Relativistic description of quark-antiquark bound states. Spin-independent treatment

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We present the results of a detailed study of light- and heavy-quark–antiquark bound states in the context of the reduced Bethe-Salpeter equation with static vector and scalar interactions. In the present paper, we consider the spin-averaged spectra. Spin effects are considered in a separate paper. We find that this approach, although apparently successful for the heavy-quark $b\bar{b}$ and $c\bar{c}$ states, fails for the $s\bar{s}$, $l\bar{l}$, and light-heavy states. The reasons for the failure are intrinsic to the method, as we discuss. Difficulties are already evident for the $c\bar{c}$ states.

I. INTRODUCTION

The discovery of the ψ and Υ resonances and their interpretation in terms of $c\bar{c}$ and $b\bar{b}$ quark-antiquark bound states led to a large number of theoretical analyses of these heavy-quark systems based on nonrelativistic potential models[1] with relativistic corrections treated as perturbations.[2] More recently, there has been considerable interest in describing these systems using wave equations which incorporate relativistic kinematics from the beginning and can therefore be applied, at least in principle, to systems containing light quarks.[3–9]

In this paper we present the results of a rather extensive analysis of the spin-averaged heavy- and light-quark $q\bar{q}$ spectra using the reduced Salpeter equation, a standard approximation to the full Bethe-Salpeter equation. Spin-dependent effects are treated in the same approximation in a separate paper.[10] We treat the equation in position space, where it is relatively easy to vary the interaction potentials between the quark and antiquark, solve the resulting nonlocal problem retaining all quark-mass-dependent effects exactly, and attempt to fit the spin-averaged data by adjusting the potentials. Although we obtain ostensibly reasonable fits to the $b\bar{b}$, $c\bar{c}$, and $s\bar{s}$ data using an interaction containing a short-range Lorentz-vector one-gluon-exchange term and a long-range Lorentz-scalar confining interaction, the results are misleading. We identify (and explain) systematic discrepancies between the data and the fit which, supplemented by calculations of light-quark ($l\bar{l}$) Regge trajectories and the spin splittings in the $c\bar{c}$ and $b\bar{b}$ spectra,[10] show that there is an *intrinsic* flaw in this approach. A new starting point is needed if one is to obtain a consistent theory of light- and heavy-quark–antiquark systems. We comment briefly on what we think is the main missing element in the present model: namely, the dynamic energy of the color fields between the quark and antiquark which is needed to describe the "stringlike" behavior of the $l\bar{l}$ states.[11]

We begin in Sec. II with a brief derivation of the reduced Salpeter equation in position space. The spin-independent terms are identified and the equation is then reduced to a radial wave equation and put in a form in which the mass eigenvalues can be calculated efficiently using a new matrix technique.[12] The results of our calculations and the detailed analysis which leads to our conclusion that a reduced Salpeter equation with scalar confinement *cannot* describe the $b\bar{b}$, $c\bar{c}$, $s\bar{s}$, and $l\bar{l}$ data adequately are given in Sec. III. We collect a number of details of the theoretical analysis in an appendix.

II. RELATIVISTIC DESCRIPTION OF $q\bar{q}$ BOUND STATES

A. The reduced Salpeter equation

The Bethe-Salpeter equation[13] describes the bound states of a two-fermion system in terms of the interaction kernel specific to the field theory in question. The full equation can be written in momentum space as

$$(\eta_1 \not{P} + \not{p} - m_1)\chi(p)(\eta_2 \not{P} - \not{p} + m_2) = i \int \frac{d^4p'}{(2\pi)^4} V(p,p';P)\chi(p') , \quad (1)$$

where $\eta_1+\eta_2=1$, χ is the momentum-space wave function for the quark-antiquark system, P is the four-momentum for the bound state, and V is an interaction kernel which acts on χ. The short-distance behavior of V can be calculated in QCD using perturbation theory. However, the long-distance behavior of V involves nonperturbative effects, and is not known in detail except in the case of static quarks where it can be determined in the lattice approximation to QCD (Refs. 14–16). We will therefore treat the form of V in part phenomenologically. We will also make several approximations which have become customary in the treatment of the Bethe-Salpeter equation. Although we cannot check the approximations in detail as in QED because of the uncertainties in the form of V, we think they are generally reasonable, and

are unlikely to affect our conclusions.

The first approximation we make is to assume an instantaneous interaction: $V(p,p',P)\rightarrow V(\mathbf{p},\mathbf{p}')$. This is equivalent to using a position-space description in which the interaction potential is calculated with the equal-time restriction $x_0=x_0'$ and neglects the retarded nature of the interaction. After some standard mathematical manipulations we arrive at the Salpeter equation[17]

$$\phi(\mathbf{p})=\frac{1}{(2\pi)^3}\int d^3p'\left[\frac{\Lambda^+(\mathbf{p})\gamma_0 V(\mathbf{p},\mathbf{p}')\phi(\mathbf{p}')\gamma^0\Lambda^-(-\mathbf{p})}{M-\omega_1-\omega_2}-\frac{\Lambda^-(\mathbf{p})\gamma_0 V(\mathbf{p},\mathbf{p}')\phi(\mathbf{p}')\gamma^0\Lambda^+(-\mathbf{p})}{M+\omega_1+\omega_2}\right]. \tag{2}$$

Here M is the mass of the bound state, $\omega_i=(\mathbf{p}^2+m_i^2)^{1/2}$, $\Lambda^{\pm}(\mathbf{p})$ are the projection operators

$$\Lambda^{\pm}(p)=\frac{\omega\pm(\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p}+\gamma^0 m)}{2\omega} \tag{3}$$

and

$$\phi(\mathbf{p})=\int dp_0\chi(\mathbf{p},p_0)\ . \tag{4}$$

Our second approximation is the dropping of the second term in Eq. (2). This approximation is usually justified for heavy-quark systems on the grounds that

$$M+\omega_1+\omega_2\approx 4m_q\gg M-\omega_1-\omega_2\approx V\ , \tag{5}$$

so that the second term can be ignored relative to the first. The approximation is also reasonable on the average for $\langle\omega_1+\omega_2\rangle\gg\frac{1}{2}\langle V\rangle$. It corresponds to dropping contributions to the wave function in which an outgoing quark is replaced by its antiquark in an incoming state, contributions which do not have a clear wave-function interpretation, and leads to a well-defined eigenvalue problem. With this approximation we obtain the standard reduced Salpeter equation used in a number of studies of relativistic bound states:[4,6–9]

$$(M-\omega_1-\omega_2)\phi(\mathbf{p})=\frac{1}{(2\pi)^3}\int d^3p'\Lambda^+(\mathbf{p})\gamma_0 V(\mathbf{p},\mathbf{p}')\phi(\mathbf{p}')\gamma^0\Lambda^-(-\mathbf{p})\ . \tag{6}$$

The formal product of $V\phi$ in Eq. (6) represents a sum of scalar potentials V_i and bilinear covariants:

$$V\phi\rightarrow\sum_i V_i(\mathbf{p},\mathbf{p}')\mathbf{O}_i\phi\mathbf{O}_i\ , \tag{7}$$

where $\mathbf{O}_i$ is a Dirac matrix. The proper choice of the Lorentz structure of the interaction is very important in fitting the spin dependence of the $q\bar{q}$ bound states. The short-distance interaction between a quark and an antiquark is expected to be vector, $\mathbf{O}=\gamma_\mu$, corresponding to the perturbative one-gluon-exchange interaction, a result which is supported by nonperturbative calculations of the heavy-quark interaction in lattice QCD (Ref. 16). The nature of the coupling at large $q\bar{q}$ separations can be determined for heavy quarks from lattice calculations by determining the rate of decrease of the spin-dependent part of the potential. Recent calculations of this spin dependence[16] are accurate enough to establish clearly that the large-r coupling is scalar as naively expected, $\mathbf{O}=1$. The change in the nature of the coupling as one moves from small to large values of r leads to some complication in the choice of coupling for the intermediate region of the potential. We will assume that it is a mixture of scalar and vector terms.

B. Reduction of the relativistic wave equation

We next develop the formalism necessary to put the reduced Salpeter equation in a form which can be solved numerically.[6,9] As noted above, the coupling at short distances is expected to be vector, while the long-distance coupling is expected to be scalar. We will assume that the interaction kernel involves only these two Lorentz structures. The reduced Salpeter equation is then

$$(M-\omega_1-\omega_2)\phi(\mathbf{p})=\Lambda^+(\mathbf{p})\gamma_0\int\frac{d^3p'}{(2\pi)^3}[V_V(|\mathbf{p}-\mathbf{p}'|)\gamma_\mu\phi(\mathbf{p}')\gamma^\mu+V_S(|\mathbf{p}-\mathbf{p}'|)\phi(\mathbf{p}')]\gamma^0\Lambda^-(-\mathbf{p})\ , \tag{8}$$

where V_V and V_S are scalar potentials, and the wave function ϕ is a 4×4 matrix.

We will represent ϕ in block matrix form as

$$\phi=\begin{bmatrix}\phi^{+-} & \phi^{++}\\ \phi^{--} & \phi^{-+}\end{bmatrix}, \tag{9}$$

where each component is a 2×2 matrix. Relations implied by the projection operators in Eq. (8) allow us to express the "small components" of ϕ in terms of the large component ϕ^{++},

$$\begin{aligned}\phi^{+-}&=\phi^{++}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega_2+m_2}\ ,\\ \phi^{-+}&=\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega_1+m_1}\phi^{++}\ ,\\ \phi^{--}&=\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega_1+m_1}\phi^{++}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega_2+m_2}\ ,\end{aligned} \tag{10}$$

and to reduce Eq. (8) to an equation for ϕ^{++} alone. Some details of this reduction are given in the Appendix. If we convert to a normal spin basis using the relation

$$\phi_{m_1 m_2}(\mathbf{p})=[\phi^{++}(\mathbf{p})i\sigma_2]_{m_1 m_2} \tag{11}$$

given in Eq. (A8), the resulting matrix equation for the wave function $\phi_{m_1 m_2}$ with spin projections m_1, m_2 has the form

$$(M-\omega_1-\omega_2)\phi(\mathbf{p}) = \int \frac{d^3p'}{(2\pi)^3} \sum_{i=S,V} F_i(\mathbf{p},\mathbf{p}',\sigma_1,\sigma_2) V_i(|\mathbf{p}-\mathbf{p}'|)\phi(\mathbf{p}') \, . \tag{12}$$

The functions F_V and F_S which appear in Eq. (12) are

$$F_V(\mathbf{p},\mathbf{p}',\sigma_1,\sigma_2)=\frac{1}{4\omega_1\omega_2}\left[(\omega_1+m_1)(\omega_2+m_2)+(\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_i)_1(\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_i)_2+\frac{\omega_1+m_1}{\omega_2'+m_2}(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_2 \right.$$
$$+\frac{\omega_2+m_2}{\omega_2'+m_2}(\boldsymbol{\sigma}\mathbf{p}\sigma_i)_1(\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}')_2+\frac{\omega_2+m_2}{\omega_1'+m_1}(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_1+\frac{\omega_1+m_1}{\omega_1'+m_1}(\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}')_1(\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_i)_2$$
$$\left.+\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_1(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_2}{(\omega_1'+m_1)(\omega_2'+m_2)}+\frac{(\omega_1+m_1)(\omega_2+m_2)}{(\omega_1'+m_1)(\omega_2'+m_2)}(\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}')_1(\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}')_2\right] \tag{13a}$$

and

$$F_S(\mathbf{p},\mathbf{p}',\sigma_1,\sigma_2)=\frac{1}{4\omega_1\omega_2}\left[(\omega_1+m_1)(\omega_2+m_2)-\frac{\omega_2+m_2}{\omega_1'+m_1}(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_1-\frac{\omega_1+m_1}{\omega_2'+m_2}(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_2+\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_1(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_2}{(\omega_1'+m_1)(\omega_2'+m_2)}\right], \tag{13b}$$

where matrices labeled 1 and 2 act on the first and second spin indices of $\phi_{m_1 m_2}$. The terms containing more than one Pauli matrix can be reduced. The results of the reductions are given in the Appendix.

In this paper we will neglect the spin-dependent parts of the interaction in Eq. (12), and deal with spin-averaged data for the $q\bar{q}$ bound states. The effects of spin will be considered in detail in a separate paper.[10] It is straightforward to separate out the spin-independent parts of the F's using the identity

$$F_i^{\mathrm{si}}(\mathbf{p},\mathbf{p}')=\tfrac{1}{4}\mathrm{Tr}_1\mathrm{Tr}_2 F_i(\mathbf{p},\mathbf{p}',\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) \, . \tag{14}$$

The results are

$$F_V^{\mathrm{si}}(\mathbf{p},\mathbf{p}')=\frac{1}{4\omega_1\omega_2}\left[(\omega_1+m_1)(\omega_2+m_2)+p^2+\left[\frac{\omega_1+m_1}{\omega_2'+m_2}+\frac{\omega_2+m_2}{\omega_1'+m_1}+\frac{\omega_1+m_1}{\omega_1'+m_1}+\frac{\omega_2+m_2}{\omega_2'+m_2}\right]\mathbf{p}\cdot\mathbf{p}'\right.$$
$$\left.+\frac{(\mathbf{p}\cdot\mathbf{p}')^2}{(\omega_1'+m_1)(\omega_2'+m_2)}+\frac{(\omega_1+m_1)(\omega_2+m_2)}{(\omega_1'+m_1)(\omega_2'+m_2)}\mathbf{p}'^2\right] \tag{15a}$$

and

$$F_S^{\mathrm{si}}(\mathbf{p},\mathbf{p}')=\frac{1}{4\omega_1\omega_2}\left[(\omega_1+m_1)(\omega_2+m_2)-\left[\frac{\omega_1+m_1}{\omega_2'+m_2}+\frac{\omega_2+m_2}{\omega_1'+m_1}\right]\mathbf{p}\cdot\mathbf{p}'+\frac{(\mathbf{p}\cdot\mathbf{p}')^2}{(\omega_1'+m_1)(\omega_2'+m_2)}\right] . \tag{15b}$$

Using the spin-independent F's in Eq. (12) we obtain an equation in which $\phi_{m_1 m_2}(\mathbf{p})$ can be factored into the product of a scalar function $\phi(\mathbf{p})$ and a standard singlet- or triplet-spin function. The singlet and triplet states are degenerate and we will deal henceforth only with $\phi(\mathbf{p})$ and the associated wave equation

$$(M-\omega_1-\omega_2)\phi(\mathbf{p})=\int \frac{d^3p'}{(2\pi)^3} \sum_{i=S,V} F_i^{\mathrm{si}}(\mathbf{p},\mathbf{p}') V_i(|\mathbf{p}-\mathbf{p}'|)\phi(\mathbf{p}') \, . \tag{16}$$

Equation (16) gives a well-defined eigenvalue problem for the masses M of the $q\bar{q}$ bound states. The momentum dependence of the interaction is treated exactly within our (standard) approximations. It is not necessary to make a further expansion of the operators F_i^{si} in powers of p/m as is frequently done in treatments of heavy-quark systems.[2] Use of the complete expression is in fact essential if one wishes to investigate $s\bar{s}$ or light-quark systems.

C. Reduction to a position-space radial equation

It will be convenient in fitting $q\bar{q}$ bound states to transform Eq. (16) to position space where the potentials V_i are local and easy to handle. A Fourier transform gives the equation

$$(M-E_1-E_2)\psi(\mathbf{r})=\sum_{i=V,S}\int d^3r'\int d^3r''\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}e^{i\mathbf{p}'\cdot(\mathbf{r}'-\mathbf{r}'')}\sum_{i=S,V}F_i^{\mathrm{si}}(\mathbf{p},\mathbf{p}')V_i(r')\psi(\mathbf{r}'')\ , \tag{17}$$

where E_i is the nonlocal operator $(-\nabla^2+m_i^2)^{1/2}$ obtained as the formal Fourier transform of $\omega_i=(\mathbf{p}^2+m_i^2)^{1/2}$, $V_i(r')$ is the Fourier transform of $V_i(|\mathbf{p}-\mathbf{p}'|)$, and ψ is the Fourier transform of $\phi(\mathbf{p})$.

Equation (17) is intractable as it stands because of the multiple integrations in the interaction term. However, when we write $F_i^{\mathrm{si}}(\mathbf{p},\mathbf{p}')$ as a sum of products of functions which involve $\mathbf{p}$ or $\mathbf{p}'$ separately,

$$F_i(\mathbf{p},\mathbf{p}')=\sum_j A_{ij}(\mathbf{p})B_{ij}(\mathbf{p}')\ , \tag{18}$$

we can rewrite Eq. (17) as

$$(M-E_1-E_2)\psi(\mathbf{r})=\sum_{i=S,V}\sum_j\int d^3r'\int d^3r''\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}A_{i,j}(-i\nabla_r)e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}V_i(r')B_{ij}(-i\nabla_{r'})e^{i\mathbf{p}'\cdot(\mathbf{r}'-\mathbf{r}'')}\psi(\mathbf{r}'')\ , \tag{19}$$

where we have replaced $\mathbf{p}$ and $\mathbf{p}'$ in A and B by gradient operators.

The integrations can now be performed, and we are left with a simple but nonlocal expression:

$$(M-E_1-E_2)\psi(\mathbf{r})=\sum_{i=S,V}\sum_j A_{ij}(-i\nabla_r)V_i(r)B_{ij}(-i\nabla_r)\psi(\mathbf{r})\ , \tag{20}$$

where the nonlocality introduced by the operator $E=(-\nabla^2+m^2)^{1/2}$ appears in the "potential" on the right-hand side of the equation as well as in the kinetic energy operators on the left. The operator E can be handled by a matrix method which is sketched in Sec. III, and described in detail elsewhere.[12]

There is no coupling of different orbital angular momenta in Eq. (20). We can therefore extract the angular dependence of ψ in a single spherical harmonic:

$$\psi(\mathbf{r})=R_l(r)Y_{lm}(\hat{\mathbf{r}})\ . \tag{21}$$

The consistency of this construction can be seen by pushing the spherical harmonic Y_{lm} through the differential operators in A and B using the identities

$$\begin{aligned}
&\nabla^2F(r)Y_{lm}(\hat{\mathbf{r}})=Y_{lm}(\hat{\mathbf{r}})\nabla_l^2F(r),\quad(-\nabla^2+m^2)^{1/2}F(r)Y_{lm}(\hat{\mathbf{r}})=Y_{lm}(\hat{\mathbf{r}})(-\nabla_l^2+m^2)^{1/2}F(r)\ ,\\
&[\nabla_iV(r)\nabla_i]F(r)Y_{lm}(\hat{\mathbf{r}})=Y_{lm}(\hat{\mathbf{r}})\left[\frac{dV(r)}{dr}\frac{d}{dr}+V(r)\nabla_l^2\right]F(r)\ ,\\
&[\nabla_i\nabla_jV(r)\nabla_i\nabla_j]F(r)Y_{lm}(\hat{\mathbf{r}})=Y_{lm}(\hat{\mathbf{r}})\left[\frac{d^2V(r)}{dr^2}\frac{d^2}{dr^2}+\frac{dV(r)}{dr}\left[\nabla_l^2\frac{d}{dr}+\frac{d}{dr}\nabla_l^2\right]+V(r)\nabla_l^2\nabla_l^2\right]F(r)\ ,
\end{aligned} \tag{22}$$

where

$$\nabla_l^2=\frac{1}{r}\frac{d^2}{dr^2}r-\frac{l(l+1)}{r^2}\ . \tag{23}$$

We will restrict the analysis of Eq. (20) to the case of equal quark and antiquark masses for the remainder of this section. The generalization to systems with unequal masses is straightforward. With this restriction we can rewrite Eq. (20) as a radial equation:

$$\begin{aligned}
(M-2E_l)R_l(r)=\frac{1}{4E_l^2}\Bigg\{&(E_l+m)^2[V_S(r)+V_V(r)]\\
&+(E_l+m)\left[\left[2\frac{dV_S(r)}{dr}-4\frac{dV_V(r)}{dr}\right]\frac{d}{dr}+[2V_S(r)-4V_V(r)]\nabla_l^2\right]\frac{1}{E_l+m}\\
&+\left[\left[\frac{d^2V_S(r)}{dr^2}+\frac{d^2V_V(r)}{dr^2}\right]\frac{d^2}{dr^2}+\left[\frac{dV_S(r)}{dr}+\frac{dV_V(r)}{dr}\right]\left[\nabla_l^2\frac{d}{dr}+\frac{d}{dr}\nabla_l^2\right]\right.\\
&\left.+[V_S(r)+V_V(r)]\nabla_l^2\nabla_l^2\right]\frac{1}{(E_l+m)^2}-\nabla_l^2V_V(r)-(E_l+m)^2V_V(r)\frac{\nabla_l^2}{(E_l+m)^2}\Bigg\}R_l(r)\ ,
\end{aligned} \tag{24}$$

where E_l is a nonlocal operator. In the nonrelativistic limit, Eq. (24) reduces to the usual local Schrödinger equation

$$\left[M-2m+\frac{\nabla_l^2}{m}\right]R_l(r)=[V_S(r)+V_V(r)]R_l(r)\ . \tag{25}$$

D. The quark-antiquark interaction

The general features of the $q\bar{q}$ interaction in QCD are well known. The details are not, especially for light quarks. For large quark-antiquark separations, both lattice QCD (Ref. 14) and hadronic string models[11] predict an asymptotically linear interaction between heavy quarks:

$$V(r)\approx Br-\frac{\beta}{r}+\cdots\ . \tag{26}$$

This confining part of the potential is expected to have a Lorentz-scalar structure, a result confirmed by the spin dependence of V found in lattice calculations.[16]

The $1/r$ or Lüscher term[18] in Eq. (26) arises from the transverse zero-point oscillations of the string, and can be identified as a Casimir energy. For a standard Nambu string, the most appropriate string to identify with a QCD flux tube, $12\beta/\pi=\frac{1}{2}(d-2)=1$ in $d=4$ dimensions. Otto and Stack indeed obtain $12\beta/\pi=0.95\pm0.08$ in a lattice SU(3) calculation.[15] We will take $\beta=\pi/12$.

For small quark-antiquark separations, the $q\bar{q}$ interaction is described by perturbative QCD, and is given in the approximation we will use by a static, Lorentz-vector single-gluon-exchange potential including vacuum-polarization corrections:[19,20]

$$V_{1\text{ gluon}}(r)=-\frac{4}{3}\frac{\alpha_s(\mu)}{r}\left[1+\frac{\alpha_s(\mu)}{\pi}[b_0\ln\mu r+A(r)]+O\left[\frac{\alpha_s^2}{\pi^2}\right]\right]\ . \tag{27}$$

Here μ is the mass parameter introduced in the dimensional regularization scheme. $A(r)$ is the Fourier transform of the $q\bar{q}$ vacuum-polarization function, given to good approximation through the $b\bar{b}$ bound-state region by[20]

$$A(r)\approx\tfrac{25}{6}\gamma_E+\tfrac{197}{333}+\tfrac{1}{3}[\gamma_E+\ln m_c re^{5/6}+E_1(m_c re^{5/6})]\ , \tag{28}$$

where $\gamma_E=0.5772\ldots$ is Euler's constant and $E_1(z)$ is an exponential integral function.[21]

We can identify the logarithmic term in Eq. (27) as the first term in a perturbative expansion of a running coupling constant with an r-dependent momentum scale $Q(r)$. Following Hagiwara *et al.*,[20] we choose this scale to eliminate the entire $O(\alpha_s)$ term within the square brackets in Eq. (27), and define $Q(r)$ as

$$Q(r)=\frac{1}{r}e^{-A(r)/b_0}\ . \tag{29}$$

With this choice

$$V_{1\text{ gluon}}(r)=-\frac{4}{3}\frac{\alpha_s(r)}{r}\ , \tag{30}$$

where the r dependence of $\alpha_s(r)\equiv\alpha_s(Q(r))$ is determined for $Q(r)/\Lambda_{\overline{\text{MS}}}^{(n_f)}\gg1$ by the QCD renormalization-group equation, with Q given to next-to-leading order by

$$b_0\ln\left[\frac{Q(r)}{\Lambda_{\overline{\text{MS}}}^{(n_f)}}\right]^2=\frac{2\pi}{\alpha_s}+\frac{2b_1}{b_0}\ln\left[\frac{b_0\alpha_s}{2\pi}\Big/\left[1+\frac{b_1\alpha_2}{b_0\pi}\right]\right]\ . \tag{31}$$

Here

$$b_0=\tfrac{11}{2}-\tfrac{1}{3}n_f,\quad b_1=\tfrac{51}{4}-\tfrac{19}{12}n_f\ , \tag{32}$$

for a color-singlet $q\bar{q}$ system with n_f active quarks, and $\Lambda_{\overline{\text{MS}}}^{(n_f)}$ is the corresponding QCD scale factor in the modified minimal subtraction scheme. The expression for $\alpha_s(r)$ obtained from Eq. (31) diverges for

$$Q(r)=\Lambda_{\overline{\text{MS}}}^{(n_f)}\left[\frac{b_0^2}{2b_1}\right]^{b_1/b_0^2}\approx1.12\Lambda_{\overline{\text{MS}}}^{(4)},\quad n_f=4\ , \tag{33}$$

e.g., for $r\approx1.50\ \text{GeV}^{-1}=0.3$ fm for $\Lambda_{\overline{\text{MS}}}^{(4)}=200$ MeV. This Landau singularity is unphysical, as the expression in Eq. (31) is only valid for $\alpha_s/\pi\ll1$. We have therefore regularized $\alpha_s(r)$ by making the replacement[22]

$$\left[\frac{Q(r)}{\Lambda_{\overline{\text{MS}}}}\right]^2\to\left[\frac{Q(r)}{\Lambda_{\overline{\text{MS}}}}\right]^2+\left[\frac{b_0^2}{2b_1}\right]^{2b_1/b_0^2}\ . \tag{34}$$

This change has the effect of moving the singularity in $\alpha_s(r)$ to $r=\infty$ for any choice of $\Lambda_{\overline{\text{MS}}}$, but changes $\alpha_s(r)$ very little for r small. The exact form of the regularization at large r will not be important in our final potential.

The results above restrict the form of the $q\bar{q}$ interaction and its Lorentz structure for $r\to0$ and $r\to\infty$, but not for intermediate values of r. The heavy-quark potential derived from lattice QCD calculations is consistent with a simple superposition of the two types of behavior, but rather little is known about the intermediate range potential otherwise. We will therefore use a fairly flexible parametrization of the potentials in fitting the data in $q\bar{q}$ systems, and take the interaction as a sum of scalar and vector terms with

$$V_V(r)=-\frac{4}{3}\frac{\alpha_s(r)}{r}e^{-\mu' r}+\delta(-\beta/r+Br)(1-e^{-\mu r}) \tag{35}$$

and

$$V_S(r)=(1-\delta)(-\beta/r+Br)(1-e^{-\mu r})+V_0+(C_0+C_1r+C_2r^2)(1-e^{-\mu r})e^{-\mu r}\ , \tag{36}$$

where $\beta=\pi/12$. The vector term incorporates the expected short-distance behavior from single-gluon exchange, but with a damping factor $e^{-\mu' r}$ to eliminate this term at large r where the r^{-1} dependence is associated with the Lüscher term in V_S. We have also included a multiple of the long-range interaction in V_V to see if we can determine the vector-scalar nature of the confining interaction. We expect $\delta=0$. V_S includes the expected

long-range interaction and a purely phenomenological intermediate-range term. In the parametrization above, μ^{-1} acts as a confinement radius, around which the nature of the $q\bar{q}$ interaction changes. A different way of introducing an effective confinement radius by splicing together the short- and long-range interactions has been considered by Nickisch[6] and by Fulcher.[23] We find the method above to be simpler and more flexible.

III. FITS TO $q\bar{q}$ BOUND STATES

A. Data and procedures

The data which we used in our studies consisted of the spin-averaged masses of the $b\bar{b}$, $c\bar{c}$, and $s\bar{s}$ states given in the first column in Table I (Ref. 24), with

$$\bar{M}_l=\frac{1}{4(2l+1)}\sum_j(2j+1)M(j,l,s)\ . \tag{37}$$

The averaging eliminates spin-dependent contributions to the masses to first order in the usual spin-spin, spin-orbit, and tensor interactions between the quarks. The data are incomplete; the $n\,{}^1S_0$ and $2\,{}^1P_1$ $b\bar{b}$ and $c\bar{c}$ masses are not known, and the situation with respect to excited ${}^1S_0$ $s\bar{s}$ states is unclear. We have therefore used the results of previous spin-dependent fits to the data to estimate the centers of gravity of incomplete multiplets, but have increased the uncertainties in the masses accordingly.

Our theoretical calculations of the $q\bar{q}$ spectra were performed using Eq. (24) with the vector and scalar potentials in Eqs. (35) and (36). The crux of the calculation was the generation of matrix representations for the nonlocal operators which appear. The method used is described in detail elsewhere.[12] We note here that for equal-mass $q\bar{q}$ systems we can scale the particle masses out of such operators as E_l+m. We then choose a finite basis $\{\mathcal{L}_j(x),\ j=1,\ldots,n\}$ of associated Laguerre functions (these are appropriate for the solution of Coulomb-type problems), and construct a matrix $\mathbf{D}_l$ which gives the exact action of $(-\nabla_l^2+1)$ on the basis functions at the zeros $\{x_{n+1,i},\ i=1,\ldots,n\}$ of $\mathcal{L}_{n+1}(x)$. The matrix representation of the square-root operator $(-\nabla_l^2+1)^{1/2}$ is then constructed as $\mathbf{S}_l=\mathbf{D}_l^{1/2}=\mathbf{U}_l\lambda_l^{1/2}\mathbf{U}_l^{-1}$, where λ_l is the diagonal matrix of eigenvalues of $\mathbf{D}_l$, and $\mathbf{U}_l$ is the matrix of eigenvectors. Equation (24) can then be reduced to a standard matrix eigenvalue problem. The wave functions R_l are represented by their values at the points x_i, that is, by column vectors, and operators such as E_l, E_l^{-1}, or $(E_l+m)^{-1}$ are represented by $\mathbf{S}_l$, $\mathbf{S}_l^{-1}$, or $(\mathbf{S}_l+1)^{-1}$. This method for solving Eq. (24) converges much more rapidly with increasing matrix size than that used in our earlier work.[6] It was sufficient in the calculations reported here to use 25×25 matrices which can be manipulated quickly. The method is also considerably more flexible and convenient than momentum-space methods[4,7,8] for problems such as the present one, where one wishes to vary positron-space potentials within known constraints.

B. Results

1. Form of the $q\bar{q}$ interaction

In an attempt to determine the extent to which the spin-averaged data on the $b\bar{b}$, $c\bar{c}$, and $s\bar{s}$ systems in Table I restrict the form of the quark-antiquark interaction, we fit those data using the full, nonlocal relativistic wave equation in Eq. (24) with variable mixtures of scalar and vector confining terms, and also using the local relativistic equation used in many other works:[3,6]

$$(M-2E_l)R_l(r)=[V_S(r)+V_V(r)]R_l(r)\ . \tag{38}$$

The potentials in these equations are defined in Eqs. (35) and (36). The potential parameters and the quark masses were determined using a nonlinear least-squares fitting routine. Because of the theoretical uncertainties in the

TABLE I. Fits to the spin-averaged $s\bar{s}$, $c\bar{c}$, and $b\bar{b}$ spectra using the full relativistic interaction in Eq. (24) with scalar confinement ($\delta=0$) and an equal mixture of scalar and vector confining potentials ($\delta=\frac{1}{2}$), and using the local potential approximation in Eq. (38). The masses used in the spin averaging were taken from Ref. 24.

State	Data (GeV)	$\delta=0$ (GeV)	$\delta=\frac{1}{2}$ (GeV)	Local (GeV)
		$s\bar{s}$ states		
$1S$	1.004	1.004	1.004	1.004
$2S$	1.624	1.556	1.660	1.623
		$c\bar{c}$ states		
$1S$	3.068	3.079	3.070	3.077
$2S$	3.663	3.657	3.662	3.658
$1P$	3.525	3.514	3.524	3.519
		$b\bar{b}$ states		
$1S$	9.436	9.423	9.436	9.431
$2S$	10.013	10.014	10.014	10.014
$3S$	10.341	10.345	10.343	10.351
$1P$	9.899	9.896	9.900	9.896
$2P$	10.261	10.257	10.260	10.257
χ^2 (10 points)		5.06	0.24	2.85

model and further uncertainties in the spin averaging, we did not use the quoted experimental errors in the masses to weight the data used in the fit, but rather weighted all states except the $s\bar{s}$ $2S$ state equally with assigned uncertainties of 10 MeV. The $s\bar{s}$ $2S$ state, which is poorly determined experimentally, was assigned an uncertainty of 100 MeV. The values of χ^2 for the three fits given in Table I correspond to these uncertainties.

The fits given in Table I correspond to the theoretically favored case of pure scalar confinement ($\delta=0$), to a value $\delta=\frac{1}{2}$ near the optimum for this parameter, and to the case of the local interaction, Eq. (38) which is independent of δ. There is a clear preference in these fits for the nonlocal wave equation with an approximately equal mixture of scalar and vector couplings ($\delta=\frac{1}{2}$). However, in a separate analysis of the full, spin-independent spectra[10] we found that smaller values of δ are mildly preferred by fits to the spin splittings. While the quality of the $\delta=0$ fit to the spin-averaged data given in Table I is not as high as that of the $\delta=\frac{1}{2}$ fit, it is certainly acceptable, with a value of χ^2 per point of 0.5. We conclude here only that the Lorentz structure of the confining interaction cannot be determined using the spin-averaged data alone. We will henceforth restrict our attention to the case of pure scalar confinement at large distances ($\delta=0$), as expected theoretically, and examine the characteristics of the corresponding model in some detail. We will conclude ultimately that this model is not acceptable and that our starting point, the reduced Salpeter equation, is deficient.

The parameters for the $\delta=0$ fit given in Table I are

$$
\begin{aligned}
&m_s=0.482\pm0.059\ \text{GeV}\ ,\\
&m_c=1.636\pm0.042\ \text{GeV}\ ,\\
&m_b=4.962\pm0.042\ \text{GeV}\ ,\\
&\Lambda=0.430\pm0.042\ \text{GeV}\ ,\\
&B=0.177\pm0.023\ \text{GeV}^2\ ,\\
&V_0=-0.366\pm0.093\ \text{GeV}\ ,\\
&C_0=2.45\pm0.54\ \text{GeV}\ ,\\
&C_1=-0.074\pm0.405\ \text{GeV}^2\ ,\\
&C_2=0.343\pm0.217\ \text{GeV}^3\ ,\\
&\mu=0.933\pm0.177\ \text{GeV}\ ,\\
&\mu'=0.740\pm0.101\ \text{GeV}\ .
\end{aligned}
\tag{39}
$$

The shape of the corresponding local potential $V=V_S+V_V$, and the probability densities for the low-lying $s\bar{s}$, $c\bar{c}$, and $b\bar{b}$ states are shown in Fig. 1. We note that $s\bar{s}$ and excited $c\bar{c}$ wave functions extend well into the linear confining region, and are the most important in determining the slope parameter B in the confining term. The $b\bar{b}$ wave functions are concentrated at small r, and give the strongest constraints on the short-distance interaction.

The parameters given above appear to be quite reasonable. For example, the value of the slope parameter,

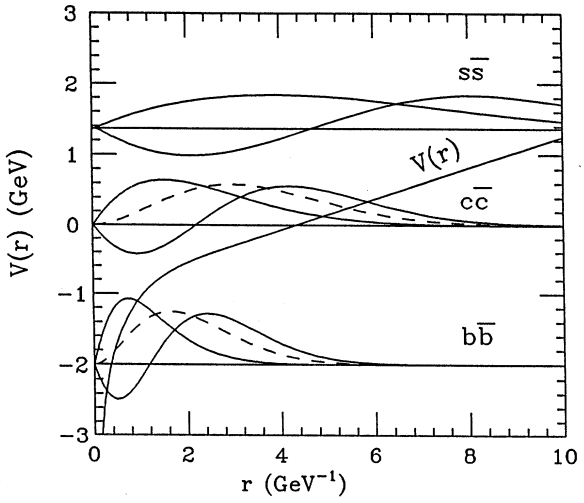

FIG. 1. Plots of the low-lying $s\bar{s}$, $c\bar{c}$, and $b\bar{b}$ wave functions as functions of the interquark separation r for the local potential approximation, Eq. (38). Comparison of the curves with the local potential $V=V_S+V_V$ indicates the regions of the interactions to which the different states are most sensitive. Solid lines: $1S$ and $2S$ states. Dashed lines: $1P$ states. The wave functions obtained using the full nonlocal spin-independent interaction in Eq. (24) are similar.

$B\approx0.177\pm0.023$ GeV^2 is in remarkable agreement with the value $1/2\pi\alpha'=0.177$ expected in a string model for the confining interaction if one uses the observed slope $\alpha'=0.9$ GeV^2 of the ρ Regge trajectory. However, this is accidental, as we will show in the next section. The value of the QCD parameter, $\Lambda=0.430\pm0.042$ GeV, is somewhat larger than that determined from high-energy experiments, but it is not unreasonable. The b- and c-quark masses are well determined, with the mass difference $m_b-m_c=3.326\pm0.030$ GeV better determined than the separate masses. Finally, the parameters μ and μ' are such that the Lorentz-vector single-gluon-exchange term is damped out, and the scalar confining term becomes important, for $q\bar{q}$ separations greater than ~1.2 GeV$^{-1}\approx0.2$ fm. This transition is evident in the potential curve in Fig. 1, which becomes quite linear outside the transition region.

Close examination of the $\delta=0$ results in Table I shows a disturbing systematic trend: namely, that the separations between the $1S$ ground state and the excited S and P states are too large in the $b\bar{b}$ system, but markedly too small for the $c\bar{c}$ and $s\bar{s}$ systems. This effect is hidden in Table I by the tendency of the fitting program to split the error between the $1S$ and the $2S$ and $1P$ states to minimize the value of χ^2, but is clearly evident in Table II. In that table we give the results of a fit in which all the $b\bar{b}$ states, but only the $1S$ states in the $c\bar{c}$ and $s\bar{s}$ systems were fitted. The parameters are quite similar to those given in Eq. (39).

We have found no way of eliminating the systematic trends evident in Tables I and II. The difficulty is even

TABLE II. Results of a fit to the spin-averaged $b\bar{b}$ data and the $s\bar{s}$ and $c\bar{c}$ ground states using the full relativistic interaction in Eq. (24) with scalar confinement ($\delta=0$). The states marked with asterisks were not included in the fitting procedure.

State	Data (GeV)	Fit (GeV)	Error in fit (MeV)
		$s\bar{s}$ states	
$1S$	1.004	1.004	0
$2S^*$	1.624	1.567	−57
		$c\bar{c}$ states	
$1S$	3.068	3.068	0
$2S^*$	3.663	3.645	−18
$1P^*$	3.525	3.499	−26
		$b\bar{b}$ states	
$1S$	9.436	9.436	0
$2S$	10.013	10.015	+2
$3S$	10.341	10.343	+2
$1P$	9.899	9.900	+1
$2P$	10.261	10.259	−2

more striking if we extend the calculations to light-quark systems as discussed in the next subsection, and point to a fundamental flaw in our (standard) procedure.

2. *Regge trajectories*

In Fig. 2 we show the Regge trajectories calculated for light-quark ($l\bar{l}$) systems using the full relativistic wave equation with $m_l=200$ MeV and scalar confinement ($\delta=0$), and compare the results with the observed Regge trajectories for the spin-triplet and spin-singlet $l\bar{l}$ mesons. The slopes of the calculated trajectories are strikingly large compared to those observed for the ρ, ω, and π trajectories. The $L=0$ Regge intercept for the leading trajectory corresponds to a mass of the lowest $1S$ $l\bar{l}$ state of 540 MeV, somewhat below the spin average of the ρ and π masses, 613 MeV. A change in the light-quark mass to $m_l\approx 320$ MeV removes this discrepancy, but still leaves the Regge trajectories much too steep, with slopes greater than twice the observed slopes. The large slopes correspond to overly close spacings of masses with increasing L. Thus the first two spacings on the leading trajectory are calculated as 440 MeV and 290 MeV, to be compared with the observed spacings $M(a_2)-M(\rho_1)=550$ MeV and $M(\rho_3)-M(a_2)=370$ MeV on the ρ trajectory.

It is customary to determine the asymptotic slope of the scalar confining potential $V_S\approx Br$ for r large by using the string theory result,[11] $B=1/2\pi\alpha'$, where α' is the slope of the (approximately linear) Regge trajectories. The value of B given in Eq. (39), $B\approx 0.177$ GeV^2, agrees essentially exactly with the string model result for the observed slope of the ρ trajectory, $\alpha'\approx 0.9$ GeV^{-2}, but has no relation to the calculated slopes of the $l\bar{l}$ trajectories in Fig. 2. We have found, in fact, that there is no value of B, reasonable or unreasonable, which will lead to an $l\bar{l}$ Regge slope consistent with experiment.[25] The light-quark systems clearly satisfy stringlike dynamics, but not the relativistic potential dynamics considered here.

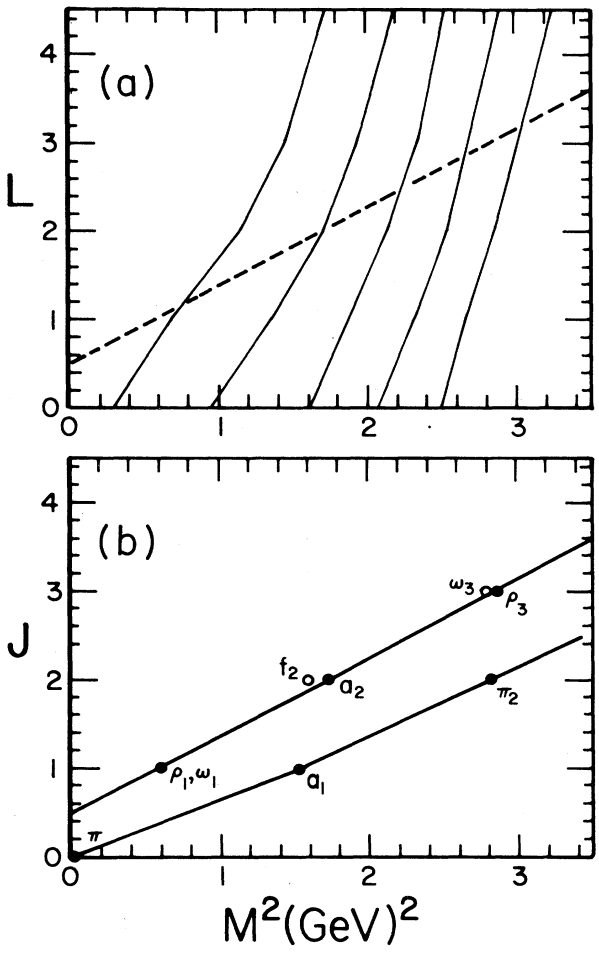

FIG. 2. (a) Regge trajectories calculated for light-quark ($l\bar{l}$) systems ($m_l=200$ MeV) using the pure scalar confining interaction ($\delta=0$) in the spin-independent relativistic wave equation, Eq. (24). The dashed curve indicates the slopes of the ρ Regge trajectory. (b) The observed Regge trajectories for the spin-triplet (upper curve) and -singlet (lower curve) $l\bar{l}$ mesons. The meson masses are from Ref. 24.

The calculated separation of the $1S$ and $2S$ $l\bar{l}$ states of 440 MeV given above continues the trend evident in the $\delta=0$ column in Table I of a *decrease* in the separation of the $1S$ and $2S$ states with decreasing quark mass, i.e., 591, 576, 552, and 440 MeV for the $b\bar{b}$, $c\bar{c}$, $s\bar{s}$, and $l\bar{l}$ systems. This trend is opposite to that in the spin-averaged $b\bar{b}$, $c\bar{c}$, and $s\bar{s}$ data where the $1S$-$2S$ separations are 577, 595, and 624 MeV, hence *increasing* with decreasing quark mass.

The foregoing problems appear to be intrinsic to the relativistic wave equation in Eq. (24) with scalar confinement. In an earlier investigation[26] of the Regge trajectories using the local interaction in Eq. (38), we found that it was possible to fit the spin-averaged $s\bar{s}$, $c\bar{c}$, and $b\bar{b}$ spectra and at the same time to get good slopes for the $l\bar{l}$ trajectories, although not good values for the ground-state $l\bar{l}$ mass. The effect of including the mass-dependent terms in the full interaction in Eq. (24) has dis-

rupted those results in a way which apparently cannot be compensated by a change in the potentials provided we keep a scalar confining interaction.

3. *Limit of zero quark mass*

To see the origin of the problems above, it is useful to consider the limit of zero quark mass. In this limit, the spin-independent interaction operators in Eqs. (15) reduce to

$$F_V^{\rm si}(\mathbf{p},\mathbf{p}')=\frac{1}{4}\left[3+4\frac{\mathbf{p}\cdot\mathbf{p}'}{pp'}+\frac{(\mathbf{p}\cdot\mathbf{p}')^2}{p^2p'^2}\right],\quad m_q=0\ , \tag{40a}$$

$$F_S^{\rm si}(\mathbf{p},\mathbf{p}')=\frac{1}{4}\left[1-\frac{\mathbf{p}\cdot\mathbf{p}'}{pp'}\right]^2,\quad m_q=0\ . \tag{40b}$$

The limits of these quantities are quite different for collinear and anticollinear quark momenta:

$$F_V^{\rm si}(\mathbf{p},\mathbf{p})=2,\quad F_S^{\rm si}(\mathbf{p},\mathbf{p})=0,\quad \mathbf{p}'=\mathbf{p}\ , \tag{41a}$$

and

$$F_V^{\rm si}(\mathbf{p},-\mathbf{p}')=0,\quad F_S^{\rm si}(\mathbf{p}-\mathbf{p}')=1,\quad \mathbf{p}'=-\mathbf{p}\ . \tag{41b}$$

In the nonrelativistic limit $p,p'\ll m_q$, $F_V^{\rm si}$ and $F_S^{\rm si}$ both reduce to unit operators.

Because of these very different limits, vector and scalar interactions have quite different effects on the $q\bar{q}$ spectrum for different quark masses and potential types. All of the components of the vector and scalar potentials in Eqs. (35) and (36) are sharply peaked in momentum space for $\mathbf{p}'\approx\mathbf{p}$, e.g., the gluon-exchange and linear potentials fall off roughly as $|\mathbf{p}-\mathbf{p}'|^{-2}$ and $|\mathbf{p}-\mathbf{p}'|^{-4}$ for $|\mathbf{p}-\mathbf{p}'|$ large. Thus, equal collinear momenta are favored by the forms of the potentials V_V and V_S. However, the effective magnitude of V_S in Eq. (16) is reduced for $\mathbf{p}'\approx\mathbf{p}$ in the case of light quarks because of the vanishing of $F_S^{\rm si}(\mathbf{p},\mathbf{p})$, while the effective magnitude of V_V is enhanced. In particular, light quarks will see a weaker confining potential than heavy quarks at large distances if the confinement is scalar.

The effects will of course be reduced somewhat when the angular averages which enter the partial-wave projections to states of definite L are taken into account, but will not disappear entirely. The residual effects seem, in fact, to account for the systematic trends discussed in the preceding subsection.[9] As may be seen from Fig. 1, light-quark systems are much more sensitive than heavy-quark systems to the behavior of the interaction at large distances. The progressive weakening (or flattening) of the effective long-range confining interaction with decreasing quark mass accounts for both the compression of the energy spacings with decreasing quark mass found in our calculations, and the steepness of the $l\bar{l}$ Regge trajectories. The corresponding strengthening of the vector gluon-exchange interaction at short distances is relatively less important as the lighter quarks are not especially sensitive to this region of the potential.

The effects discussed above can be eliminated to a considerable extent by taking an equal mixture of scalar and vector contributions to the linear confining potential, that is, choosing $\delta\approx\frac{1}{2}$ in Eqs. (35) and (36). The effective interaction then involves the sum of $F_S^{\rm si}$ and $F_V^{\rm si}$, a quantity which remains near unity for $|\mathbf{p}-\mathbf{p}'|$ small, and the strength of the confining interaction is roughly independent of the quark mass. The $\delta=\frac{1}{2}$ fit to the $s\bar{s}$, $c\bar{c}$, and $b\bar{b}$ data is in fact much better than the $\delta=0$ fit (see Table I), but $\delta\approx\frac{1}{2}$ is ruled out by the spin dependence of the static $q\bar{q}$ potential found in lattice gauge theory.[16]

C. Conclusions

The results of this work are discouraging with respect to the utility of the reduced Salpeter equation for the description of light- or strange-quark systems. We conclude, in fact, that this approach is fundamentally flawed.[27] The problems recited above—the incorrect systematic trends in mass differences, and the drastically incorrect slopes of Regge trajectories—are *intrinsic* to an approach based on the use of the reduced Salpeter equation with static scalar confinement. However, the failure of the model to reproduce the observed "stringy" behavior of the $l\bar{l}$ Regge trajectories suggests that the problems would be eliminated in a theory which included the *dynamical* energy of the color field between quark and antiquark as well as the *static* field energy represented by the static one-gluon exchange and confining interactions.

The kinetic energy terms of Eqs. (16) and (24) refer only to the motion of the quarks. The energy of the color fields is entirely in the interaction terms. The static field energy found in lattice calculations[15] in fact matches quite well the (Schrödinger) potential between nonrelativistic heavy quarks: i.e., the $b\bar{b}$ potential. One can reasonably expect the color field to adjust adiabatically to slow motions of the heavy quarks and the kinetic energy in the fields to remain small, and, hence, a nonrelativistic Schrödinger or Salpeter picture of the dynamics with a static potential to work well for sufficiently heavy quarks.

At the opposite extreme, the relativistic string model[11] gives a good description of the Regge spectrum of light-quark systems which involves no kinetic or mass energy for the quarks. The energy of the string is essentially the energy of a moving flux tube, with the quarks serving only to carry the appropriate internal quantum numbers. A realistic model would presumably lie in between, and take account of the kinetic energy of the quarks, and both the configurational and kinetic energies associated with color fields which change as the quarks move. Such an approach would require a change in starting point, e.g., the use of the full Bethe-Salpeter equations with a so-far-unknown nonstatic kernel, or a Hamiltonian approach in which the color fields appear explicitly, perhaps in a flux-tube approximation. As a step in the last direction, LaCourse and Olsson[28] have recently investigated a quantized version of the classical model of a rigid string with quarks at the ends, and find, not surprisingly, that the string-model results for the slopes of Regge trajectories are reproduced, although other problems remain. In any case, a description based on the reduced Salpeter equation emphasizes only the dynamics of the quarks and is not appropriate for light-quark systems.

We remark that we would expect the kinetic energy in

a rapidly changing color field to increase the spacing between mass states in light quark systems relative to the spacing in heavy-quark systems, a change in the correct direction, but one which we cannot quantify.

We end with a word of caution. The systematic problems discussed in the preceding sections extend to the $c\bar{c}$ system. Our attempt to improve the spin-independent calculations relative to those performed using the simple relativistic wave equation in Eq. (38), or even the Schrödinger equation, has made matters worse. We conclude that the mass-dependent corrections to the interaction which appear in the reduced Salpeter equation or its expanded form relative to Eq. (38) are unreliable. There must be other dynamical effects of the same general magnitude in a complete theory. Thus, despite the time we have invested in its solution, we conclude that the reduced Salpeter equation should *not* be used for phenomenological investigations of $q\bar{q}$ systems, except possibly for the spin dependence, which we discuss in a separate paper.[10] This conclusion unfortunately leaves the theory of light quark systems unsettled.

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APPENDIX

This appendix describes some of the details involved in the reduction of the 4×4 matrix wave equation for ϕ to a 2×2 form. We consider the case of Lorentz-vector and -scalar couplings as in Eq. (8). The relevant interaction matrices are

$$\mathcal{V}=\Lambda^{+}(\mathbf{p})\gamma_0\gamma_\mu\phi(\mathbf{p}')\gamma^\mu\gamma^0\Lambda^{-}(-\mathbf{p}) \tag{A1}$$

for the vector coupling and

$$\mathcal{S}=\Lambda^{+}(\mathbf{p})\gamma_0\phi(\mathbf{p}')\gamma^0\Lambda^{-}(-\mathbf{p}) \tag{A2}$$

for the scalar coupling. We use the standard representation for the gamma matrices:

$$\gamma_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \boldsymbol{\gamma}=\begin{bmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{bmatrix}, \tag{A3}$$

where $\boldsymbol{\sigma}$ is the usual representation of the 2×2 Pauli matrices. The projection operators $\Lambda^{\pm}$ are defined in Eq. (3). The components of the 4×4 matrix ϕ are defined by

$$\phi=\begin{bmatrix}\phi^{+-} & \phi^{++}\\ \phi^{--} & \phi^{-+}\end{bmatrix}. \tag{A4}$$

We wish to reduce Eq. (8) to an equation for ϕ^{++}, the large component of ϕ. Straightforward matrix operations give the $++$ component of $\mathcal{V}$:

$$\begin{aligned}\mathcal{V}^{++}=\frac{1}{4\omega_1\omega_2}[&(\omega_1+m_1)\phi^{++}(\mathbf{p}')(\omega_2+m_2)+\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_i\phi^{++}(\mathbf{p}')\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}+(\omega_1+m_1)\phi^{+-}(\mathbf{p}')\boldsymbol{\sigma}\cdot\mathbf{p}\\ &+\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_i\phi^{+-}(\mathbf{p}')\sigma_i(\omega_2+m_2)+\boldsymbol{\sigma}\cdot\mathbf{p}\phi^{-+}(\mathbf{p}')(\omega_2+m_2)+(\omega_1+m_1)\sigma_i\phi^{-+}(\mathbf{p}')\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}\\ &+\boldsymbol{\sigma}\cdot\mathbf{p}\phi^{--}(\mathbf{p}')\boldsymbol{\sigma}\cdot\mathbf{p}+(\omega_1+m_1)\sigma_i\phi^{--}(\mathbf{p}')\sigma_i(\omega_2+m_2)]\,.\end{aligned} \tag{A5}$$

The small components of ϕ can now be expressed in terms of ϕ^{++} by using the relations in Eqs. (10), with the result

$$\begin{aligned}\mathcal{V}^{++}=\frac{1}{4\omega_1\omega_2}\Bigg[&(\omega_1+m_1)\phi^{++}(\mathbf{p}')(\omega_2+m_2)+\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_i\phi^{++}(\mathbf{p}')\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}\\ &+\frac{\omega_1+m_1}{\omega_2'+m_2}\phi^{++}(\mathbf{p}')\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}+\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_i\phi^{++}(\mathbf{p}')\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_i\frac{\omega_2+m_2}{\omega_2'+m_2}\\ &+\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}'\phi^{++}(\mathbf{p}')\frac{\omega_2+m_2}{\omega_1'+m_1}+\frac{\omega_1+m_1}{\omega_1'+m_1}\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}'\phi^{++}(\mathbf{p}')\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}\\ &+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}'\phi^{++}(\mathbf{p}')\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{(\omega_1'+m_1)(\omega_2'+m_2)}+\frac{\omega_1+m_1}{\omega_1'+m_1}\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}'\phi^{++}(\mathbf{p}')\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_i\frac{\omega_2+m_2}{\omega_2'+m_2}\Bigg],\end{aligned} \tag{A6}$$

where $\omega_i'=(\mathbf{p}'^2+m_i^2)^{1/2}$. The corresponding result for the scalar coupling is

$$\begin{aligned}\mathcal{S}^{++}=\frac{1}{4\omega_1\omega_2}\Bigg[&(\omega_1+m_1)(\omega_2+m_2)\phi^{++}(\mathbf{p}')-\frac{\omega_2+m_2}{\omega_1'+m_1}\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}'\phi^{++}(\mathbf{p}')\\ &-\frac{\omega_1+m_1}{\omega_2'+m_2}\phi^{++}(\mathbf{p}')\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}'\phi^{++}(\mathbf{p}')\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{(\omega_1'+m_1)(\omega_2'+m_2)}\Bigg].\end{aligned} \tag{A7}$$

The matrix wave function ϕ^{++} is connected to the standard spin wave function for two spin-$\frac{1}{2}$ particles with spin projections m_1, m_2 by a factor $i\sigma_2$ acting on the antiparticle index:

$$\phi_{m_1 m_2}(\mathbf{p}') = [\phi^{++}(\mathbf{p}') i\sigma_2]_{m_1 m_2} . \tag{A8}$$

In particular, the normalized singlet- and triplet-spin states correspond to the matrices

$$\chi(S=0) = \frac{i}{\sqrt{2}}\sigma_2, \quad \chi_m(S=1) = \frac{i}{\sqrt{2}}\sigma_m\sigma_2 . \tag{A9}$$

By using the identity

$$(-i\sigma_2)\sigma_j(i\sigma_2) = -\sigma_j^T , \tag{A10}$$

we can convert σ-dependent factors on the right of ϕ^{++} to ordinary spin operators acting on $\phi_{m_1 m_2}$ from the left, provided these terms are taken to act on the second index only. For example,

$$(\boldsymbol{\sigma}\cdot\mathbf{p}'\phi^{++}\boldsymbol{\sigma}\cdot\mathbf{p} i\sigma_2)_{m_1 m_2} = -(\boldsymbol{\sigma}\cdot\mathbf{p}')_{m_1 m_1'}(\boldsymbol{\sigma}\cdot\mathbf{p})_{m_1 m_2'}\phi_{m_1' m_2'} . \tag{A11}$$

Applying this transformation to Eqs. (A6) and (A7), we obtain the interaction operators F_V and F_S given in Eqs. (13).

The seven distinct products of Pauli matrices which occur in the F's can be reduced using the identity

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{A12}$$

to the forms

$$\begin{aligned}
&(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_2 = \mathbf{p}\cdot\mathbf{p}' + i\mathbf{p}\cdot\mathbf{p}'\times\boldsymbol{\sigma}_2 , \\
&(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_1(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_2 = (\mathbf{p}\cdot\mathbf{p}' + i\mathbf{p}\cdot\mathbf{p}'\times\boldsymbol{\sigma}_1)(\mathbf{p}\cdot\mathbf{p}' + i\mathbf{p}\cdot\mathbf{p}'\times\boldsymbol{\sigma}_2) , \\
&(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}')_1 = \mathbf{p}\cdot\mathbf{p}' + i\mathbf{p}\cdot\mathbf{p}'\times\boldsymbol{\sigma}_1 , \\
&(\boldsymbol{\sigma}\cdot\mathbf{p}\sigma^i)_1(\sigma^i\boldsymbol{\sigma}\cdot\mathbf{p}')_2 = \mathbf{p}\cdot\mathbf{p}' + i\mathbf{p}\cdot\mathbf{p}'\times(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) + \mathbf{p}\cdot\mathbf{p}'\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \mathbf{p}\cdot\boldsymbol{\sigma}_2\mathbf{p}'\cdot\boldsymbol{\sigma}_1 , \\
&(\sigma^i\boldsymbol{\sigma}\cdot\mathbf{p}')_1(\sigma^i\boldsymbol{\sigma}\cdot\mathbf{p}')_2 = \mathbf{p}'^2 - \mathbf{p}'^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + \mathbf{p}'\cdot\boldsymbol{\sigma}_1\mathbf{p}'\cdot\boldsymbol{\sigma}_2 , \\
&(\boldsymbol{\sigma}\cdot\mathbf{p}\sigma^i)_1(\boldsymbol{\sigma}\cdot\mathbf{p}\sigma^i)_2 = \mathbf{p}^2 - \mathbf{p}^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + \mathbf{p}\cdot\boldsymbol{\sigma}_1\mathbf{p}\cdot\boldsymbol{\sigma}_2 , \\
&(\sigma^i\boldsymbol{\sigma}\cdot\mathbf{p}')_1(\boldsymbol{\sigma}\cdot\mathbf{p}\sigma^i)_2 = \mathbf{p}\cdot\mathbf{p}' + i\mathbf{p}\cdot\mathbf{p}'\times(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) + \mathbf{p}\cdot\mathbf{p}'\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \mathbf{p}\cdot\boldsymbol{\sigma}_1\mathbf{p}'\cdot\boldsymbol{\sigma}_2 .
\end{aligned} \tag{A13}$$

The separation of the spin-dependent and spin-independent terms is then trivial. The spin-independent parts of the F's are given in Eq. (15), and can be extracted easily from the expressions in Eqs. (13) using the trace identity $\mathrm{Tr}\sigma_i = 0$ to eliminate the spin-dependent terms.

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