PHYSICAL REVIEW D VOLUME 41, NUMBER 4 15 FEBRUARY 1990

Ten-dimensional Lovelock-type space-times

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(Received 1 August 1989)

The correct field equations for several ten-dimensional Lovelock-type Lagrangians including terms up to fourth order in the curvature are given. They were deduced from a program which exploits the facilities of the differential package EXCALC. A first discussion of the obtained equations is presented. The justification of a conjecture made recently by Deruelle is given. Possible extensions of some previously obtained solutions of the field equations restricted to quadratic and cubic contributions are discussed. Finally, some errors in the existing literature are pointed out.

I. INTRODUCTION

Recent work in theoretical cosmology is characterized by the extension of four-dimensional general-relativistic models in two directions: Kaluza-Klein-type models with more than four dimensions, on the one hand, and models derived from Lagrangians containing nonlinear terms in the Riemann curvature tensor and its contractions, on the other hand.

These two features can even be present at the same time, in recent multidimensional unified theories of the four basic interactions. The most promising candidate of this type of theory seems to be superstring theory[1] which, in its low-energy limit, yields effective field theories of gravity, formulated in a ten-dimensional space-time. Moreover, the corresponding Lagrangian contains, besides the usual Einstein-Hilbert term linear in the scalar curvature, additional terms quadratic and of higher power in the different curvature tensors. The precise form of the corrections is however not yet known.

If we consider the additional dimensions as physical, we have to explain why low-energy space-time is now four dimensional. A possible answer, first given by Einstein and Bergmann,[2] known as "spontaneous compactification," leaves us with the problem why four among the given number of dimensions should be distinguished. There are, however, indications[3] that a nonlinear generalization of the Kaluza-Klein action, with "dimensionally continued Euler forms,"[4] could provide an answer to this question. This generalized action is often called "Lovelock action."[5] In dimensions $d>4$, it includes a series of terms corresponding to the Euler invariants in all dimensions less than d, and always yields second-order field equations. In particular, the linear and quadratic terms are, respectively, the Einstein-Hilbert and Gauss-Bonnet actions; they appear in the low-energy approximation of the heterotic superstrings.[1] In ten dimensions, cubic and quartic terms have recently been introduced in the discussion of cosmological models in the framework of superstring theories.[6,7]

The uncertainties concerning the exact form of these nonlinear terms has also led to the consideration of cosmological solutions where the coefficients of the different quadratic contributions are left arbitrary.[8] A similar problem was studied previously with the aim of avoiding the initial singularity of four-dimensional Friedmann-Robertson-Walker (FRW) and Bianchi models.[9]

Writing explicitly the field equations for general space-times in high dimensions, taking into account all the relevant terms of the Lovelock action, is a very complex task. In the case of, say, ten-dimensional models, it can take weeks to write the equations by hand and the absolute correctness of the results is not at all guaranteed. As an example of the formidable complexity of this type of calculation, let us note that the quartic part of the Lovelock action involves 25 terms, each of which constituted by the contracted product of four curvature tensors,[10] and, moreover, the resulting field equations, obtained after variation with respect to the metric tensor, are not explicitly known.

Fortunately, a very compact formulation of the Lovelock action and of the corresponding field equations in terms of differential forms has recently been given.[11,12] This formulation simplifies the explicit derivation of the corresponding field equations. Its main interest is that it is particularly well suited for the implementation on a computer, especially with the algebraic package EXCALC (Ref. 13), which masters all operations of exterior calculus and thus enables one to perform these difficult calculations more efficiently.

We have written a program[14] in EXCALC, which can compute, for a given metric, the field equations for high-dimensional space-times, taking into account all nonlinear terms, up to the quartic one, in the Lovelock action. Although the rapidity and efficiency of this program have still to be improved, we have been able to derive the explicit form of the field equations for three ten-dimensional space-times of cosmological interest: $R\times V_3\times V_6$ space-times, where V_3 and V_6 are time-dependent three- and six-dimensional maximally symmetric spaces, and spatially homogeneous and isotropic ten-dimensional models of FRW type, both filled with a perfect fluid; ten-dimensional vacuum Bianchi type-I models with a Kasner-type metric.

In the last case, we show explicitly that the form of the leading term of the field equations near the singularity for dimensions greater than 6 is equivalent to the one guessed by Deruelle[15] in an independent study of this class of models.

Finally, the ten-dimensional generalization of the Schwarzschild solution is explicitly considered; previous results obtained by Wurmser[6] are shown to be incorrect.

For all these models, possible exact solutions are also discussed.

II. EXTERIOR ANALYSIS FORMULATION OF THE FIELD EQUATIONS FOR A GENERAL LOVELOCK LAGRANGIAN

The general Lovelock Lagrangian density[5] can be expressed in the language of exterior analysis as[4,11]

$$\mathcal{L}=\sum_{m=0}^{[d/2]-1}\lambda_m\mathcal{L}_m \;, \tag{1}$$

where λ_m are constants, d is the space-time dimension, and $\mathcal{L}_m$ is given by

$$\mathcal{L}_m=\Omega_{a_1b_1}\wedge\cdots\wedge\Omega_{a_mb_m}\wedge{}^*(\theta^{a_1b_1\cdots a_mb_m})\;. \tag{2}$$

Ω^a_b are here the curvature two-forms given by

$$\Omega^a{}_b=d\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b \tag{3}$$

$$=\tfrac{1}{2}R^a{}_{bcd}\theta^c\wedge\theta^d\;, \tag{4}$$

where θ^a $(a=1,\dots,d)$ is an orthonormal coframe for the d-dimensional space-time with a Lorentzian metric η, i.e.,

$$ds^2=\eta_{ab}\theta^a\otimes\theta^b \tag{5}$$

and $\eta_{ab}=\mathrm{diag}(-1,+1,\dots,+1)$. Note that the indices $a,b,\dots$ take their values from 0 to $D-1$; they are raised and lowered with the Lorentzian metric η.

ω_{ab} are the connection one-forms, the asterisk denotes the Hodge operator and $\theta^{a_1b_1\cdots a_mb_m}$ is the wedge product:

$$\theta^{a_1b_1\cdots a_mb_m}=\theta^{a_1}\wedge\theta^{b_1}\wedge\cdots\wedge\theta^{a_m}\wedge\theta^{b_m}\;. \tag{6}$$

R^a_{bcd} are the components of the Riemann curvature tensor in the orthonormal frame. An alternative formulation of (2) without the explicit use of the Hodge operator, but totally equivalent, has been given by Müller-Hoissen.[4,11]

In what follows, we will restrict ourselves to a pure metric theory, i.e., without torsion.

$\mathcal{L}_0$ (the volume d-form) gives rise to the cosmological constant term in the field equations, $\mathcal{L}_1$ is the Einstein-Hilbert Lagrangian density and $\mathcal{L}_2$, the quadratic Gauss-Bonnet contribution.

For the ten-dimensional space-times $(d=10)$ which will be considered below, only terms $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ need to be considered in (1). The manifest advantage and great conciseness of the exterior analysis formulation appear in the process of derivation of the field equations, by variation of each of the $\mathcal{L}_m$ with respect to θ^c, while keeping ω_{ab} constant. The result for each $\mathcal{L}_m$ is given, in very compact form, by

$$\delta\mathcal{L}_m=\delta\theta_c\wedge\Omega_{a_1b_1}\wedge\cdots\wedge\Omega_{a_mb_m}\wedge{}^*(\theta^{a_1b_1\cdots a_mb_mc})\;. \tag{7}$$

Introducing the $(d-1)$-form

$$\epsilon_{a_1}=\frac{1}{(d-1)!}\epsilon_{a_1a_2\cdots a_d}\theta^{a_2}\wedge\cdots\wedge\theta^{a_d}\;, \tag{8}$$

where $\epsilon_{a_1a_2\cdots a_d}$ is totally antisymmetric, with $\epsilon_{0\cdots d-1}=\sqrt{-\det(g_{ab})}=1$, Eq. (7) can be written as

$$\frac{\delta\mathcal{L}_m}{\delta\theta^a}=P^{(m)}{}_{ab}\epsilon^b \tag{9}$$

and the corresponding Einstein-type field equations, in component form, are now

$$\sum_{m=0}^{[d/2]-1}\lambda_mP^{(m)}{}_{ab}=-16\pi T_{ab}\;. \tag{10}$$

T_{ab} denotes the stress-energy tensor and the physical constants G and c are set equal to 1.

Thanks to the fact that all operations of exterior calculus are implemented in EXCALC (Ref. 13), it is rather easy to compute from Eqs. (7) and (9) the left-hand side of the field equations.[10] On the other hand, the programming effort to deduce these results is reduced to a minimum since EXCALC's syntax is very close to the usual way of writing the corresponding formulas on a sheet of paper. An example of this programme for a seven-dimensional generalized FRW model is given in Ref. 14. Of course, the CPU time needed for such calculations is rapidly growing with the dimension of space-time, and for ten-dimensional models it is typically of the order of one hour on an IBM 3090/180E.

III. THE FIELD EQUATIONS FOR A TEN-DIMENSIONAL COSMOLOGICAL MODEL OF TYPE $R\times V_3\times V_6$

The ten-dimensional cosmological models we consider here have the metric[4]

$$ds^2=\eta_{ab}\theta^a\otimes\theta^b$$

$$=-(\theta^0)^2+R^2(t)\delta_{ij}\tilde\theta^i\otimes\tilde\theta^j+S^2(t)\delta_{\alpha\beta}\theta^\alpha\otimes\theta^\beta\;, \tag{11}$$

where

$$\theta^0 = dt,\quad \theta^i = R(t)\tilde{\theta}^i,\quad \text{and}\quad \theta^\alpha = S(t)\tilde{\theta}^\alpha \tag{12}$$

with a,b running from 0 to 9, i,j from 1 to 3, and α,β from 4 to 9. $\tilde{\theta}^i$ and $\tilde{\theta}^\alpha$ denote orthonormal coframes on maximally symmetric three- and six-dimensional Riemannian spaces, respectively, given by

$$\tilde{\theta}^i = \frac{dx^i}{1+\dfrac{k\sum_{j=1}^{3}(x^j)^2}{4}} \tag{13}$$

and

$$\tilde{\theta}^\alpha = \frac{dx^\alpha}{1+\dfrac{k_1\sum_{\beta=4}^{9}(x^\beta)^2}{4}} \tag{14}$$

with k and $k_1 = -1$, 0, or $+1$.

The space part of these models can be viewed as the direct product of two FRW models, respectively, with three and six dimensions, and with curvature constants k and k_1.

For each of these ten-dimensional models, we consider the following Lagrangian density [cf. (1)]:

$$\mathcal{L} = \lambda_0\mathcal{L}_0 + \lambda_1\mathcal{L}_1 + \lambda_2\mathcal{L}_2 + \lambda_3\mathcal{L}_3 + \lambda_4\mathcal{L}_4\ , \tag{15}$$

where λ_0, λ_1, λ_2, λ_3, and λ_4 are constants and each of the $\mathcal{L}_m$ is given by (2).

The writing of the left-hand side of the field equations (10), with the aid of EXCALC, begins with the evaluation of the connection one-forms $\omega^a{}_b$ and of the curvature two-forms $\Omega^a{}_b$, which enable us to find finally, from (7) to (9), the explicit form of the components $P^{(m)}{}_{ab}$.

As regards the stress-energy tensor, we choose it, in the neighborhood of the initial singularity, as that of a closed or heterotic superstring perfect gas, with the following equation of state, deduced by Matsuo[16]:

$$\begin{aligned} p &= \frac{1}{n-1}\rho = \frac{\rho}{3}\quad \text{(physical space)}\ ,\\ q &= 0\quad \text{(internal space)}\ , \end{aligned} \tag{16}$$

where p and q are the pressures of the gas for the physical ($n=4$) and internal spaces, respectively, and ρ is the total matter density. The physical space is thus, accordingly, radiation dominated, close to the singularity.

The field equations (10), including the quartic terms, take then the form

$$\begin{aligned}\sum_{m=0}^{4}\lambda_m P^{(m)}{}_{00} = {}&\lambda_0 + 6\lambda_1\left[P+5Q+6\frac{\dot R}{R}\frac{\dot S}{S}\right] + 72\lambda_2\left[5Q^2+5PQ+10\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+2P\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+20Q\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]\right]\\ &+720\lambda_3\left[Q^3+8\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^3+9PQ^2+18Q^2\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+12PQ\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+36Q\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2\right]\\ &+17\,280\lambda_4\left[PQ^3+6Q^2\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+6PQ^2\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+8Q\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^3\right] = 16\pi\rho\ ,\end{aligned} \tag{17}$$

$$\begin{aligned}\sum_{m=0}^{4}\lambda_m P^{(m)}{}_{ii} = {}&\lambda_0 + 2\lambda_1\left[P+15Q+12\frac{\dot R}{R}\frac{\dot S}{S}+2\frac{\ddot R}{R}+6\frac{\ddot S}{S}\right]\\ &+24\lambda_2\left[15Q^2+10\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+5PQ+40Q\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+20Q\frac{\ddot S}{S}+2P\frac{\ddot S}{S}+10Q\frac{\ddot R}{R}\right.\\ &\left.\qquad+20\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+4\frac{\ddot R}{R}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]\right]\\ &+720\lambda_3\left[Q^3+3Q^2P+12Q^2\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+6Q^2\frac{\ddot S}{S}+6Q^2\frac{\ddot R}{R}+12Q\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2\right.\\ &\left.\qquad+4PQ\frac{\ddot S}{S}+8\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+24Q\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+8Q\frac{\ddot R}{R}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]\right]\\ &+5760\lambda_4\left[2Q^3\frac{\ddot R}{R}+12Q^2\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+12Q^2\frac{\ddot R}{R}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+6PQ^2\frac{\ddot S}{S}\right.\\ &\left.\qquad+24Q\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+PQ^3+6Q^2\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2\right] = -16\pi p\quad (i=1,2,3)\ ,\end{aligned} \tag{18}$$

$$\sum_{m=0}^{4}\lambda_m P^{(m)}{}_{\alpha\alpha}=\lambda_0+2\lambda_1\left[3P+10Q+15\frac{\dot R}{R}\frac{\dot S}{S}+3\frac{\ddot R}{R}+5\frac{\ddot S}{S}\right]$$
$$+24\lambda_2\left[5Q^2+20\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+10PQ+5P\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+P\frac{\ddot R}{R}+30Q\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]\right.$$
$$\left.+10Q\frac{\ddot R}{R}+5P\frac{\ddot S}{S}+10Q\frac{\ddot S}{S}+10\frac{\ddot R}{R}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+20\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]\right]$$
$$+720\lambda_3\left[4\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^3+3PQ^2+3Q^2\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+3Q^2\frac{\ddot R}{R}+2Q^2\frac{\ddot S}{S}+6PQ\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]\right.$$
$$+12Q\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+2PQ\frac{\ddot R}{R}+6PQ\frac{\ddot S}{S}+4\frac{\ddot R}{R}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+12\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2$$
$$\left.+4P\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+12Q\frac{\ddot R}{R}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+12Q\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]\right]$$
$$+5760\lambda_4\left[6Q^2\frac{\ddot R}{R}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+3PQ^2\frac{\ddot S}{S}+12Q\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2+12PQ\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]\right.$$
$$+3PQ^2\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]+4Q\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^3+3PQ^2\frac{\ddot R}{R}+12Q\frac{\ddot R}{R}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^2$$
$$\left.+8\frac{\ddot S}{S}\left[\frac{\dot R}{R}\frac{\dot S}{S}\right]^3\right]=-16\pi q\quad(\alpha=4,5,6,7,8,9)\ , \tag{19}$$

where we have introduced the abbreviations[4]

$$P=\left[\frac{\dot R}{R}\right]^2+\frac{k}{R^2},\quad Q=\left[\frac{\dot S}{S}\right]^2+\frac{k_1}{S^2}\ . \tag{20}$$

The overdot denotes, as usual, the time derivative.

Note that, due to the isotropy of the FRW models, one space component of the corresponding field equations only is independent and, accordingly, sufficient to characterize each of the two constituting FRW spaces.

The equation which expresses energy-momentum conservation can be written, with the same notations, as

$$\dot\rho+3(\rho+p)\frac{\dot R}{R}+6(\rho+q)\frac{\dot S}{S}=0\ . \tag{21}$$

Equations (17) to (19) have been computed using EXCALC. The contributions coming from the terms up to third order agree with the ones calculated by Müller-Hoissen [Eqs. (3.4)–(3.11) of Ref. 4 with $M=3$, $N=6$]. The correctness of the fourth-order contribution has been checked by a hand calculation supported by a computer evaluation, made outside EXCALC, of the coefficients of the various wedge products present in (7).

As pointed out by Müller-Hoissen,[4] the inclusion of quartic terms could lead to essential modifications of results obtained for dimensions less than nine, based on a Lagrangian density containing at most terms of third order in the curvature. The importance of quartic curvature terms (not necessarily of Lovelock form) in the low-energy effective action in the heterotic superstring theory has also recently been stressed.[7,17] Moreover, if one is interested in the behavior of the model very close to the singularity, the leading terms to consider in the field equations are those with the highest power in $(1/t)$, i.e., those obtained from the highest-order terms in the Lagrangian density (the quartic terms for ten-dimensional models).

Without any simplifying assumptions on $S(t)$, the system of differential equations (17)–(19) is extremely difficult to solve. A simplifiying hypothesis, very frequently adopted in this type of problem, is to assume that the scale factor of the six-dimensional internal space is constant $=S_0$ (this space could even be a six-sphere, in agreement with the "spontaneous compactification" scheme, in which case S_0 is the radius of this sphere and $k_1=1$).

Solutions of this class have been discussed by Müller-Hoissen,[4] up to the third order in the curvature. The inclusion of quartic terms explicitly given in (17)–(19) does not fundamentally modify the nature of these solutions. Equations (17)–(19) can now be simplified as

$$6c_1P+c_0=16\pi\rho\ , \tag{22}$$

$$c_0+2c_1P+4c_1\frac{\ddot R}{R}=-16\pi p\ , \tag{23}$$

$$c_2+24c_3P\frac{\ddot R}{R}+6c_4\left[P+\frac{\ddot R}{R}\right]=-16\pi q=0\ , \tag{24}$$

where the c_i's are, respectively, given by

$$c_0=\lambda_0+30\lambda_1\frac{k_1}{S_0^2}+360\lambda_2\frac{k_1^2}{S_0^4}+720\lambda_3\frac{k_1^3}{S_0^6}\ ,$$

$$c_1=\lambda_1+60\lambda_2\frac{k_1}{S_0^2}+1080\lambda_3\frac{k_1^2}{S_0^4}+2880\lambda_4\frac{k_1^3}{S_0^6}\ ,$$

$$c_2=\lambda_0+20\lambda_1\frac{k_1}{S_0^2}+120\lambda_2\frac{k_1^2}{S_0^4}\ , \tag{25}$$

$$c_3=\lambda_2+60\lambda_3\frac{k_1}{S_0^2}+720\lambda_4\frac{k_1^2}{S_0^4}\ ,$$

$$c_4=\lambda_1+40\lambda_2\frac{k_1}{S_0^2}+360\lambda_3\frac{k_1^2}{S_0^4}\ .$$

It is possible to replace one of the field equations (22)–(24), for example, (23) by the equation of conservation of energy-momentum (21) which reads now

$$\dot{\rho}+3(\rho+p)\frac{\dot{R}}{R}=0\ . \tag{26}$$

Adopting Matsuo's equation of state of matter (16) very near the singularity, the system of independent field equations to be solved can be written as

$$\dot{R}^2=-\frac{c_0}{6c_1}R^2+\frac{8\pi D}{3c_1R^2}-k\ , \tag{27}$$

$$\rho=\frac{D}{R^4}\ , \tag{28}$$

$$c_3=0\ , \tag{29}$$

$$c_1c_2=2c_0c_4\ , \tag{30}$$

where D is a positive constant.

Equation (27) has the form of the Friedmann equation of relativistic cosmology, $[-c_0/(2c_1)]$ playing the rôle of the usual cosmological constant and $1/c_1$, that of a generalized gravitational constant in ten dimensions ($c_1>0$).

For a given theoretical framework where the values of the λ_i's would be known, it is possible to give a detailed discussion of solutions of Eq. (27) for the scale factor $R(t)$, quite parallel to the classical discussion of radiation-dominated FRW models in the presence of a cosmological constant. As regards the geometry of the internal space, it can be deduced from the relations (29) and (30) for k_1/S_0^2, which imply, on the one hand, that k_1 should be different from zero (at least if $\lambda_0\neq0$) and impose, on the other hand, a constraint among the λ_i's.

In the absence of a bare cosmological constant Λ in the Lagrangian density ($\Lambda=-\lambda_0/2$), the existence of the internal space (as far as $k_1\neq0$) induces a value of c_0, the effective cosmological constant, different from zero, even in the linear case: this gives rise, via Eq. (27), to the very rich class of FRW-type solutions with a cosmological constant.

If we insist on the condition that the theory should admit a vacuum solution of the form $M^4\times S^6$, where M^4 denotes the Minkowski space-time, and S^6, the six-sphere, we are led to the relations[4] (not modified by the presence of quartic terms)

$$c_0=0,\quad c_2=0\ , \tag{31}$$

which satisfy (30).

Conditions (29) and (31) leave then, among the λ_i's, two independent constants only. In this case, the bare cosmological constant term, λ_0, is exactly counterbalanced by the contributions at diverse orders of the internal geometry.

Inflationary-type solutions of Eqs. (17)–(19) for the physical space, e.g., de Sitter metric, can also be obtained, as far as specific conditions on the c_i's derived from (22) and (24) are satisfied.

Of course, in order to ensure the self-consistency of the ansatz used here, i.e., zero pressure and a constant radius for the internal space, a stability analysis of the corresponding field equations for this particular ansatz should be given.

We are presently studying the general issue (with a variable radius of the internal space): the relevant field equations, given by the system (17)–(19), will then deviate markedly from the usual Friedmann equations and will be, in general, impossible to solve analytically.

A possible outcome to this difficulty would be to resort to a numerical treatment or to the qualitative techniques of study of dynamical systems, adopted, for instance, by Kripfganz and Perlt,[18] in their investigation of ten-dimensional models containing as internal space a Calabi-Yau space, the effective action being chosen as that of the heterotic superstring theory, including the Gauss-Bonnet contribution.[19]

We are presently considering these qualitative methods in the case k and $k_1\neq0$, where they could be successful: this work, which is outside the scope of the present paper, could shed some light on the stability of the possible solutions, including the particular case $S=S_0$, considered here.[20]

IV. THE FIELD EQUATIONS FOR A TEN-DIMENSIONAL LOVELOCK-TYPE SPATIALLY HOMOGENEOUS AND ISOTROPIC COSMOLOGICAL MODEL

For a spatially homogeneous and isotropic ten-dimensional space-time, the metric can be put into the form

$$ds^2=\eta_{ab}\theta^a\otimes\theta^b=-(\theta^0)^2+R^2(t)\delta_{ij}\tilde{\theta}^i\otimes\tilde{\theta}^j\ , \tag{32}$$

where

$$\theta^0=dt\quad\text{and}\quad\theta^i=R(t)\tilde{\theta}^i \tag{33}$$

with a,b running from 0 to 9, and i,j from 1 to 9.

The $\tilde{\theta}^i$'s characterize the orthonormal coframe of a maximally nine-dimensional Riemannian space, i.e.,

$$\tilde{\theta}^i=\frac{dx^i}{1+\frac{k\sum_{j=1}^{9}(x^j)^2}{4}} \tag{34}$$

with $k=-1$, 0, or $+1$.

The corresponding field equations for a Lagrangian density of the form (15), calculated with EXCALC, can then be written as

$$\lambda_0+72\lambda_1\frac{F}{R^2}+3024\lambda_2\left[\frac{F}{R^2}\right]^2+60\,480\lambda_3\left[\frac{F}{R^2}\right]^3+362\,810\lambda_4\left[\frac{F}{R^2}\right]^4=16\pi\rho\ , \quad (35)$$

$$\lambda_0+8\lambda_1\frac{2\ddot{R}R+7\dot{R}^2+7k}{R^2}+336\lambda_2\frac{4\ddot{R}\dot{R}^2R+4k\ddot{R}R+5\dot{R}^4+10k\dot{R}^2+5k^2}{R^4}$$
$$+20\,160\lambda_3\frac{2\ddot{R}\dot{R}^4R+4k\ddot{R}\dot{R}^2R+2k^2\ddot{R}R+\dot{R}^6+3k\dot{R}^4+3k^2\dot{R}^2+k^3}{R^6}$$
$$+40\,320\lambda_4\frac{8\ddot{R}\dot{R}^6R+24k\ddot{R}\dot{R}^4R+24k^2\ddot{R}\dot{R}^2R+8k^3\ddot{R}R+\dot{R}^8+4k\dot{R}^6}{R^8}$$
$$+40\,320\lambda_4\frac{6k^2\dot{R}^4+4k^3\dot{R}^2+k^4}{R^8}=-16\pi p \quad (36)$$

with

$$F(t)=\dot{R}^2+k\ . \quad (37)$$

The left-hand side of (36) can be put in the form of a total derivative

$$\frac{d}{dt}\left[\lambda_0\frac{R^9}{9}+8\lambda_1FR^7+336\lambda_2F^2R^5+6720\lambda_3F^3R^3+40\,320\lambda_4F^4R\right]=-R^8\dot{R}(16\pi p)\ . \quad (38)$$

Using the equation of conservation of energy-momentum in the form

$$\frac{d}{dt}(R^9\rho)=-9R^8\dot{R}p \quad (39)$$

it is easy to check the equivalence between (35) and (38).

The equation to be solved is then

$$\lambda_0+72\lambda_1x+3024\lambda_2x^2+60\,840\lambda_3x^3+362\,810\lambda_4x^4=16\pi\rho \quad (40)$$

with

$$x(t)=\frac{F(t)}{R^2(t)} \quad (41)$$

and ρ, evaluated in terms of R, for a given equation of state $p=p(\rho)$, from (39), e.g.,

$$\rho=\frac{A}{R^9}\ \text{ for } p=0\ ,$$
$$\rho=\frac{B}{R^{10}}\ \text{ for } p=\frac{\rho}{9}\ \text{ [cf. (16)]}\ , \quad (42)$$

where A and B are constants.

The nonlinear differential equation (40) for $R(t)$ cannot, in general, be integrated in closed form for nonvanishing ρ. However, it is possible to generalize the solution given by Wheeler[10] for $\rho=0$, and restricted to a Lagrangian density containing linear and quadratic terms only. For λ_0, λ_1, λ_2, λ_3, and λ_4 given and $\rho=0$, it is possible to solve numerically the quartic equation (40) for x. Let us consider its real solutions (if any) and denote them generically by $x_s(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, with x_s positive, negative, or null. $R(t)$ is then the solution of the differential equation [from (37) and (41)]

$$\dot{R}^2+k=x_sR^2\ . \quad (43)$$

To $x_s=0$, which is a solution of (40) in the absence of cosmological constant, λ_0, correspond the Minkowski space-time (for $k=0$) and a Milne-type model with $R \div t$, for $k=-1$.

The case $x_s>0$ leads to three types of models: (a) de Sitter-type model: $R\div\exp(\sqrt{x_s}t)$ $(k=0)$; (b) Bouncing model at $t=0$: $R=\sqrt{\kappa}\cosh(\sqrt{x_s}t)$ $(k=+1)$ $(R^2>\kappa$, where $\kappa=k/x_s)$; (c) Singular model at $t=0$: $R=\sqrt{|\kappa|}\sinh(\sqrt{x_s}t)$ $(k=-1)$.

If $x_s<0$, there is only a "big-bang" type model for $k=-1$, with R given by

$$R=\sqrt{\kappa}\sin(\sqrt{|x_s|}t)\quad (R^2<\kappa)\ .$$

All these solutions are known from traditional relativistic cosmology, but, there, their existence is crucially dependent on a nonvanishing cosmological constant. Here, on the contrary, even if $\lambda_0=0$, the presence of nonlinear terms in the Lagrangian density gives rise to an effective cosmological constant, given by one of the real roots of the quartic equation (40), with $\rho=0$ (if any).

V. THE CASE OF A VACUUM LOVELOCK-TYPE BIANCHI TYPE-I MODEL

We consider now Kasner-type solutions of a vacuum ten-dimensional Bianchi I model, whose corresponding metric is given by

$$ds^2=\eta_{ab}\theta^a\otimes\theta^b=-(\theta^0)^2+\eta_{ij}t^{2p_i}dx^i\otimes dx^j\ , \quad (44)$$

where

$$\theta^0=dt\ \text{ and }\ \theta^i=t^{p_i}dx^i \quad (45)$$

with i,j running from 1 to 9. The p_i's $\in\mathcal{R}$ are the Kasner exponents.[21]

For a metric of the form (44), the corresponding field equations reduce to algebraic equations for the p_i's. We have obtained with EXCALC the explicit form of these equations, up to the fourth order in the curvature tensor. We will give here only the result for the quartic term which should be the dominant one in the closest neighborhood of the singularity, where it is of leading order in $(1/t)$, i.e., in $(1/t^8)$:

$$\sum_{i_1<i_2<\cdots<i_8} p_{i_1}p_{i_2}\cdots p_{i_8}=0 \tag{46}$$

[(0,0) component of (10)], the values of $i_1, i_2, \ldots, i_8$ being chosen among $(1,2,\ldots,9)$.

The (i,i) component of the quartic part of (10) has a more complicated expression but its combination with the (0,0) component (46) leads to a very compact form

$$\left[\sum_{j=1}^{9} p_j-7\right]\left[\sum_{\substack{i_1<i_2<\cdots<i_7\\ i_1,i_2,\ldots,i_7\neq i}} p_{i_1}p_{i_2}\cdots p_{i_7}\right]=0 \tag{47}$$

the values of $i_1, i_2, \ldots, i_7$ being chosen again among $(1,2,\ldots,9)$.

Introducing Deruelle's[15] notations, i.e.,

$$a_1=\sum_{j=1}^{9} p_j, \ldots, a_m=\sum_{i_1<i_2<\cdots<i_m} p_{i_1}p_{i_2}\cdots p_{i_m} , \tag{48}$$

one easily shows that the field equations (46) and (47) are identical with Deruelle's[15] "guess" of the general form of the field equations for dimensions greater than 6, i.e., if $d=2m+2$, with $m=4$, in the ten-dimensional case

$$a_{2m}=0 , \tag{49}$$

$$[a_1-(2m-1)]\sum_{k=0}^{2m-1}(-1)^k p_i{}^k a_{2m-1-k}=0 . \tag{50}$$

In particular, one can check that

$$\sum_{\substack{i_1<i_2<\cdots<i_7\\ i_1,i_2,\ldots,i_7\neq i}} p_{i_1}p_{i_2}\cdots p_{i_7}=a_7-p_i a_6+p_i{}^2a_5-p_i{}^3a_4 +p_i{}^4a_3-p_i{}^5a_2+p_i{}^6a_1-p_i{}^7 . \tag{51}$$

We have thus justified with the help of EXCALC Deruelle's form [(49) and (50)] [equivalent to (46) and (47)] of the field equations for multidimensional Kasner-type metrics. Incidentally, we note that we have also very easily derived, as a matter of check, the field equations for any dimension smaller than 10, including the odd dimensions, for which these equations take a somewhat different form.

Equations (46) and (47) form the basis for a study of ten-dimensional Kasner-type metrics which are solutions to the dominant contributions of the Lovelock field equations near the singularity, i.e., the quartic one. This study constitutes a first step in the analysis of the influence of the nonlinear terms of a Lovelock-type Lagrangian density on the nature (chaotic or nonchaotic) of the general behavior of a multidimensional spatially homogeneous model, very close to the initial singularity.[22]

VI. THE VACUUM SCHWARZSCHILD SOLUTION IN A TEN-DIMENSIONAL LOVELOCK THEORY

The gravitational field in the vacuum outside a static spherically symmetric body can be described, in the framework of a ten-dimensional theory, by the metric

$$ds^2=\eta_{ab}\theta^a\otimes\theta^b=-(\theta^0)^2+(\theta^1)^2+r^2\eta_{kl}\tilde\theta^k\otimes\tilde\theta^l , \tag{52}$$

where

$$\theta^0=\sqrt{B(r)}dt , \quad \theta^1=\sqrt{A(r)}dr , \quad \theta^k=r\tilde\theta^k \tag{53}$$

with k,l running from 2 to 9 and $\tilde\theta^k$ denoting the orthonormal coframe of a maximally symmetric eight-dimensional Riemannian space, given by

$$\tilde\theta^k=\frac{dx^k}{1+\frac{\sum_{m=2}^{9}(x^m)^2}{4}} . \tag{54}$$

For the metric (52) and a Lovelock-type theory, we obtain the independent field equations

$$\lambda_0+8\lambda_1(2k_3+7k_4)+336\lambda_2k_4(4k_3+5k_4)+20\,160\lambda_3k_4{}^2(2k_3+k_4)+40\,320\lambda_4k_4{}^3(8k_3+k_4)=0 , \tag{55}$$

$$-\lambda_0+8\lambda_1(2k_2-7k_4)+336\lambda_2k_4(4k_2-5k_4)+20\,160\lambda_3k_4{}^2(2k_2-k_4)+40\,320\lambda_4k_4{}^3(8k_2-k_4)=0 \tag{56}$$

with

$$k_1(r)=\frac{1}{2\sqrt{AB}}\frac{d}{dr}\left[\frac{1}{\sqrt{AB}}\frac{dB}{dr}\right], \quad k_2(r)=\frac{1}{2rAB}\frac{dB}{dr} , \quad k_3(r)=\frac{1}{2rA^2}\frac{dA}{dr}, \quad k_4(r)=\frac{1-1/A}{r^2} . \tag{57}$$

Comparison of Eqs. (55) and (56) with similar equations derived, partially by hand, by Wurmser[6] shows that all the numerical coefficients of the k_i's in Wurmser's expressions for the quadratic and cubic contributions are incorrect.

Integration of the two independent field equations (55) and (56) for $A(r)$ and $B(r)$ leads to the result

$$A=\frac{1}{B}=\frac{1}{1-\frac{r^2}{D(r)}} \tag{58}$$

with $D(r)$ given by the real solutions (if any) of the algebraic equation

$$\lambda_0 D^4 + 72\lambda_1 D^3 + 3024\lambda_2 D^2 + 60\,480\lambda_3 D + 362\,880\lambda_4 = \frac{D^4}{r^9} . \quad (59)$$

The ten-dimensional generalization of the usual four-dimensional Scharzschild solution, corresponding to the linear case ($\lambda_1 \neq 0$, $\lambda_0 = \lambda_2 = \lambda_3 = \lambda_4 = 0$) is given by

$$A = \frac{1}{B} = \frac{1}{1 - \frac{C}{r^7}} , \quad (60)$$

where C is a constant.

For a series of λ_i's given, (59) enables us to evaluate $D(r)$ in terms of r and (58) can then be used as the basis of the study of the existence of horizons for these Lovelock-type ten-dimensional black holes. Such a study has recently been undertaken by Myers and Simon[23] and Whitt,[24] on the basis of a general solution of the multidimensional spherically symmetric case, equivalent to (58) and (59) and independently derived by Wheeler.[25]

VII. CONCLUSIONS

The complexity of the study of realistic multidimensional models in nonlinear theories of gravity, already at the level of the explicit writing of the field equations, makes the use of computer algebra programming unavoidable.

The program we have developed which exploits the differential geometry package EXCALC, enables one to obtain in the language of exterior analysis the correct expression of the field equations for any Lovelock-type Lagrangian density.

We have considered here some ten-dimensional Lovelock-type models of cosmological and astrophysical interest and have derived the corresponding field equations up to the fourth order in the curvature. For some of these models, the quartic term had not yet been written, due to the complexity of the calculations, especially if one does not adopt the language of exterior differential forms: the general expression of the variation of the fourth-order contribution to the Lovelock Lagrangian density has even not yet been obtained.

On the other hand, mistakes are frequent in the literature devoted to these models: for instance, for the ten-dimensional Schwarschild space-time, the published field equations[6] have been shown to be partially incorrect. This illustrates the point that it is difficult, in this type of problem, to trust results obtained by hand, even if the models are very symmetric.

The equations obtained in this paper constitute the basis for a study of the geometrical and physical solutions of the corresponding models. We have discussed some possible solutions of these field equations.[26]

Using present computer algebra facilities, and in view of the increase of their capabilities, we hope to be able to analyze more realistic and less symmetric multidimensional models. Already it becomes possible to attempt to take explicitly into account the supergravity content of these theories as well as to include several important ingredients of string theories (as, e.g., coset[3] and Calabi-Yau spaces[27] as models of the internal space).

ACKNOWLEDGMENTS

We would like to thank F. Müller-Hoissen and E. Schrüfer for fruitful discussions. This work has been partially supported by the Belgian Fonds National de la Recherche Scientifique (F.N.R.S.). One of us (D.P.) is especially grateful to the Institute of Astrophysics of Liège for the hospitality and the F.N.R.S. for financial aid.

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