_i=\prod_{j=0}^{n-1}\frac{d\rho_j}{\rho_j}dw\ , \qquad (2.26)$$

$$\int_0^1\prod_{i=0}^{n}\frac{dx_i}{x_i}=\int_0^1\prod_{j=0}^{n-1}\theta(\rho_j-\rho_{j+1})\frac{d\rho_j}{\rho_j}\frac{dw}{w}\ , \qquad (2.27)$$

and,

$$\prod_{j=0}^{n-1}\frac{d\rho_j}{\rho_j}\frac{dw}{w}=\prod_{j=0}^{n-1}dv_j(-\ln w)^n dw\ , \qquad (2.28)$$

$$\int_0^1\prod_{j=0}^{n-1}\theta(\rho_j-\rho_{j+1})\frac{d\rho_j}{\rho_j}\frac{dw}{w}=\int_0^1\prod_{j=0}^{n-1}dv_j\theta(v_{j+1}-v_j)\int_0^\infty v^n dv\ . \qquad (2.29)$$

Note that the $\tilde{v}_r$ can be identified with the Feynman x_i in reverse order.

At this stage, it would be convenient to shift $p'\to p$ where

$$\text{(a) } p'=p-\sum_{j=1}^{n}k_jx_j,\quad \text{(b) } p'=p-\sum_{r=0}^{n}k_r\tilde{v}_r\ . \qquad (2.30)$$

This would give

$$S_\epsilon=\frac{n!}{(2\pi)^{2n}}\int d\Lambda_{\rm FY}(p,y;\epsilon)\mathcal{T}_{PT}(p') \qquad (2.31)$$

or

$$S_\epsilon=\frac{1}{2^{n+1}(2\pi)^{2n}}\int d\Lambda_{\rm ST}(p,v_r,v;\epsilon)\mathcal{T}_{PT}(p')\ , \qquad (2.32)$$

where

$$d\Lambda_{FY}(p,y;\epsilon)=d^{2n}p\prod_{j=0}^{n}dy_j\frac{\delta\left[\sum_{l=0}^{n}y_l-1\right]}{[p^2+D+i\epsilon]^{n+1}}\ , \qquad (2.33)$$

$$d\Lambda_{ST}(p,v_r,v;\epsilon)$$

$$=d^{2n}p\prod_{i=0}^{n}\frac{dx_i}{x_i}w^{p^2/2}e^{-v[D(k,\tilde{v}_r)+i\epsilon]/2}$$

$$=d^{2n}p\prod_{j=0}^{n-1}\theta(v_{j+1}-v_j)dv_jv^ndv$$

$$\times w^{p^2/2}e^{-v[D(k,\tilde{v}_r)+i\epsilon]/2}\ . \qquad (2.34)$$

In fact, the linearly divergent pieces of the Feynman integral in (2.33) give rise to finite surface terms upon shifting. These end up in the A terms of the Adler-Rosenberg expansion (2.2). The remainder of the integral is less-than-linearly divergent and shifting is allowed. For the string integral in (2.25), let us perform the v integration first in order to determine if shifting is allowed. This gives[14]

$$S_\epsilon = n!\int \frac{d^{2n}p}{(2\pi)^{2n}} \int_0^1 \prod_{i=0}^{n-1} d\nu_i \theta(\nu_{i+1}-\nu_i) \frac{\mathcal{T}_{PT}(p')}{\left[\left[p' + \sum_{r=1}^{n} k_r \tilde{\nu}_r\right]^2 + D + i\epsilon\right]^{n+1}} . \tag{2.35}$$

Since this has the same finite form as the Feynman result, shifting is allowed. The surface terms which arise from shifting the linearly divergent pieces[12] contribute to the A terms.

After changing variable in (2.25) according to (2.30), the usual procedure of performing the p-integration *before* the ν-integration[5] gives integrals like

$$\int d^{2n}p \, w^{p^2/2} , \tag{2.36}$$

which for a Minkowski spacetime are infinite. This contradicts the finite result of (2.35) and means that it is invalid to naively interchange the p and ν integrals.[15] Note that *ad hoc* Euclideanization will give acceptable results and this is popular among many string theorists. Below it will be seen that this Euclideanization *arises naturally* from the correct interchange of integration and so is justified in the point case.

The derivation of the anomaly using Feynman parameters is well known;[12,13] however, since it will be generalized to strings, let us reiterate it here for the integral (2.35). Since the ν integral of (2.35) is finite, the p' and ν integrals may be interchanged. The B terms in the Adler-Rosenberg expansion may then be extracted from the trace $\mathcal{T}_{PT}(p')$ by using "symmetric integration." To be specific, if the $\pm\infty$ limits of the integrals are taken separately and the techniques of analytic integration are used, it turns out that

$$\begin{aligned}\lim_{\epsilon\to 0}\int \frac{d^{2n}p}{(p^2+D+i\epsilon)^{n+1}} &= \lim_{\epsilon\to 0}(-1)^{n+1} i \,\mathrm{sgn}D \int \frac{d^{2n}p_E}{(p_E^2+|D|+i\epsilon\,\mathrm{sgn}D)^{n+1}} \\ &= \frac{(-1)^{n+1} i\pi^n}{n!D} ,\end{aligned} \tag{2.37}$$

$$\lim_{\epsilon\to 0}\int \frac{p^\mu d^{2n}p}{(p^2+D+i\epsilon)^{n+1}} = 0 , \tag{2.38}$$

for any $D\neq 0$. The contributions from the p integral to the A terms come from the logarithmically divergent integrals

$$\int \frac{p^\mu p_\nu d^{2n}p}{(p^2+D+i\epsilon)^{n+1}} = \delta^{\mu\nu}\infty , \tag{2.39}$$

and the integrals

$$\begin{aligned}\int \frac{p^\mu p^\nu p^\rho d^{2n}p}{(p^2+D+i\epsilon)^{n+1}} &\propto -\tfrac{1}{3}\delta^{\mu\nu}\delta^{\nu\rho} \lim_{a,b\to+\infty} [(2|D|-b)\sqrt{|D|+b} \\ &\qquad -(2|D|-a)\sqrt{|D|+a}\,] \\ &= \delta^{\mu\nu}\delta^{\nu\rho} \lim_{\alpha,\beta\to+\infty} (\alpha-\beta) = \text{ambiguous} .\end{aligned} \tag{2.40}$$

However, (2.4) shows that the infinites in the A terms cancel[12] under the various regularization techniques leaving a constant but technique-dependent remainder.

On the domain given by $D>0$, Eq. (2.37) shows that the shifted p-integral is a factor $(-1)^{n-1}i$ times the corresponding Euclideanized integral. Since the Euclidean integral is finite, the p and ν integrations may be interchanged in this context and, working backwards through (2.35) and (2.25), it can be seen that

$$\begin{aligned}&\int \frac{d^{2n}p'}{(2\pi)^{2n}} \frac{1}{2^n} \int_0^1 \prod_{i=0}^{n} \frac{dx_i}{x_i} x_i^{(p_i^2+i\epsilon)/2} \mathcal{T}_{PT}(p') \\ &\quad = (-1)^{n-1} i \int_0^1 \prod_{i=0}^{n} \frac{dx_i}{x_i} \int \frac{d^{2n}p_E'}{(2\pi)^{2n}} x_i^{(p_{E_i}^2+i\epsilon)/2} \mathcal{T}_{PT}(p_E') .\end{aligned} \tag{2.41}$$

Thus, the interchange of integrals in (2.24) requires a simultaneous spatial Wick rotation. This is a very important point that will be explored for superstring amplitudes in the next section.

Using the above, the B coefficients are found to have the form

$$^r B_l = \frac{2i^n}{(2\pi)^n} \int_0^1 \prod_{i=0}^{n-1} d\nu_i \theta(\nu_{i+1}-\nu_i) \frac{f_{rl}(k;\tilde{\nu})}{D} , \tag{2.42}$$

as $\epsilon\to 0$, where

$$f_{pl}(\tilde{x}) = \begin{cases} \tilde{x}_p(1-\tilde{x}_l), & l<p , \\ \tilde{x}_l(1-\tilde{x}_p), & p\le l . \end{cases} \tag{2.43}$$

The corresponding alternating sum is

$$\sum_{r=0}^{n} (-1)^r S(\xi_0,\ldots,k_r,\ldots,\xi_n;k) = \frac{2i^n}{(2\pi)^n n!}\epsilon(\xi,k) , \tag{2.44}$$

which corresponds to the previously calculated value. This result is dependent *only* on the $\epsilon\to 0$ limit definition of the anomaly in (2.6).

Let us finally note that the ambiguity inherent in (2.1) is reflected in the X_r's. The physics comes from the fact that (2.5) is independent of the X_r's—in a restricted sense. If (2.5) is evaluated directly from (2.3) and the result is compared to (2.44), a compatibility relation is found, viz.,

$$\sum_{r=0}^{n} X_r = 1 . \tag{2.45}$$

Note that this did not have to be imposed on the regularization technique as before, but arises from an intrinsic analysis. Similarly, the value of the anomaly (2.44) can be easily derived from the compatibility. Geometrically, (2.45) describes an n-dimensional hyperplane V in the $(n+1)$-dimensional linear X_r space. Reference 12 shows that regularization techniques correspond to linear subsets of V. Regularization independence is expressed by the constancy of physical quantities such as (2.5) on V.

III. STRING EXTENSION OF THE POINT AMPLITUDE

In the amplitude (2.44), ξ_j is the polarization of the jth vertex with incoming momentum k_j. Note that there is only one label per vertex and one momentum propagating between vertices.[12] These physically describe point particles. Let us now modify the amplitude to correspond to extended stringlike particles.

Let $V(j)$ be an operator giving the jth emission around the loop and let Δ be an operator representing the propagation between emissions. If Γ_{2n+1} is a generalization of γ_{2n+1}, the planar loop given by Green and Schwarz[16] corresponding to (2.44) is the parity-violating piece of

$$T=\int d^{2n}p'\mathrm{Tr}[\Delta V(1)\Delta\cdots\Delta V(n+1)\tfrac{1}{2}(1+\Gamma_{2n+1})]\ , \tag{3.1}$$

where the explicitly written loop-momentum integral may be put implicitly into the trace. Let us use the expressions for Δ, V, and Γ_{2n+1} from the old covariant superstring formalism as given in Ref. 10 so that the planar anomaly becomes

$$-T^{\mathrm{P}}=\mathrm{Tr}\left[\prod_{j=1}^{n+1}V(j)\frac{1}{F_0}\Gamma_{2n+1}\right]\ , \tag{3.2}$$

where the reversal of the order of the factors so as to imitate (2.44) introduces a minus sign. The Möbius loop $T^{\mathcal{M}}$ and the nonplanar loop T^{NP} have the same form as (3.2) except with one and two factors of $(-1)^N$, respectively, introduced in a nontrivial way.[10] In detail, (3.2) is

$$T^P=\mathrm{Tr}\left[\prod_{j=1}^{n+1}(\not\xi_j+i\sqrt{2}\gamma_{2n+1}\xi_jC)V_0(j)\times\frac{\not p/i\sqrt{2}+\gamma_{2n+1}B}{L_0+i\epsilon}\gamma_{2n+1}\Gamma_d\right]\ . \tag{3.3}$$

The planar amplitude naturally breaks into a sum of four nontrivial pieces which shall be considered in the next section. Each term may be reexpressed as the integral of a Dirac trace times a string trace by replacing $p'\to -p'$. As an example, one term is

$$\int\frac{d^{2n}p'}{(-i\sqrt{2})^{n+1}}\mathrm{Tr}\left[\prod_{i=1}^{n+1}\not\xi_i\not p_i\gamma_{2n+1}\right]\times\mathrm{Tr}\left[\prod_{j=1}^{n+1}V_0(j)\frac{1}{L_0+i\epsilon}\Gamma_d\right]\ . \tag{3.4}$$

This piece contains the string-vacuum part of (3.3), which correspond to point field theory. After standard separation of the zero-mode factors, (3.4) takes a form similar to (2.24):

$$S_\epsilon=\int\frac{d^{2n}p'}{(i\sqrt{2})^{n+1}}\int_0^1\prod_{i=0}^{n}\frac{dx_i}{x_i}x_i^{(p_i^2+i\epsilon)/2}\mathcal{T}_{PT}(p')\mathcal{T}^P(x;k)\ . \tag{3.5}$$

Here $\mathcal{T}_{PT}(p')$ is defined in (2.7) and

$$\mathcal{T}^P=\mathrm{Tr}\left[w^N\prod_{i=1}^{n+1}\tilde{V}_0(k_i,p_i)\Gamma_d\right] \tag{3.6}$$

as in (A.19a) of Ref. 10. It was shown in the last section that the p' integration cannot be arbitrarily performed first. For the remainder of this section, let us investigate the conditions under which an interchange of integrals will alter the p' integration from Minkowski to Euclidean for a general integral of the form (3.5).

Naively, if $\mathcal{T}^P$ is analytic, i.e., if it has the convergent expansion

$$\mathcal{T}^P(x;k)=\sum_{m=0}^{\infty}a_m(k,\tilde{\nu})\nu^m\ , \tag{3.7}$$

where a_m is continuous in $\tilde{\nu}$, then, following (2.25), assuming that the sum commutes with the integrals, (3.5) becomes

$$\int\frac{d^{2n}p'}{(i\sqrt{2})^{n+1}}\mathcal{T}_{PT}(p')\int_0^1\prod_{i=0}^{n-1}\theta(\nu_{i+1}-\nu_i)d\nu_i\int_0^\infty d\nu\,\nu^n\exp\left\{-\frac{\nu}{2}\left[\left[p'+\sum_{r=1}^{n}k_r\tilde{\nu}_r\right]^2+D(k,\tilde{\nu}_r)+i\epsilon\right]\right\}\mathcal{T}^P(x;k) \tag{3.8}$$

$$=\int\frac{d^{2n}p'}{(i\sqrt{2})^{n+1}}\mathcal{T}_{PT}(p')\int_0^1\prod_{i=0}^{n-1}\theta(\nu_{i+1}-\nu_i)d\nu_i\times\sum_m a_m\int_0^\infty d\nu\,\nu^{m+n}\exp\left\{-\frac{\nu}{2}\left[\left[p'+\sum_{r=1}^{n}k_r\tilde{\nu}_r\right]^2+D(k,\tilde{\nu}_r)+i\epsilon\right]\right\} \tag{3.9}$$

$$=\frac{1}{(i\sqrt{2})^{n+1}}\int_0^1\prod_{i=0}^{n-1}\theta(\nu_{i+1}-\nu_i)d\nu_i\sum_m(m+n)!a_m(k,\tilde{\nu})\int\frac{d^{2n}p'\mathcal{T}_{PT}(p')}{\left[\left[p'+\sum_{r=1}^{n}k_r\tilde{\nu}_r\right]^2+D+i\epsilon\right]^{m+n+1}}\ . \tag{3.10}$$

Finiteness and power counting show that the integrals may be interchanged and shifting is allowed.

Equation (2.37) is derived for $X=D>0$ by analytically continuing the $|\mathbf{p}|$ integral to an imaginary integral, where $p=(p^0,\mathbf{p})$. This equates (2.37) with the Euclideanized integral which can then be easily evaluated. The same Euclideanization is valid for the integrals

$$\lim_{\epsilon\to 0}\int \frac{(p^\mu)^k d^{2n}p}{(p^2+X+i\epsilon)^{m+n+1}}=(-1)^{n-1}i\lim_{\epsilon\to 0}\int \frac{(p_E^\mu)^k d^{2n}p}{(p_E^2+X+i\epsilon)^{m+n+1}}\ , \tag{3.11}$$

where $p_E^2=\delta_{\mu\nu}p_E^\mu p_E^\nu$. These integrals vanish if $k<2(m+n)+1$ is odd and they are finite for $k\le 2m$ even—otherwise they are at best ambiguous. As a result, working backwards through (3.8)–(3.10), we see that (3.5) becomes

$$\int \frac{d^{2n}p'}{(i\sqrt{2})^{n+1}}\int_0^1\prod_{i=0}^{n}\frac{dx_i}{x_i}x_i^{(p_i^2+i\epsilon)/2}\mathcal{T}_{PT}(p')\mathcal{T}^P(x;k)$$
$$=(-1)^{n-1}i\int_0^1\prod_{i=0}^{n}\frac{dx_i}{x_i}\int\frac{d^{2n}p_E'}{(i\sqrt{2})^{n+1}}x_i^{(p_{Ei}^2+i\epsilon)/2}\mathcal{T}_{PT}(p_E')\mathcal{T}^P(x;k) \tag{3.12}$$

using the Euclidean equivalent of (A4). Thus Euclideanization is naively effected by the intercharge of the momentum and string parameter integrals.

The exact condition for the superstring anomaly graph to have equivalent Euclidean and Minkowski expressions is that the non-point-like factors (3.7) be uniformly convergent in ν on $[0,\infty)$ for all $\tilde{\nu}$ and k. Appendix B contains a rigorous proof of this general "anti-Fubini" theorem. It is useful for deciding the validity of Wick rotation in superstring loop amplitudes.

IV. LOCATION OF THE INFINITIES

An important point to clarify is that superstring loop amplitudes are not finite. In fact, in the usual formalism they are not even well defined. Green and Schwarz have stated that an unregulated calculation of the parity-violating open-superstring amplitude leads to infinities.[16] A hint as to where infinities can arise is found in Ref. 7 which gives an operator-product expansion for adjacent vertex operators. The ill-defined nature of the expansion is dealt with by legislating apparent infinities away with an analyticity requirement. Clavelli, Cox, and Harms (CCH) have found that the chiral anomaly amplitude is finite in the same sense as the point anomaly discussed in Sec. II as long as terms which have adjacent vertex operators are subtracted out.[6] Let us consider this in more detail.

With vertex operators of the form[10]

$$V_{F_2}(i)=V(k_i,1)=\zeta_i\cdot\Gamma(1){:}\exp[ik_i\cdot X(1)]{:}\ , \tag{4.1}$$

the operator product

$$V_{F_2}(0)V_{F_2}(1)=\zeta_0\cdot\Gamma(1)\zeta_1\cdot\Gamma(1)V_0(0)V_0(1) \tag{4.2}$$

explicitly involves the factor

$$\zeta_0\cdot\Gamma(1)\zeta_1\cdot\Gamma(1)=\not{\zeta}_0\not{\zeta}_1+i\sqrt{2}(\not{\zeta}_0\zeta_{1\mu}\not{\zeta}_1\zeta_{0\mu})\sum_{n=-\infty}^{\infty}d_n^\mu$$
$$-2\zeta_{0\mu}\zeta_{1\nu}\sum_{m,n=-\infty}^{\infty}d_m^\mu d_n^\nu\ . \tag{4.3}$$

If this expression were assumed to be finite, an inconsistency would be found. For example, in the case that $\zeta_0=\zeta_1$ the last term becomes

$$-\zeta_{0\mu}\zeta_{0\nu}\sum_{m,n=-\infty}^{\infty}\{d_m^\mu,d_n^\nu\}=-\zeta_0^2\sum_{n=-\infty}^{\infty}1\ , \tag{4.4}$$

which diverges. If (4.3) were conditionally convergent, a divergent rearrangement of the infinite series would exist. In order to make sense of (4.2), some kind of regularization is required.

Vertex operator products of the form (4.2) are the only source of infinite behavior in the anomaly since, as found by CCH, the amplitudes are finite after terms involving them are subtracted away. This would also give the infinite current divergences found earlier.[10] CCH use the "canceled propagator argument" which is defined to be "the 'fact' that a term with a canceled propagator vanishes"[7] (single quotes mine). This is indeed a fact for regulated amplitudes since the implicit point-splitting limit can be straightforwardly interchanged with the integrals.[15] However, for an unregulated amplitude such as (3.2), which is studied by CCH, the limit would be considered as a regularization technique in itself.[5,17] They are erroneous, therefore, in concluding that the amplitude is finite. Unfortunately, straightforward regularization is found to give inconsistent results.[10]

Let us examine where the divergence arise in the unsubtracted amplitude. First of all, let us list the terms of (3.3). In addition to (3.5) there is

$$\frac{-1}{(i\sqrt{2})^{n-1}}\sum_{r<s}\int d^{2n}p'\,\mathrm{Tr}(\not{\xi}_1\not{p}_1\cdots[\xi_r]\cdots[\xi_s]\cdots\not{\xi}_{n+1}\not{p}_{n+1}\gamma_{2n+1})$$
$$\times\left[\prod_{j=1}^{r-1}V_0(j)\frac{1}{L_0+i\epsilon}\xi_rC\prod_{l=r}^{s-1}V_0(l)\frac{1}{L_0+i\epsilon}\xi_sC\prod_{m=s}^{n+1}V_0(m)\frac{1}{L_0+i\epsilon}\Gamma_d\right]\ , \tag{4.5}$$

$$\frac{-1}{(i\sqrt{2})^{n-1}}\sum_{r<s}\int d^{2n}p'\,\mathrm{Tr}[\not{\xi}_1\not{p}_1\cdots[p_r]\cdots[p_s]\cdots\not{\xi}_{n+1}\not{p}_{n+1}\gamma_{2n+1}]$$
$$\times\left[\prod_{j=1}^{r}V_0(j)\frac{1}{L_0+i\epsilon}B\prod_{l=r+1}^{s}V_0(l)\frac{1}{L_0+i\epsilon}B\prod_{m=s+1}^{n+1}V_0(m)\frac{1}{L_0+i\epsilon}\Gamma_d\right]\ , \tag{4.6}$$

and,

$$\frac{1}{(i\sqrt{2})^{n-1}}\sum_{r,s}\int d^{2n}p'\int_0^1\prod_{i=0}^{n}\frac{dx_i}{x_i}x_i^{(p_i^2+i\epsilon)/2}\mathcal{T}_{rs}^{D}\mathcal{T}_{rs}^{S}\ , \tag{4.7}$$

where the Dirac trace

$$\mathcal{T}_{rs}^{D}=\begin{cases}\mathrm{Tr}(\not\xi_1\not p_1\cdots[\xi_r]\cdots[p_s]\cdots\not\xi_{n+1}\not p_{n+1}\gamma_{2n+1}), & r\le s\\ \mathrm{Tr}[\not\xi_1\not p_1\cdots[p_s]\cdots[\xi_r]\cdots\not\xi_{n+1}\not p_{n+1}\gamma_{2n+1}], & r>s\end{cases} \tag{4.8}$$

and the oscillator trace

$$\mathcal{T}_{rs}^{S}=\begin{cases}\left[w^N\prod_{j=1}^{r-1}\tilde V_0(k_j,\rho_j)\xi_r C\prod_{l=r}^{s}\tilde V_0(k_l,\rho_l)B\prod_{m=s+1}^{n+1}\tilde V_0(k_m,\rho_m)\Gamma_d\right], & r\le s\ ,\\ \left[w^N\prod_{j=1}^{s}\tilde V_0(k_j,\rho_j)B\prod_{l=s+1}^{r-1}\tilde V_0(k_l,\rho_l)\xi_r C\prod_{m=r}^{n+1}\tilde V_0(k_m,\rho_m)\Gamma_d\right], & r\le s\ .\end{cases} \tag{4.9}$$

Note that all the terms may be written in the form of (3.5) as required by the general anti-Fubini theorem of the preceding section.

Now let us assume that the $\mathcal{T}^P$ factors are uniformly convergent as in (B1). Further, assume the $\mathcal{T}^P$ factors for (4.5)–(4.7) vanish for $\nu=0$. Then, Wick-rotating to Euclidean space, the momentum integration may be performed first. It has already been seen that (3.5) is finite. Also, since the remaining Dirac traces are at most first degree in p', symmetric integration,

$$\int d^{2n}p\ x^{p_E^2/2}=\left[\frac{\pi}{2\nu}\right]^{1/2}, \tag{4.10}$$

$$\int d^{2n}p\ p^{\mu}x^{p_E^2/2}=0\ , \tag{4.11}$$

leaves regularization-independent finite integrals. As a result, the existence of infinities in (3.1) implies that the $\mathcal{T}^P$'s are not uniformly convergent and hence contain poles.[15] Similar analysis can be applied to the Möbius and nonplanar graphs as well.

The point anomaly has been calculated in a manner independent of regularization.[13] On the other hand, the various topologies of chiral-anomaly graphs give regularization-dependent amplitudes.[10] The difference between the point and superstring one-loop graphs is found in the nonzero mode $\mathcal{T}^P$ factors. Thus, it must be these that give rise to the above regularization dependence. These must somehow prevent dimensional limits from being finite. String parameter integrals over the p-independent poles in $\mathcal{T}^P$ may either be ambiguous or infinite. In the latter case, since they would be unaffected by the analytic continuation $p_0\to ip_0$, they would give essential singularities in the integrand with respect to p.[18,1] This agrees with a preliminary unregulated calculation. Ambiguity dependence and hence regularization dependence can be expected when dealing with such infinities.

In conclusion, we are now in a position to test the validity of Wick rotation for any integer-dimension regularization of (3.1). On the other hand, if ill-defined canceled propagator terms of the unregulated amplitudes are "subtracted" following CCH, the resultant expression is finite and Wick rotation is not required. This subtraction, however, is a regularization technique and at this stage must be considered as good as others that do not give a vanishing anomaly. Further study of the validity of Wick rotation for dimension-changing regulators would be in order.

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APPENDIX A: POINT MISCELLANEA

Let us use the same conventions and definitions as Ref. 10 with the following modifications. First of all, it is convenient to use different symbols for the metrics

$$g=\mathrm{diag}(1,-1,\ldots,-1)\ \text{ and }\ \eta=\mathrm{diag}(-1,1,\ldots,1)\ . \tag{A1}$$

Now define

$$\gamma_{2n+1}=i^{n+1}\gamma^0\cdots\gamma^{2n-1}\ , \tag{A2}$$

so that $\gamma_{2n+1}^2=1$ and

$$\mathrm{Tr}(\gamma_{2n+1}\gamma_{\mu_1}\cdots\gamma_{\mu_{2n}})=(-2)^n i^{n-1}\epsilon_{\mu_1\cdots\mu_{2n}}\ . \tag{A3}$$

Note that the replacements $\gamma^{\mu}\to\gamma_{2n+1}\gamma^{\mu}$, and $\gamma_{2n+1}\to i^{n-1}\gamma_{2n+1}$ relate our conventions to the conventions of Ref. 5.

Next, let p and p' be the integration variables representing loop momentum. It is useful to define

$$p_r^\mu \equiv p'^\mu + \sum_{i=1}^{r} k_i^\mu \ , \tag{A4}$$

where

$$p_{n+1}^\mu = p_0^\mu \equiv p'^\mu, \quad \xi_0 = \xi_{n+1}, \quad \text{and} \quad k_{n+1}^\mu \equiv k_0^\mu = -\sum_{i=1}^{n} k_i^\mu \ . \tag{A5}$$

A compact epsilon trace is

$$\epsilon_r'(\xi_1 \cdots \xi_n, k_1 \cdots k_n) = \epsilon_{\alpha_0\alpha_1\rho_1 \cdots [\alpha_r] \cdots \alpha_n\rho_n} \xi_1^{\alpha_0} \cdots \xi_r^{\alpha_{r-1}} \xi_{r+1}^{\alpha_{r+1}} \cdots \xi_n^{\alpha_n} k_1^{\rho_1} \cdots k_n^{\rho_n} \ . \tag{A6}$$

APPENDIX B: THE GENERAL ANTI-FUBINI THEOREM

It is of interest to determine the conditions under which interchange of integrals gives the Wick rotation. Some of the theorems of infinite integrals on which the following reasoning is based are found in Ref. 15. Let us begin with a useful corollary of (3.7).

Equation (2.25) is convergent. Therefore, if (3.7) is uniformly convergent in ν on $[0,\infty)$ for all $\tilde{\nu}$ and k, then (3.9) is justified. The infinite range of convergence also forces the inverse Borel transform[15]

$$\int d\nu\, \nu^n e^{-\nu} \mathcal{T}^P(x\nu) = \sum_m a_m(k,\tilde{\nu})(m+n)!x^m \tag{B1}$$

to be entire in x and uniformly convergent in $\tilde{\nu}$ for all $k,n \geq 0$. Then, if $\epsilon > 0$,

$$\int_0^1 \prod_{i=0}^{n-1} \theta(\nu_{i+1}-\nu_i)d\nu_i \sum_m \frac{a_m(k,\tilde{\nu})(m+n)!}{\left[\left[p' + \sum_{r=1}^{n} k_r\tilde{\nu}_r\right]^2 + D + i\epsilon\right]^{m+n+1}}$$
$$= \sum_m \int_0^1 \prod_{i=0}^{n-1} \theta(\nu_{i+1}-\nu_i)d\nu_i \frac{a_m(k,\tilde{\nu})(m+n)!}{\left[\left[p' + \sum_{r=1}^{n} k_r\tilde{\nu}_r\right]^2 + D + i\epsilon\right]^{m+n+1}} \tag{B2}$$

converges since the integrals are all finite. Also,

$$\int d^{2n}p' \sum_m \frac{a_m(k,\tilde{\nu})(m+n)!}{\left[\left[p' + \sum_{r=1}^{n} k_r\tilde{\nu}_r\right]^2 + D + i\epsilon\right]^{m+n+1}} \mathcal{T}_{PT}(p') = \sum_m a_m(m+n)! \int \frac{d^{2n}p' \mathcal{T}_{PT}(p')}{\left[\left[p' + \sum_{r=1}^{n} k_r\tilde{\nu}_r\right]^2 + D + i\epsilon\right]^{m+n+1}} \tag{B3}$$

may be verified by looking at term-by-term interchanges of sums and integrals. Then, if (B3) converges uniformly in ν_i, (3.10) converges to (3.8) as desired and Euclideanization may be verified for a one-loop superstring anomaly process.

Let us now examine (B3). First of all, consider

$$\int dp_0' \sum_m \frac{a_m(k,\tilde{\nu})(m+n)!}{\left[\left[p' + \sum_{r=1}^{n} k_r\tilde{\nu}_r\right]^2 + D + i\epsilon\right]^{m+n+1}} \mathcal{T}_{PT}(p') = \sum_m a_m(m+1)! \int \frac{dp_0' \mathcal{T}_{PT}(p')}{\left[\left[p' + \sum_{r=1}^{n} k_r\tilde{\nu}_r\right]^2 + D + i\epsilon\right]^{m+n+1}} \ . \tag{B4}$$

This holds if $\sum_m g_m(x,y)$ is uniformly convergent in $x \leq 0$ and $y \geq 0$ on $|x|,|y| \geq M$ for M sufficiently large, where

$$g_m(x,y) = a_m(m+n)! \int_x^y \frac{dp_0' \mathcal{T}_{PT}(p')}{\left[\left[p' + \sum_{r=1}^{n} k_r\tilde{\nu}_r\right]^2 + D + i\epsilon\right]^{m+n+1}} \ . \tag{B5}$$

In fact,

$$\int_{-\infty}^{\infty} \frac{x^k dx}{(x^2+X+i\epsilon)^{m+n+1}} = \frac{\Gamma(1\pm\frac{1}{2})\Gamma(m+n\pm\frac{1}{2})}{\Gamma(m+n+1)(X+i\epsilon)^{m+n\pm1/2}} \tag{B6}$$

for $k=0,2$, respectively,[19] while it vanishes for $k=1,3$; therefore,

$$\int_{-\infty}^{\infty}\frac{dp_0'\mathcal{T}_{PT}(p')}{\left[\left[p'+\sum_{r=1}^{n}k_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n+1}}=\int_{-\infty}^{\infty}\frac{dp_0\mathcal{T}_{PT}(p')}{(p_0^2+X+i\epsilon)^{m+n+1}}$$

$$=\frac{c_0\alpha_{m+n}(\frac{1}{2})}{(X+i\epsilon)^{m+n+1/2}}+\frac{c_2\alpha_{m+n}(\frac{3}{2})}{(X+i\epsilon)^{m+n-1/2}}\ , \quad \text{(B7)}$$

where,

$$X=-\left[\mathbf{p}+\sum\mathbf{k}_r\tilde{v}_r\right]^2+D,\quad \alpha_{m+n}(s)=\frac{\Gamma(s)\Gamma(m+n+1-s)}{\Gamma(m+n+1)}\ ,$$

and

$$\mathcal{T}_{PT}(p')=\sum_{k=0}^{3}c_kp_0^k\ .$$

Also, if $\rho=\sum k_r^0v_r$, $|x|\geq M$ and

$$\alpha_r^{(m+n)}=\frac{[2(m+n)]!}{[(m+n)!]^2}\frac{2r!(r-1)!}{[4(X+i\epsilon)]^{m+n-r+1}(2r)!}\quad\text{and}\quad\beta^{(m+n)}=\frac{[2(m+n)]!}{4^{m+n}[(m+n)!]^2}\ , \quad \text{(B8)}$$

then,

$$\int_{-\infty}^{-x}\frac{dp_0'\mathcal{T}_{PT}(p')}{\left[\left[p'+\sum_{r=1}^{n}k_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n+1}}=\int_{-\infty}^{-x+\rho}\frac{dp_0\mathcal{T}_{PT}(p')}{(p_0^2+X+i\epsilon)^{m+n+1}} \quad \text{(B9)}$$

$$=\sum_{k=0}^{3}c_k(-1)^k\int_{x-\rho}^{\infty}\frac{x^kdx}{(x^2+X+i\epsilon)^{m+n}}\ , \quad \text{(B10)}$$

where the relevant integrals are

$$\int_{\xi}^{\infty}\frac{x^kdx}{(x^2+X+i\epsilon)^{m+n+1}}$$

$$=\begin{cases}\sum_{r=1}^{m+n}\alpha_r^{(m+n)}\frac{\xi}{(\xi^2+X)^r}-\frac{\beta^{(m+n)}}{X^{m+n+1/2}}\arctan\frac{\xi}{\sqrt{X}},\quad k=0,\ X\neq0\ , & \text{(B11)}\\ \frac{1}{[2(m+n)+1]\xi^{2(m+n)+1}},\quad k=0,\ X=0\ , & \text{(B12)}\\ \frac{A_{m+n}}{(\xi^2+X)^{m+n}},\quad k=1,\ \text{any}\ X\ , & \text{(B13)}\\ \frac{A_{m+n}\xi}{(\xi^2+X)^{m+n}}+\frac{1}{2(m+n)}\left[\sum_{r=1}^{m+n-1}\alpha_r^{(m+n-1)}\frac{\xi}{(\xi^2+X)^r}-\frac{\beta^{(m+n-1)}}{X^{m+n-1/2}}\arctan\frac{\xi}{\sqrt{X}}\right],\quad k=2,\ X\neq0\ , & \text{(B14)}\\ \frac{A_{m+n}\xi}{(\xi^2+X)^{m+n}}+\frac{1}{4(m+n)[(m+n)+1]\xi^{2(m+n)-1}},\quad k=2,\ X=0\ , & \text{(B15)}\\ \frac{A_{m+n-1}}{(\xi^2+X)^{m+n-1}}-\frac{A_{m+n}X}{(\xi^2+X)^{m+n}},\quad k=3,\ \text{any}\ X\ . & \text{(B16)}\end{cases}$$

The coefficients A_{m+n} are given in Ref. 19. Since $|A_{m+n}|<1$, using (B1) it is seen that

$$\sum_m g_m(-x,y)=\sum_m a_m(m+n)!\{[(y+\rho)^2+X]^{-m-n+1}+[(x-\rho)^2+X]^{-m-n+1}$$

$$+|X|[(y+\rho)^2+X]^{-m-n}+|X|[(x-\rho)^2+X]^{-m-n}\} \quad \text{(B17)}$$

and (B1) is absolutely convergent everywhere. As a result, (B6) shows that only the $k=0,2$ terms of $\mathcal{T}_{PT}(p')$ contribute to (B3). In fact, it is clear from (B14)–(B15) that (B17) will be uniformly convergent for both $k=0$ and 2 if it is for $k=0$ generically. Clearly a divergence exists if $X=0$. If $X\neq 0$, the expression in (B12) must be considered. Now $r!(r-1)!\leq(2r)!$, and

$$\frac{[2(m+n)]!}{[(m+n)!]^2}\leq 4^{m+n}\;; \tag{B18}$$

therefore,

$$\sum_r \frac{\alpha_r^{(m+n)}\xi}{(\xi^2+X)^r}\leq\frac{\xi}{X}\left[\frac{4}{X}\right]^{m+n}\sum_r\left[\frac{\xi^2}{X}+1\right]^{-r}=\frac{1}{\xi}\left[\left[\frac{4}{X}\right]^{m+n}-\left[\frac{4}{(\xi^2+X)}\right]^{m+n}\right].$$

Also,

$$\frac{\beta^{(m+n)}}{X^{m+n+1/2}}\arctan\frac{\xi}{\sqrt{X}}\leq\frac{\pi}{2X^{m+n+1/2}}\;. \tag{B19}$$

Thus, the sum

$$\sum_m a_m(m+n)!\int_\xi^\infty\frac{dx}{(x^2+X+i\epsilon)^{m+n+1}} \tag{B20}$$

will be uniformly convergent since each of its terms will be bounded by a uniformly convergent series and will therefore themselves be uniformly convergent.

It is now clear from (B13)–(B16) that (B5) will be uniformly convergent and (B4) holds. In fact,

$$\int dp_0'\sum_m\frac{a_m(k,\tilde{v})(m+n)!}{\left[\left[p'+\sum_{r=1}^n k_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n+1}}\mathcal{T}_{PT}(p')$$

$$=c_0\sum_m\frac{a_m(m+n)!\alpha_{m+n}(\frac{1}{2})}{(X+i\epsilon)^{m+n+1/2}}+c_2\sum_m\frac{a_m(m+n)!\alpha_{m+n}(\frac{3}{2})}{(X+i\epsilon)^{m+n-1/2}}\;, \tag{B21}$$

but using standard properties of Γ functions,[19] it can be shown that $\alpha_{m+n}(s)<\pi$ for $0<s<m+n+1$, since s not a positive integer. By the comparison test, then, each of the series of (B21) are convergent and the total series is uniformly convergent everywhere.

The next thing to check is that

$$\sum_{k=0,2}c_k\int dp_1'\sum_m\frac{a_m(m+n)!\alpha_{m+n}\left[\frac{k+1}{2}\right]}{\left[-\left[\mathbf{p}'+\sum_{r=1}^n\mathbf{k}_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n+(1-k)/2}}$$

$$=\sum_{k=0,2}c_k\sum_m a_m(m+n)!\alpha_{m+n}\left[\frac{k+1}{2}\right]\int\frac{dp_1'}{\left[-\left[\mathbf{p}'+\sum_{r=1}^n\mathbf{k}_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n+(1-k)/2}}\;. \tag{B22}$$

As before, this requires $\sum_m h_m(x,y)$ to be uniformly convergent in x and y where

$$h_m(x,y)=a_m(m+n)!\alpha_{m+n}\left[\frac{k+1}{2}\right]\int_x^y\frac{dp_1'}{\left[-\left[\mathbf{p}'+\sum_{r=1}^n\mathbf{k}_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n+(1-k)/2}}\;. \tag{B23}$$

Again, it is seen that shifting gives

$$\int_{-\infty}^\infty\frac{dp_1'}{\left[-\left[\mathbf{p}'+\sum_{r=1}^n\mathbf{k}_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n+(1-k)/2}}=\int_{-\infty}^\infty\frac{dx}{(-x^2+X+i\epsilon)^{m+n+(1-k)/2}}\;, \tag{B24}$$

where $X=-(\hat{p}\,'+\sum_r \hat{k}_r\tilde{v}_r)^2+D$, $\hat{p}\,'=(p^2,\ldots,p^n)$, etc. This expression is then, for $X\neq 0$,

$$\int_{-\infty}^{\infty}\frac{dx}{(-x^2+X+i\epsilon)^{m+n+(1-k)/2}}=\frac{\Gamma(\frac{1}{2})\Gamma(m+n-k/2)}{\Gamma(m+n+(1-k)/2)iX^{m+n-k/2}}\ , \tag{B25}$$

and it blows up for $X=0$. Now, if $\sigma=\sum k_r^1 v_r$, shifting takes $\int_{-\infty}^{-x}\to\int_{x-\sigma}^{\infty}$ [see (B4)] and, for $X>0$ (which can be arranged),

$$\int_{\xi}^{\infty}\frac{dx}{(-x^2+X+i\epsilon)^{m+n+(1-k)/2}}$$
$$=\frac{i}{2}\frac{(m+n-k/2)!(m+n-k/2-1)!}{[2(m+n-k/2)]!}\left[\frac{4}{X}\right]^{m+n-k/2}\left[1-\frac{\xi}{\sqrt{\xi^2-X}}\sum_{r=0}^{m+n-k/2-1}\frac{(2r)!}{4^r(1-\xi^2/X)^r(r!)^2}\right]. \tag{B26}$$

Therefore,

$$\sum_m h_m(-x,y)=\sum_m a_m(m+n)!\alpha_{m+n}\left[\frac{k+1}{2}\right]\frac{\beta_{m+n}(k/2)}{X^{m+n-k/2}}\frac{i\xi}{\sqrt{\xi^2-X}}\sum_{r=0}^{m+n-k/2-1}\frac{(2r)!}{4^r\left[1-\frac{\xi^2}{X}\right]^r(r!)^2}\ , \tag{B27}$$

where,

$$\beta_{m+n}(s)=B(\tfrac{1}{2},m+n-s)<1\ , \tag{B28}$$

the usual β function.[19] Now,

$$\sum_{r=0}^{N}\frac{(2r)!}{4^r(r!)^2}\left[1-\frac{\xi^2}{X}\right]^{-r}\le\sum_{r=0}^{N}\left|1-\frac{\xi^2}{X}\right|^{-r}=\frac{1-|1-\xi^2/X|^{-N-1}}{1-|1-\xi^2/X|^{-1}}\ , \tag{B29}$$

so that,

$$\sum_m h_m(-x,y)\le\frac{i\pi\xi}{\sqrt{\xi^2-X}}\frac{1-\xi^2/X}{2-\xi^2/X}\sum_m\frac{a_m(m+n)!}{X^{m+n-k/2}}\left[1-\frac{1}{|1-\xi^2/X|^{m+n-k/2-1}}\right]\ , \tag{B30}$$

which is clearly uniformly convergent. Thus $\sum h_m$ is uniformly convergent and (B22) holds. As a result,

$$\int\int dp_0'dp_1'\sum_n\frac{a_m(k,\tilde{v})(m+n)!}{\left[\left[p'+\sum_{r=1}^{n}k_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n+1}}=\sum_{k=0}^{1}c_{2k}\sum_m a_m(m+n)!\frac{\alpha_{m+n}\left[\frac{k+1}{2}\right]B(\frac{1}{2},m+n-k)}{i\left[-\left[\hat{p}\,'+\sum\hat{k}_r\tilde{v}_r\right]^2+D+i\epsilon\right]^{m+n-k}}\ . \tag{B31}$$

Since (B1) bounds this series, (B31) will also be uniformly convergent and the above process for evaluating the double integrals may be repeated n times if $m\neq 0$ to verify (B3). In the zero-mode ($m=0$) case, the superficially divergent integral corresponds to the point anomaly where it has been shown to be finite.[12,13] It has been pointed out that this argument may be made more elegant with the use of powerful results of measure theory and analysis.[20]

[1] D. J. Gross and V. Perival, Phys. Rev. Lett. **60**, 2105 (1985).

[2] D. A. Eliezer and R. P. Woodard, Nucl. Phys. **B325**, 389 (1989).

[3] Y. Katsuki *et al.*, Phys. Lett. B **227**, 381 (1989).

[4] A. Matheson, in *'88 Electroweak Interactions and Unified Theories,* proceedings of the XXIIIrd Rencontre de Moriond, Les Arcs, Savoie, France, 1989, edited by J. Tran Thanh Van (Editions Frontières, Gif-sur-Yvette, France, 1988); K. A. Olive *ibid*; T. R. Taylor, *ibid.*

[5] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).

[6] L. Clavelli, P. H. Cox, and B. Harms, Phys. Rev. D **35**, 1908 (1987). More recent work based on this is Z.-H. Lin, *ibid.* **39**, 3023 (1989).

[7] Green, Schwarz, and Witten, *Superstring Theory* (Ref. 5), Sec. 7.1.5.

[8] D. A. Kirzhnits, Usp. Fiz. Nauk **90**, 129 (1966) [Sov. Phys. Usp. **9**, 692 (1967)].

[9] N. Marcus, Phys. Lett. B **219**, 265 (1989).

[10] R. B. Mann, Nucl. Phys. **B303**, 99 (1988).

[11] String parameters are closely related to Schwinger parameters. See D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena,* revised 2nd ed. (World Scientific, Singapore, 1984).

[12]R. B. Mann, Nucl. Phys. **B266**, 125 (1986); A. M. Chowdhury, G. McKeon, and R. B. Mann, Phys. Rev. D **33**, 3090 (1986).

[13]R. McArthur and R. B. Mann, Mod. Phys. Lett. A **2**, 269 (1987).

[14]N. G. de Bruijn, *Asymptotic Methods in Analysis* (North-Holland, Amsterdam, 1958), formula (4.1.8).

[15]T. J. I'a. Bromwich and T. M. MacRobert, *An Introduction to the Theory of Infinite Series,* 2nd ed. (Macmillan, London, 1926), Appendix III; K. Knopp, *Infinite Sequences and Series* (Dover, New York, 1956); also Ref. 17.

[16]M. Green and J. Schwarz, Phys. Lett. **149B**, 117 (1984); Nucl. Phys. **B255**, 93 (1985).

[17]For example, if both criteria of theorem 7.11 in W. Rudin, *Principles of Mathematical Analysis,* 3rd ed. (McGraw-Hill, New York, 1976) hold, then *both* double limits must be finite. If either one of the criteria break down, as is allowed with the application of canceled propagators to the amplitude of Ref. 6, then the limits may not be naively interchanged.

[18]J. W. Moffat, Can. J. Phys. **64**, 561 (1986); J. W. Moffat, in *New Frontiers in Particle Physics,* proceedings of the Lake Louise Winter Institute, Lake Louise, Canada, 1986, edited by J. M. Cameron, B. A. Campbell, A. N. Kamal, and F. C. Khanna (World Scientific, Singapore, 1986).

[19]S. M. Selby, *CRC Standard Mathematical Tables,* 21st ed. (Chemical Rubber Co., Cleveland, 1973), Sec. 615.

[20]J. Levy (private communication).