Casimir energy for a piecewise uniform string

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The Casimir energy for the transverse oscillations of a piecewise uniform closed string is calculated. The string consists of two parts I and II, endowed in general with different tensions and mass densities, although adjusted in such a way that the velocity of sound always equals the velocity of light. The dispersion equation is worked out under general conditions and the frequency spectrum is determined in special cases. When the ratio L_{II}/L_I between the string lengths is an integer, it is in principle possible to determine the frequency spectrum through solving algebraic equations of increasingly high degree. The Casimir energy relative to the uniform string is in general found to be negative, although in the special case $L_I=L_{II}$ the energy is equal to zero. Delicate points in the regularization procedure are discussed; in particular, it turns out that a straightforward use of the Riemann ζ-function regularization method leads to an incorrect expression for the Casimir energy.

I. INTRODUCTION

Valuable insight into the nature of the Casimir effect[1]—as well as the more general aspects of periodicity in field theories—may be obtained through a consideration of very simple physical models. In this paper we intend to calculate the Casimir energy of the transverse oscillations of a string of length L when the string is composed of two parts of lengths L_I and L_{II}. The tensions T_I and T_{II}, and the mass densities ρ_I and ρ_{II}, are in general different, although we require the sound velocity to be equal to the light velocity:

$$v_s=(T_I/\rho_I)^{1/2}=(T_{II}/\rho_{II})^{1/2}=1 \tag{1}$$

(with $\hbar=c=1$). The situation is sketched in Fig. 1. [The condition (1) in fact is analogous to the condition $\epsilon\mu=1$ in phenomenological electrodynamics, with ϵ denoting the permittivity and μ the permeability of the medium. This condition requires the velocity of propagation of the photons in the medium to be equal to the velocity of light.[2]]

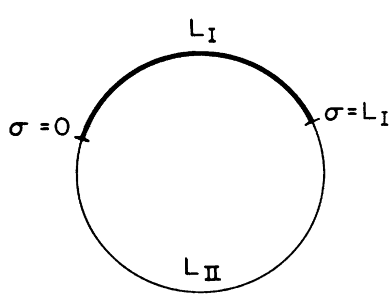

FIG. 1. Basic features of the string.

As far as we are aware, this composite string model has not been considered before. In the next section we make use of the basic equations of motion for transverse waves on the string to derive the dispersion equation. This equation will not be solved in full generality; we solve it in some special cases. First, in Sec. III we consider the special case of a uniform string; we consider the case where the tension ratio $x\equiv T_I/T_{II}\rightarrow 0$, and also the situation where the pieces L_I and L_{II} are equal. In the last case the Casimir energy is zero, irrespective of the value of the tension ratio. Thereafter, in Sec. IV we investigate the case of odd integer values of $s\equiv L_{II}/L_I$. The structure of the frequency spectrum is investigated and the formal expression for the Casimir energy is derived. For low values of s, $s=3,5,7$, the calculations are carried out in detail. Section V contains a parallel development of the theory when s is an even integer. Also in this case the lowest values of s, $s=2,4,6$, are investigated in detail. Integer values of s imply that the frequency spectrum can be evaluated through solving algebraic equations for the trigonometric functions. Our main results as regards the Casimir energy are given in Tables II and IV and shown graphically in Fig. 2.

The divergent zero-point energies are regularized by means of an exponential cutoff. The regularization procedure in the present problem is more delicate than what one might expect beforehand. Thus a straightforward application of the Riemann ζ-function regularization method would lead to an incorrect result. This point is discussed in more detail in Sec. IV D. Finally it ought to be remarked that, although the present paper is concerned with a specific string model, our results may be given a more general physical interpretation as the energy of the vacuum state of quantum field theory in a two-dimensional space-time endowed with special properties. Thus, the string model may be regarded as a convenient working tool rather than the essential physical element in the theory.

II. THE DISPERSION EQUATION

Let $\psi=\psi(\sigma,\tau)$ be the transverse displacement of a point with spatial coordinate σ from its equilibrium position at time τ. If we consider only motion in one plane, the field ψ can be regarded as a scalar, one-dimensional, field. Taking into account right- and left-moving waves in regions I and II, we have the general forms

$$\begin{aligned}\psi_{\rm I}&=\xi_{\rm I}e^{i\omega(\sigma-\tau)}+\eta_{\rm I}e^{-i\omega(\sigma+\tau)}\ ,\\ \psi_{\rm II}&=\xi_{\rm II}e^{i\omega(\sigma-\tau)}+\eta_{\rm II}e^{-i\omega(\sigma+\tau)}\ ,\end{aligned} \tag{2}$$

with the ξ and η being constants. These general expressions satisfy the fundamental wave equation

$$\left[\frac{\partial^2}{\partial\sigma^2}-\frac{\partial^2}{\partial\tau^2}\right]\psi(\sigma,\tau)=0\ . \tag{3}$$

In the following we separate off the common time factor $e^{-i\omega\tau}$ from (2). There are two different kinds of boundary conditions. First, the transverse displacement itself must be continuous across the two junctions:

$$\psi_{\rm I}(0)=\psi_{\rm II}(L),\quad \psi_{\rm I}(L_{\rm I})=\psi_{\rm II}(L_{\rm I})\ , \tag{4}$$

leading to

$$\begin{aligned}&\xi_{\rm I}+\eta_{\rm I}=\xi_{\rm II}e^{i\omega L}+\eta_{\rm II}e^{-i\omega L}\ ,\\ &\xi_{\rm I}e^{i\omega L_{\rm I}}+\eta_{\rm I}e^{-i\omega L_{\rm I}}=\xi_{\rm II}e^{i\omega L_{\rm I}}+\eta_{\rm II}e^{-i\omega L_{\rm I}}\ .\end{aligned} \tag{5}$$

Second, the transverse elastic force on the string must also be continuous across the junctions:

$$\begin{aligned}T_{\rm I}\frac{\partial\psi_{\rm I}}{\partial\sigma}\bigg|_{\sigma=0}&=T_{\rm II}\frac{\partial\psi_{\rm II}}{\partial\sigma}\bigg|_{\sigma=L}\ ,\\ T_{\rm I}\frac{\partial\psi_{\rm I}}{\partial\sigma}\bigg|_{\sigma=L_{\rm I}}&=T_{\rm II}\frac{\partial\psi_{\rm II}}{\partial\sigma}\bigg|_{\sigma=L_{\rm I}}\ ,\end{aligned} \tag{6}$$

which imply

$$\begin{aligned}&T_{\rm I}(\xi_{\rm I}-\eta_{\rm I})=T_{\rm II}(\xi_{\rm II}e^{i\omega L}-\eta_{\rm II}e^{-i\omega L})\ ,\\ &T_{\rm I}(\xi_{\rm I}e^{i\omega L_{\rm I}}-\eta_{\rm I}e^{-i\omega L_{\rm I}})=T_{\rm II}(\xi_{\rm II}e^{i\omega L_{\rm I}}-\eta_{\rm II}e^{-i\omega L_{\rm I}})\ .\end{aligned} \tag{7}$$

The compatibility condition for the set of equations (5) and (7) for the four unknowns $\xi_{\rm I},\eta_{\rm I},\xi_{\rm II},\eta_{\rm II}$ is that the determinant of the coefficients vanishes. It is convenient to define x as the ratio between the two tensions:

$$x=T_{\rm I}/T_{\rm II}\ . \tag{8}$$

Some calculation results in the equation

$$(1-x)^2\cos(\omega L-2\omega L_{\rm I})-(1+x)^2\cos\omega L+4x=0\ . \tag{9}$$

This is the dispersion equation, determining the frequencies ω of the possible transverse oscillations once the quantities x, L, and $L_{\rm I}$ are given. Note that the equation is invariant under the substitution $x\to 1/x$. For this reason we shall only consider x in the interval $0<x\le 1$ in the following.

A characteristic feature of (9) is that it allows the frequency spectrum to be calculated in principle in terms of algebraic equations if the ratio between $L_{\rm II}$ and $L_{\rm I}$ is an integer. This will be the subject of our study in Secs. IV and V. First, however, we shall in the next section consider some simple special cases. In general, the Casimir energy for the composite string is calculated as the zero-point energy $E_{\rm I+II}$ for parts I+II, summed over all modes, minus the zero-point energy for the *uniform* string:

$$E=E_{\rm I+II}-E_{\rm uniform}\ . \tag{10}$$

III. SIMPLE SPECIAL CASES

A. Uniform string

This case corresponds to $x=1$. From (9) we have

$$\cos\omega L=1\ . \tag{11}$$

This equation admits the formal solution $\omega_n L=2\pi n$, with n being a positive or negative integer. It is however physically meaningful to associate ω_n with a physical quantity only. [Left-moving waves are associated with negative *wave numbers* (not frequencies). Waves of this type are incorporated in the fundamental form (2).] For our purpose—Casimir energy—the zero-frequency mode $\omega_n=0$ is of no interest. We thus have for the uniform string

$$\omega_n=2\pi n/L,\quad n=1,2,3,\ldots\ . \tag{12}$$

The set of equations (5) and (7) is satisfied for arbitrary values of $\xi_{\rm I}=\xi_{\rm II}$ and $\eta_{\rm I}=\eta_{\rm II}$, meaning that the amplitudes of the right- and left-moving waves are completely unspecified.

The basic expression for the Casimir energy is

$$E_{\rm uniform}=2\times\tfrac{1}{2}\sum_{n=1}^{\infty}\omega_n\ , \tag{13}$$

where the prefactor 2 takes into account that the modes (12) are degenerate. One may associate this degeneracy with the right-left symmetry on the uniform string. Expression (13), as it stands, is infinite. As in Ref. 3, we regularize it by means of a high-frequency cutoff function f:

$$E_{\rm uniform}\to\sum_{n=1}^{\infty}\omega_n f(\omega_n)\ , \tag{14}$$

where f satisfies the conditions

$$f(0)=1,\quad f(\infty)=f'(\infty)=f''(\infty)=\cdots=0\ . \tag{15}$$

As regards the dependence of f on ω_n, the following physical argument[3] may be given. On physical grounds one expects that it is possible to estimate the value of n at the high-frequency cutoff by a *local* experiment, i.e., at an arbitrary point on the string without information about the string's remote parts. Choosing f to be, for instance, a function of n alone would lead to a conflict with this requirement, since a small observer placed at some point on the string only observes the frequencies ω_n on the string but does not otherwise see how the string is and so is unable to estimate for how high n the modes are excited.

The fundamental form (14) is seen to be compatible with this physical argument about local observability.

In the following we choose the simplest imaginable form for f: namely, $f=e^{-\alpha\omega_n}$, with α a small positive parameter. We obtain

$$E_{\text{uniform}}=\frac{L}{2\pi\alpha^2}-\frac{\pi}{6L}+O(\alpha^2)\ . \tag{16}$$

If we simply omit the cutoff divergent term in the limit $\alpha\to 0$, we can interpret the finite term $-\pi/6L$ as the Casimir energy of the uniform string. It is physically suggestive to rewrite the finite term as

$$-\frac{\pi}{6L}=-\frac{L}{\pi}\int_0^\infty\frac{d\omega\,\omega}{e^{L\omega}-1}\ , \tag{17}$$

and interpret it as an integral over a thermal spectrum with a temperature equal to $1/L$. However, one has to be somewhat cautious about this point. Ford[4] has recently examined the Casimir energy of a uniform string, using spectral weight functions. According to his analysis, there are only limited circumstances in which the quantum fluctuations are similar to thermal fluctuations.

Expression (16) for the uniform string plays in our theory an important role: it is the energy to be subtracted from the zero-point energy of the composite string; cf. (10). As we shall see, the cutoff terms in the Casimir energy for the composite string thereby drop out automatically.

B. The case $x\to 0$

This case implies that $T_{\text{I}}\to 0$, if T_{II} is assumed finite. Recall that the energy density is adjusted correspondingly, so that condition (1) is always satisfied. We impose no condition on the ratio $L_{\text{I}}/L_{\text{II}}$. Equation (9) reduces to

$$\sin\omega L_{\text{I}}\sin\omega L_{\text{II}}=0\ , \tag{18}$$

so that the modes can be associated with part I or part II of the string separately. We get the two sequences

$$\omega_n=\pi n/L_{\text{I}}\ , \tag{19a}$$

$$\omega_n=\pi n/L_{\text{II}}\ , \tag{19b}$$

with $n=1,2,3,\ldots$, as before. There exists a notable physical difference between the modes (19a) and (19b) under the present conditions. Using (19a) in Eqs. (5) and (7) we find, as long as ω_n is *different from* any of the eigenfrequencies $2\pi n/L$ of the uniform string, that $\xi_{\text{II}}=\eta_{\text{II}}=0$. That is, modes of this kind are unable to penetrate into region II. Only those modes that are associated with the string as a whole can propagate. The modes (19b) behave differently, in that they do not lead to an analogous restriction on the coefficients ξ and η; the modes can propagate in both regions I and II. The total zero-point energy of regions I+II is

$$\begin{aligned}E_{\text{I+II}}&=\frac{\pi}{2L_{\text{I}}}\sum_{n=1}^\infty n\exp\left[-\frac{\pi\alpha n}{L_{\text{I}}}\right]+\frac{\pi}{2L_{\text{II}}}\sum_{n=1}^\infty n\exp\left[-\frac{\pi\alpha n}{L_{\text{II}}}\right]\\&=\frac{L}{2\pi\alpha^2}-\frac{\pi}{24}\left[\frac{1}{L_{\text{I}}}+\frac{1}{L_{\text{II}}}\right]+O(\alpha^2)\ .\end{aligned} \tag{20}$$

Equation (10) now yields the Casimir energy. We choose to express it in terms of L and the quantity s, where the latter is defined as the ratio between the lengths of the two pieces:

$$s=L_{\text{II}}/L_{\text{I}}\ . \tag{21}$$

For definiteness we take L_{I} to be the smaller of the two lengths, so that $s\geq 1$. We obtain, in the limit of $\alpha\to 0$,

$$E=-\frac{\pi}{24L}\left[s+\frac{1}{s}-2\right]\ . \tag{22}$$

The cutoff divergent terms are seen to drop out. It should be stressed that when subtracting E_{uniform} it is irrelevant whether the uniform string is made up of type I, or type II, material. In any case the cutoff function $f=\exp(-2\pi\alpha n/L)$ depends on the total length L as the single physical string parameter. Consequently, the expression (16) is independent of material type.

It should be mentioned that the two branches in (19a) and (19b) are not degenerate, so that there is no extra factor of 2 appearing in (20).

The Casimir energy (22) is never positive. Its maximum value is zero, corresponding to $s=1$. For increasing values of s (>1) the energy decreases monotonically. The $s=1$ result can be elucidated if we return to Eqs. (19) which yield $\omega_n=2\pi n/L$ in this case. This is actually the same spectrum as for the *uniform string*. Therefore, when we calculate the zero-point energy for the composite string in the case $s=1$ (still maintaining $x\to 0$) and subtract off the uniform string energy, we must necessarily obtain zero as result. The case $s=1$ moreover serves to illustrate why the branches (19) are nondegenerate: the *sum* of the two terms in the first two lines of Eq. (20) is equal to the single expression (13) for the uniform string, in which the degeneracy factor of 2 has been imposed explicitly.

As regards the *sign* of the Casimir energy for general values of s, there seems to exist no simple principle explaining why the sign is negative. The situation is in this respect analogous to that of the Casimir energy evaluated for different geometries. The sign depends on which geometry is chosen, without there being any simple clue to the understanding of the resulting sign in each case.

It ought to be mentioned that instead of adopting an exponential cutoff function, as above, we might alternatively have used the analytic extension of Riemann's ζ function in the regularization of the energy. Such a procedure consists in writing simply

$$E=\frac{\pi}{2}\left[\frac{1}{L_{\text{I}}}+\frac{1}{L_{\text{II}}}-\frac{4}{L}\right]\sum_{n=1}^\infty n\ , \tag{23}$$

and thereafter replacing the divergent sum by the finite expression[5] $\zeta(-1)=-\frac{1}{12}$ for the analytic continuation of $\zeta(s)$:

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s} . \tag{24}$$

This latter method is however rather formal in nature.

As a final point, we refer to the conceptually possible interpretation of the finite term (17) as a thermal term. We may conceive of applying analogous interpretations to the finite terms $-\pi/24L_{\rm I}$ and $-\pi/24L_{\rm II}$ associated with parts I and II of the composite string. The "temperatures" $1/L_{\rm I}$ and $1/L_{\rm II}$ of the two parts are however in general different. In pictorial terms, we may associate part I with "our" Universe and part II with a "mirror" universe. Thermal equilibrium between the two universes is thus seen to be present only in the special case $L_{\rm I}=L_{\rm II}$.

C. The case $L_{\rm I}=L_{\rm II}$

In this special case the formalism becomes significantly simplified. (The value $x\rightarrow 0$ was treated above.) For general x, the dispersion equation (9) reduces to $\cos\omega L=1$. That is, the frequency spectrum becomes identical to that holding for the uniform string; cf. (12). It follows that for $s=1$ the Casimir energy is equal to zero, for any value of x.

IV. s BEING AN ODD INTEGER

In the following we shall investigate the dispersion equation in detail when s runs through the lowest integer values, $s=2,3,\ldots,7$ (the case $s=1$ was treated above). When s is an integer, it is in principle straightforward to calculate the spectrum by solving algebraic equations although in practice the calculations become increasingly heavy when s becomes large. For general s it is convenient to rewrite (9) in the form

$$\sin\omega L_{\rm I}\sin(s\omega L_{\rm I})+F(x)\sin^2\left[\frac{1+s}{2}\omega L_{\rm I}\right]=0 , \tag{25}$$

where we have defined F as

$$F(x)=4x/(1-x)^2 . \tag{26}$$

In the present section we consider henceforth only odd integers s. It is convenient to regard $F(x)$, instead of x, as the input parameter. As mentioned previously, x is restricted to lie in the region $0<x\leq 1$. It corresponds to F lying in the region $0<F<\infty$.

For $s=3,5,7,\ldots$ the general structure of the frequency spectrum is the following. First, the dispersion equation (25) has one degenerate branch, given by

$$\sin\omega L_{\rm I}=0,\quad \omega L_{\rm I}=\pi n . \tag{27}$$

Next, there are $\frac{1}{2}(s-1)$ nondegenerate double branches, obtained by solving an algebraic equation of degree $\frac{1}{2}(s-1)$ in $\sin^2\omega L_{\rm I}$. Each double branch corresponds to a definite solution for $\sin^2\omega L_{\rm I}$. The frequency spectrum corresponding to such a branch can always be expressed in the form

$$\omega L_{\rm I}=\begin{cases}\pi(\beta+n) ,\\ \pi(1-\beta+n) ,\end{cases} \tag{28}$$

where $n=0,1,2,\ldots$ and where β is a number in the interval $0<\beta\leq\frac{1}{2}$. The value of β is found by explicit calculation in each case. For the double branch there are in general four solutions for $\omega L_{\rm I}$ in the region between 0 and 2π, viz. $\pi\beta$, $\pi(1-\beta)$, $\pi(1+\beta)$, and $\pi(2-\beta)$.

We now consider the lowest odd values of s separately.

A. $s=3$

Equation (25) yields, in this case,

$$z^2[4(1+F)z^2-4F-3]=0 , \tag{29}$$

where we for simplicity have defined $z\equiv\sin\omega L_{\rm I}$. The root $z=0$ gives the degenerate branch, as stated in (27). There is only one nondegenerate double branch in this case. It is determined by

$$z^2=\frac{3+4F}{4(1+F)} , \tag{30}$$

from which β is calculated once F is given. For small values of F (i.e., small x) we see that

$$\beta(F\ll 1)=\frac{1}{3}+\frac{\sqrt{3}}{6\pi}F , \tag{31}$$

giving coincidence with the situation discussed in Sec. III B when $F\rightarrow 0$. At the other extreme, for large values of F, we obtain correspondingly

$$\beta(F\gg 1)=\frac{1}{2}-\frac{1}{2\pi\sqrt{F}} . \tag{32}$$

When $F\rightarrow\infty$ (i.e., $x\rightarrow 1$) we recover the uniform string, as discussed in Sec. III A. Note that the complete spectrum in this case, as calculated from (28) in conjunction with (27), is identical with the uniform string spectrum (12).

B. $s=5$

The dispersion equation yields

$$z^2[16(1+F)z^4-4(5+6F)z^2+5+9F]=0 . \tag{33}$$

In addition to the degenerate branch, there are $\frac{1}{2}(5-1)=2$ double branches, determined by

$$z^2=\frac{5+6F\pm\sqrt{5+4F}}{8(1+F)} . \tag{34}$$

There are two values of β: $\beta_i=(\beta_1,\beta_2)$. It is useful to write down the approximate expressions for β in the limiting cases. First,

$$\beta_1(F\ll 1)=\frac{2}{5}-\frac{F}{10\pi}(5-2\sqrt{5})^{1/2} ,$$
$$\beta_2(F\ll 1)=\frac{1}{5}+\frac{F}{10\pi}(5+2\sqrt{5})^{1/2} . \tag{35}$$

Second,

$$\beta(F\gg 1)=\frac{1}{3}\pm\frac{1}{6\pi}\sqrt{3/F} . \tag{36}$$

C. $s=7$

The dispersion equation is

$$z^2[64(1+F)z^6-16(7+8F)z^4+8(7+10F)z^2-7-16F]=0\ , \quad (37)$$

showing that in addition to the degenerate branch there are three double branches. The cubic equation in z^2 has in general three real roots (two of these may coincide). Defining the abbreviations

$$p=-\frac{7+8F}{4(1+F)},\quad a=-\frac{7+10F+4F^2}{48(1+F)^2}\ ,$$
$$b=\frac{7-12F-33F^2-16F^3}{1728(1+F)^3},\quad \phi=\arccos\frac{-b/2}{\sqrt{-a^3/27}}\ , \quad (38)$$

we may write the solutions for the double branches z_i^2, $i=1,2,3$, in the form

$$z_i^2=-\frac{p}{3}+2\sqrt{-a/3}\cos\left[\frac{\phi}{3}+\frac{2\pi k_i}{3}\right]\ , \quad (39)$$

where the parameter k_i takes the values $k_i=\{0,1,2\}$.

Approximate analytic expressions in limiting cases will now not be given. We note, however, that when $F\rightarrow\infty$ the double branches are $z_i^2=\{1,\frac{1}{2},\frac{1}{2}\}$, so that two of the branches are coincident. These solutions, together with the degenerate solution $z=0$, naturally are in agreement with the uniform string solution (12). Table I shows β_i for the double branches calculated for three different input values of F.

D. The Casimir energy

The basic formula for the Casimir energy is (10), as before. The zero-point energy for regions I+II can be written as

$$E_{\mathrm{I+II}}=E(\text{degenerate branch})+\sum E(\text{double branches})\ . \quad (40)$$

TABLE I. Values of β_i for the double branches calculated for some given values of F. $F=\{0.1,1,100\}$ correspond to $x=\{0.0238,0.1716,0.8190\}$, respectively, in the region $0<x<1$.

F	$s=3$	$s=5$	$s=7$
0.1	0.3418	0.2089	0.1517
		0.3978	0.2847
			0.4313
1			0.1893
	0.3850	0.2500	0.2785
		0.3850	0.4468
100		0.3239	0.2441
	0.4842	0.3422	0.2553
			0.4921

In the first term here we insert the frequency spectrum (27), multiply by a factor 2 for degeneracy, and apply again the cutoff function $\exp(-\alpha\omega_n)$. Thus

$$E(\text{degenerate branch})=2\left[\frac{1}{2}\right]\frac{\pi(s+1)}{L}\sum_{n=1}^{\infty}n\exp\left[-\frac{\pi\alpha(s+1)n}{L}\right]\ . \quad (41)$$

It is convenient to introduce the abbreviation

$$t=\pi\alpha(s+1)/L\ , \quad (42)$$

and to use the Bernoulli number expansion up to third order:

$$\frac{t}{e^t-1}=1-\tfrac{1}{2}t+\tfrac{1}{12}t^2+O(t^4), \quad (43)$$

whereby

$$\sum_{n=1}^{\infty}ne^{-tn}=\frac{1}{t^2}-\tfrac{1}{12}+O(t^2)\ . \quad (44)$$

We assume that the condition $t\ll 1$ is always satisfied. We obtain

$$E(\text{degenerate branch})=\frac{1}{\alpha t}-\frac{t}{12\alpha}+O(t^2) \quad (45)$$

(note that t/α is of order unity). Consider next one of the double branches. Its zero-point energy can be written, when we take into account (28), as

$$E(\text{double branch})=\frac{t}{2\alpha}\left[[\beta e^{-t\beta}+(1-\beta)e^{-t(1-\beta)}]\sum_{n=0}^{\infty}e^{-tn}+[e^{-t\beta}+e^{-t(1-\beta)}]\sum_{n=1}^{\infty}ne^{-tn}\right]\ . \quad (46)$$

Using (44), together with the analogous expansion

$$\sum_{n=0}^{\infty}e^{-tn}=\frac{1}{t}+\tfrac{1}{2}+\tfrac{1}{12}t+O(t^3)\ , \quad (47)$$

and similar expansions of the terms in square brackets in (46), we obtain

$$E(\text{double branch})=\frac{1}{\alpha t}+\frac{t}{6\alpha}-\frac{t}{4\alpha}[\beta^2+(1-\beta)^2]+O(t^2)\ . \quad (48)$$

We now write β_i instead of β, and sum (48) over all the $\frac{1}{2}(s-1)$ double branches. Combining also with the degenerate branch result (45) we obtain

$$E_{\mathrm{I+II}}=\frac{s+1}{2\alpha t}+\frac{t(s-2)}{12\alpha}-\frac{t}{4\alpha}\sum_{i=1}^{(s-1)/2}[\beta_i^2+(1-\beta_i)^2]+O(t^2)\ . \quad (49)$$

Subtracting off the expression (16) for the uniform string, and letting $t \to 0$, we finally have

$$E = \frac{\pi s(s-1)}{12L} - \frac{\pi(s+1)}{4L} \sum_{i=1}^{(s-1)/2} [\beta_i^2 + (1-\beta_i)^2] \; ; \qquad (50)$$

the cutoff divergent terms drop out.

The presence of the β terms in (50) reflects actually a delicate point. These terms owe their existence to the fact that the sums in (47) and (44) are cutoff divergent. Thus the $1/t$ term in (47) implies that we get contribution to the β terms from an expansion of the expression in the first set of square brackets in (46) to first order in t. Analogously, the $1/t^2$ term in (44) implies that there is a contribution from the second-order terms in the second set of square brackets in (46). In (50), both these contributions are included.

As a further illustration of this point, we may consider how the ζ-function regularization method works in the present case. Giving an extra subscript ζ to mark quantities calculated with this regularization, we obtain for the uniform string, and for the degenerate branch,

$$E_{\text{uniform}}|_\zeta = -\frac{\pi}{6L} \; , \qquad E(\text{degenerate branch})|_\zeta = -\frac{t}{12\alpha} \; , \qquad (51)$$

in agreement with the finite terms in (16) and (45). For one of the double branches we have in view of the spectrum (28) the analogous construction for the energy:

$$E(\text{double branch})|_\zeta = \frac{t}{2\alpha} \sum_{n=0}^{\infty} (\beta+n) + \frac{t}{2\alpha} \sum_{n=0}^{\infty} (1-\beta+n) = \frac{t}{2\alpha}\left[1 + \sum_{n=1}^{\infty} (1+2n)\right] \to \frac{t}{2\alpha}[1+\zeta(0)+2\zeta(-1)] \; . \qquad (52)$$

Since[5] $\zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$, we have

$$E(\text{double branch})|_\zeta = \frac{t}{6\alpha} \; , \qquad (53)$$

which is seen to be different from our previous (48). The Casimir energy becomes now

$$E|_\zeta = \frac{\pi s(s-1)}{12L} \; . \qquad (54)$$

This is identical to the *first term* in (50). Thus, if we employ the ordinary Riemann ζ-function regularization, we miss the β terms in the energy. Recall that the presence of the β terms was intimately related to the cutoff divergences in the expansions (44) and (47). Does this circumstance, which at first sight may appear surprising, imply that the ζ-function regularization method as such is inapplicable in the present problem? Probably this is not so. The point is that in the above we were concerned with the *Riemann* ζ function only. The proper way to proceed in the present context is to define a ζ function (*not* the Riemann ζ function) for the wave operator. If the eigenvalues of the operator are λ_n, then the associate ζ function is $\zeta(s) = \sum \lambda_n^{-s}$. Since the eigenvalues are as given in (27) and (28), the relevant ζ function will involve a sum containing terms of the form $(\beta+n)^{-s}$ and $(1-\beta+n)^{-s}$. Consequently, the resulting expression for the Casimir energy may be expected to be a function of β and may so agree with the result of the cutoff method. We only mention this point here, without going into a detailed study. The following physical argument can in addition be given to show that (54) cannot be the correct expression for the Casimir energy. For a given value of L the expression increases without bounds when s becomes large. Suppose now that "our Universe" (part I) is small and the "mirror universe" (part II) is large. Thus $s \gg 1$. Expression (54), if it were true, would in principle allow an observer in our Universe to determine the magnitude of the mirror universe by performing a measurement of the Casimir energy. This is physically unreasonable. Recall that in the conventional evaluations of the Casimir energy the physical system is embedded into a large fictitious "box" whose magnitude drops out in the final expression for the Casimir energy.

Our expression (50), by contrast, does not suffer from the above drawback. Table II shows calculated values of $(L/\pi)E$, for various values of s and F. For $F = \{0.1, 1, 100\}$, the β terms in the Casimir energy are calculated using the data in Table I. We have for the sake of comparison included also the special cases $F=0$ and $F\to\infty$. The energy in the case $F=0(x=0)$ is calculated from expression (22). When $F\to\infty(x\to 1)$ we re-

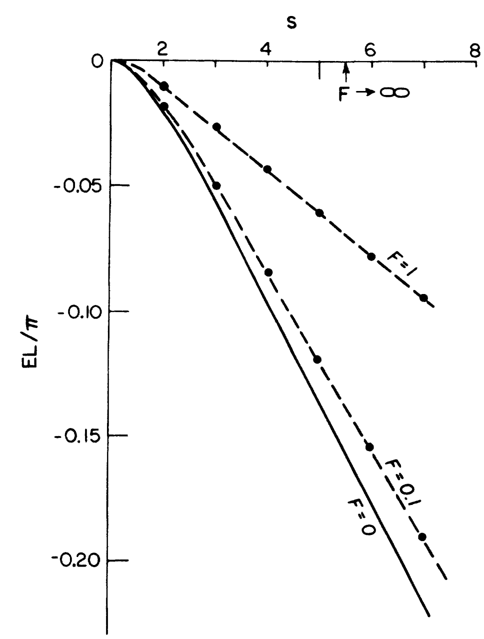

FIG. 2. Nondimensional Casimir energies calculated for various values of F and $s = L_{\text{II}}/L_{\text{I}}$.

TABLE II. Values of $(L/\pi)E$ calculated from Eq. (50) for some odd integer values of s, with F as input parameter.

F	$s=3$	$s=5$	$s=7$
0	−0.055 6	−0.133 3	−0.214 3
0.1	−0.050 1	−0.118 9	−0.189 5
1	−0.026 5	−0.060 5	−0.093 7
100	−0.000 50	−0.001 10	−0.001 68
∞	0	0	0

cover the uniform string, corresponding to zero Casimir energy. The energy, always nonpositive, is seen to vary smoothly with s and F. See also Fig. 2.

For dimensional reasons the Casimir energy of a little piece of string embedded in an essentially infinite string of a different tension has to be inversely proportional to the length L_{I} of the little string. This means that for large values of s we should get EL/π dominated by the contribution from the little string and thus

$$EL/\pi \propto s \ . \tag{55}$$

This agrees precisely with what we see from Fig. 2.

V. s BEING AN EVEN INTEGER

A. The frequency spectrum

When $s=2,4,6,\ldots$ it is in principle possible, such as in the previous case, to determine the frequency spectrum by solving algebraic equations. From the dispersion equation (25) we infer the following general properties. There exists one degenerate branch, given by

$$\cos\omega L_{\mathrm{I}}=1, \quad \omega L_{\mathrm{I}}=2\pi n \ . \tag{56}$$

Next, there are s nondegenerate simple branches, obtained by solving an algebraic equation of degree s in $q\equiv\cos\omega L_{\mathrm{I}}$. Each of these branches corresponds to

$$\omega L_{\mathrm{I}}=\begin{cases}\pi(\beta+2n) \ , \\ \pi(2-\beta+2n) \ ,\end{cases} \tag{57}$$

where $n=0,1,2,\ldots$ and where now β lies in the interval $0<\beta\leq 1$. (The reason why $\beta\leq 1$ instead of $\beta\leq\frac{1}{2}$ as in the previous section, is that the branches are now simple instead of double.) For each branch there are two solutions for ωL_{I} in the region between 0 and 2π, viz., $\pi\beta$ and $\pi(2-\beta)$.

Let us consider the lowest values of s separately. When $s=2$, we obtain, from (25),

$$(q-1)[4(1+F)q^2+4(1+F)q+F]=0 \ , \tag{58}$$

so that in addition to (56) there are two branches given by

$$q=-\frac{1}{2}\pm\frac{1}{2\sqrt{1+F}} \ . \tag{59}$$

When $s=4$, we obtain correspondingly

$$(q-1)[16(1+F)q^4+16(1+F)q^3-4(2+F)q^2 -4(2+F)q+F]=0 \ , \tag{60}$$

TABLE III. Values of β_i, corresponding to the spectrum (57), when s is an even integer.

F	$s=2$	$s=4$	$s=6$
0.1		0.2588	0.1756, 0.3319
	0.5074	0.4963	0.5025, 0.6624
	0.9312	0.7516	0.8340, 0.9608
		0.9517	
1		0.3014	0.2149, 0.3232
	0.5468	0.4750	0.5171, 0.6388
	0.8256	0.7608	0.8382, 0.9080
		0.8820	
100		0.3877	0.2783, 0.2925
	0.6487	0.4118	0.5628, 0.5804
	0.6853	0.7929	0.8535, 0.8613
		0.8078	

where the quartic equation is to be solved numerically.

Finally, when $s=6$ we obtain the dispersion equation

$$(q-1)[64(1+F)q^6+64(1+F)q^5-16(4+3F)q^4 -16(4+3F)q^3+4(3+2F)q^2+4(3+2F)q+F]=0 \ , \tag{61}$$

again necessitating numerical solution .

Table III shows the calculated values of β_i, when $s=2,4,6$. The input values for F are chosen to be the same as in Table I.

B. The Casimir energy

In analogy to (40) we have now

$$E_{\mathrm{I+II}}=E(\text{degenerate branch}) + \sum E(\text{simple branches}) \ . \tag{62}$$

In terms of the quantity t defined in (42) we obtain for the first term

$$E(\text{degenerate branch})=\frac{2t}{\alpha}\sum_{n=1}^{\infty} ne^{-2tn} =\frac{1}{2\alpha t}-\frac{t}{6\alpha}+O(t^2) \ . \tag{63}$$

For each of the remaining s simple branches we have, when we take into account the form (57),

TABLE IV. Values of $(L/\pi)E$ calculated for even integers s.

F	$s=2$	$s=4$	$s=6$
0	−0.020 8	−0.093 8	−0.173 6
0.1	−0.018 9	−0.084 4	−0.154 1
1	−0.010 2	−0.043 5	−0.077 3
100	−0.000 194	−0.000 803	−0.001 413
∞	0	0	0

E(simple branch)

$$=\frac{t}{2\alpha}\left\{[\beta e^{-t\beta}+(2-\beta)e^{-t(2-\beta)}]\sum_{n=0}^{\infty}e^{-2tn}\right.$$
$$\left.+2(e^{-t\beta}+e^{-t(2-\beta)})\sum_{n=1}^{\infty}ne^{-2tn}\right\} . \qquad (64)$$

We expand in powers of t, in the manner shown in Sec. IV D, sum over all the s branches, and combine with (63). The result is

$$E_{\mathrm{I+II}}=\frac{s+1}{2\alpha t}+\frac{t(2s-1)}{6\alpha}$$
$$-\frac{t}{8\alpha}\sum_{i=1}^{s}[\beta_i^2+(2-\beta_i)^2]+O(t^2) . \qquad (65)$$

Finally subtracting off the expression (16) for the uniform string, we obtain for the Casimir energy, in the limit as $\alpha\to 0$,

$$E=\frac{\pi s(2s+1)}{6L}-\frac{\pi(s+1)}{8L}\sum_{i=1}^{s}[\beta_i^2+(2-\beta_i)^2] . \qquad (66)$$

Again, the cutoff divergent terms drop out.

Table IV shows calculated values of $(L/\pi)E$ for the lowest even integers s. The F entries are the same as in Table II. The points in Fig. 2 give a graphical representation of all calculated energies in the region $s\in[1,7]$. The energies are seen to vary smoothly with F and s, as indicated by the dashed lines. For $F=100$ the calculated points are lying practically on the horizontal axis (corresponding to $F\to\infty$) and are not shown. The solid curve in the figure corresponds to $F=0$ and is calculated from (22).

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