On some ambiguities in the method of effective charges

G. Grunberg
Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau CEDEX, France
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In response to the preceding paper by Chýla, the status of some remaining ambiguities in the method of effective charges is reviewed and the specific example of the static potential in QCD is discussed. A criterion for convergence of perturbation theory is suggested, and shown to be satisfied by the recent $O(\alpha_s^3)$ calculation of $\sigma(e^+e^- \to \text{hadrons})$ of Gorishny, Kataev, and Larin.

In the preceding paper Chýla[1] has raised a number of criticisms concerning the method of effective charges, suggested in Ref. 2, to solve the renormalization-scheme (RS) ambiguity. His paper contains two main observations.

(1) The first one is that some alternative definitions of effective charges,[2,3] associated with a given physical quantity, are possible, thus reintroducing the RS ambiguity. I believe this remark, although correct, to be of little interest, *unless some specific motivation is given for these alternative definitions.* To make my point clear, let us recall the essence of the method of Ref. 2, as applied to a renormalizable field theory with a single dimensionless coupling constant (I shall consider for definiteness QCD with massless quarks), *in a way which does not make direct reference to the question of RS dependence and RS ambiguity.* Consider a perturbatively calculable dimensionless physical quantity,

$$\sigma(Q)=\alpha_s(1+c_1\alpha_s+c_2\alpha_s^2+\cdots)\,, \tag{1}$$

depending upon a *single* external scale Q (the latter is an important requirement, not just imposed "for simplicity": see Ref. 2 for an example of possible extensions of the method when several scales, with arbitrarily large ratios, are present). Dimensional transmutation then implies, in the continuum limit, that σ is a function $\sigma=F(Q^2/\Lambda^2)$, where the only arbitrariness is the choice of the QCD scale parameter Λ. The method of effective charges *focuses precisely on the Q^2 dependence of this function,* and, from this point of view, is closer to the spirit of the Gell-Mann–Low approach to the renormalization group (RG), rather than to that of the Stückelberg-Petermann-Callan-Symanzik approach (which focuses on the dependence of σ upon α_s and other arbitrary RS parameters, such as the renormalization point μ). The suggestion of Ref. 2 is to write down a differential equation for the Q^2 *evolution* of σ:

$$\frac{d\sigma}{d\ln Q^2}=\bar{\beta}(\sigma)\,, \tag{2}$$

where no arbitrary RS parameters appear [of course, Eq. (2) is nothing but the RG equation for the "effective charge" $\bar{\alpha}_s\equiv\sigma$]. Integration of Eq. (2) [with $\bar{\beta}(\sigma)=-\beta_1\sigma^2-\beta_2\sigma^3-\bar{\beta}_3\sigma^4+\cdots$ calculable perturbatively in powers of σ, and truncated at a finite order—which is precisely the meaning of "perturbative QCD" in this context], yields the desired function $\sigma=F(Q^2/\Lambda^2)$ [or, more precisely, the expansion of the inverse function $Q^2/\Lambda^2=F^{-1}(\sigma)$ in powers of σ]. The crucial *assumption* underlying the usefulness of Eq. (2) in perturbation theory (PT) is that the weak-coupling expansion of $\bar{\beta}(\sigma)$, truncated in low orders, gives a good estimate of the exact $\bar{\beta}(\sigma)$, for the range of σ of interest around $\sigma=0$. If this assumption is correct, the method of Ref. 2 leads to a reliable estimate of $\sigma(Q)$, which is all that can be expected from a perturbative calculation. It then matters little that other definitions of effective charges, associated with a given σ, are possible. The real problem arises only if several, *a priori* equally "natural," definitions lead, in PT, to large numerical discrepancies. One such example has already been discussed in the first paper of Ref. 2 (see Sec. VI C), and another one shall be given below. Concerning the specific, rather artificial, definitions suggested by Chýla,

$$\sigma\equiv\alpha_s^r(1+r\alpha_s^r)\,, \tag{3}$$

there is little point to consider them in general, unless some special motivation is provided (see the example below). Indeed, the assumption that PT converges well for $\bar{\beta}\equiv\beta^{r=0}$ is not compatible, unless r is small enough, with the similar assumption for β^r. The problem here is of a similar nature one would face in QED if one were to consider, instead of the expansion of the anomalous magnetic moment a_e of the electron in powers of α_{em}, that of the related quantity $\tilde{a}_e\equiv a_e(1+ra_e)$, with (say) $r=10^{10}$. Although no rigorous motivation appears to justify *a priori* the choice $r=0$, higher-order calculations have confirmed the soundness of this (intuitively obvious) choice; for a related discussion, see Ref. 4. I note in addition that the definition Eq. (3) is not compatible with a description [Eq. (2)] of σ in terms of a *single-valued* $\bar{\beta}$ function, in the case where $r<0$ (assuming such a description is possible for α_s^r), since then σ would have a nonmonotonous behavior as a function of α_s^r and hence as a function of Q, with a maximum reached for $\alpha_s^r=-1/2r$ (making the β-function description of σ break down above smaller and smaller values of α_s^r and hence below larger and larger values of Q, as $|r|\to\infty$). In some sense, the definition Eq. (3) negates the very basis of the method of Ref. 2 which, as discussed there, implies a "RG improvement" of the usual perturbation series, i.e., a partial resummation of the higher-order terms one expects to arise, when a "sound" RS is used, on the right-hand side of Eq. (3). For instance, the one-loop RG-improved formula replaces Eq. (3) by

$$\sigma=\frac{\alpha_s^r}{1-r\alpha_s^r}=\alpha_s^r[1+r\alpha_s^r+r^2(\alpha_s^r)^2+\cdots] . \tag{4}$$

The result, Eq. (4), is independent of r (with α_s^r transformed according to the one-loop RG-improved formula), but of course allowance for all the needed higher-orders corrections to Eq. (3) should be made, which, by definition, is prevented if Eq. (3) is to be interpreted as an identity.

Coming now to the promised example, where an ambiguity similar to Eq. (3) does appear in a natural way, consider the effective charge $\alpha_s^p(r)$ associated with the static potential $V(r)$ between two heavy quarks in QCD:

$$V(r)\equiv-\frac{\alpha_s^p(r)}{r} . \tag{5}$$

Alternatively, it may be useful, for a number of reasons, to deal with the force $F(r)=dV/dr$, to which one associates another effective charge $\alpha_s^f(r)$:

$$F(r)\equiv\frac{\alpha_s^f(r)}{r^2} . \tag{6}$$

The relation between α_s^p and α_s^f is

$$\alpha_s^f=\alpha_s^p+2\beta_p(\alpha_s^p) , \tag{7}$$

where β_p is the β function associated with α_s^p:

$$\frac{d\alpha_s^p}{d\ln r^2}=-\beta_p(\alpha_s^p) . \tag{8}$$

Assume now that the expansion of β_p converges well, which I replace for simplicity by the more drastic assumption that β_p reduces to its one-loop term: $\beta_p(\alpha_s^p)=-\beta_1(\alpha_s^p)^2$. Then Eq. (7) gives

$$\alpha_s^f=\alpha_s^p(1-2\beta_1\alpha_s^p) , \tag{9}$$

i.e., a relation similar to Eq. (3). Two different predictions for $F(r)$ are now available. First, integration of Eq. (8) gives $\alpha_s^p(r)$, from which $F(r)$ follows using Eqs. (6) and (9). Alternatively, one can assume that it is the β function associated with α_s^f which is given by its one-loop term:

$$\frac{d\alpha_s^f}{d\ln r^2}=-\beta_f(\alpha_s^f)=\beta_1(\alpha_s^f)^2 . \tag{10}$$

Integration of Eq. (10) yields another prediction for $F(r)$. It turns out (as can easily be checked) that there is a considerable numerical discrepancy, starting at rather small values of r, between these two *a priori* equally reasonable procedures.[5] The question arises which of these two assumptions is to be preferred, and clearly only additional information can help make a decision. This problem is further discussed in Ref. 5. I simply note here that a single-valued $\bar{\beta}$-function description cannot work for $\alpha_s^p(r)$ over the whole range of r, since $V(r)$, and hence $\alpha_s^p(r)$, have to change sign somewhere between the Coulomb (where $V\propto-1/r$) and the confining region (where $V\propto+r$), whereas the same problem does not arise for $F(r)$ and α_s^f, which may favor the use of β_f and Eq. (10). Similar ambiguities arise whenever one deals with a dimensionful physical quantity such as $V(r)$, whose dimension is carried by the external variable scale (in the present case, r). This example also shows that, although this kind of ambiguity cannot be resolved within the effective charge method itself, the latter does provide the proper framework where relevant questions can be asked, and further progress eventually made.

(2) I next turn to the more interesting remark of Chýla concerning the inadequacy of the criterion for a "well-behaved" $\bar{\beta}$ function in Ref. 2: namely, $|\bar{\beta}_3/\beta_2|\approx|\beta_2/\beta_1|$ (since Ref. 2 is not concerned with all-order behavior, it should be clear that the relation $|\bar{\beta}_{i+1}/\beta_i|\approx|\beta_2/\beta_1|$ was implicitly meant to apply only in low orders, and we consider it here in the first nontrivial order). Chýla points out that this criterion fails in the case of the quantity

$$R(Q)=\frac{\sigma(e^+e^-\to\text{hadron})}{\sigma(e^+e^-\to\mu^+\mu^-)}\equiv R_0\left[1+\frac{\bar{\alpha}_s^{e^+e^-}}{\pi}\right] , \tag{11}$$

where R_0 is a known constant. I would like to argue, however, that this fact does not necessarily mean that PT, as applied to $\bar{\beta}_{e^+e^-}$, is useless, but rather shows that the criterion of Ref. 2 is too restrictive. Indeed, both β_1 and β_2 are universal, RS-independent coefficients, so the magnitude of β_2/β_1 may be atypical for the behavior of the higher-order, process-dependent terms in the $\bar{\beta}$ functions. Instead, it is natural to consider a more general requirement: namely, that the three-loop term in $\bar{\beta}$ (which is the first nonuniversal term, and was consequently noted in Ref. 2 to give the first nontrivial indication for convergence of PT for $\bar{\beta}$) represents a small correction to the leading and universal (one+two-loop) contribution (i.e., in practice, to the one loop contribution, if one restricts oneself to cases, most often met in practice, where the two-loop contribution is small compared to one loop). This criterion is easily seen to be satisfied by $\bar{\beta}_{e^+e^-}$ in the relevant energy range. Neglecting completely for simplicity the two-loop contribution (i.e., putting $\beta_2=0$ as in Ref. 1, in which case the criterion of Ref. 2 is obviously meaningless), one gets

$$\bar{\beta}(\bar{\alpha}_s)=-\beta_1\bar{\alpha}_s^2\left[1+\frac{\bar{\beta}_3}{\beta_1}\bar{\alpha}_s^2+\cdots\right] . \tag{12}$$

The extended criterion then simply requires that $(\bar{\beta}_3/\beta_1)\bar{\alpha}_s^2$ be small compared to unity, which can be realized even for large values of $\bar{\beta}_3$, *since the correction is quadratic in* $\bar{\alpha}_s$. For $\bar{\beta}_{e^+e^-}$ one indeed gets, using the information in Ref. 6, that $\bar{\alpha}_s^{e^+e^-}(34^2\ \text{GeV}^2)=0.155$, and (for five flavors)

$$\frac{\bar{\beta}_3}{\beta_1}(\bar{\alpha}_s^{e^+e^-})^2=0.15 , \tag{13}$$

i.e., a reasonably small correction.

Alternatively, a similar conclusion can be reached from the point of view of the ordinary perturbation series for $\bar{\alpha}_s = \alpha_s(1 + c_1\alpha_s + c_2\alpha_s^2 + \cdots)$. The method of Ref. 2 indeed suggests that the relative magnitude $(c_2/c_1)\alpha_s$ of the third-order correction compared to the second-order one is of little significance, since one can always put $c_1 = 0$ by a change of the renormalization point μ [the "fastest apparent convergence" (FAC) choice], without changing the renormalization convention[1] (RC), which leaves us with the formula[2]

$$\bar{\alpha}_s = \alpha_s \left[1 + \frac{\bar{\beta}_3 - \beta_3}{\beta_1} \alpha_s^2 + \cdots \right] . \tag{14}$$

It was argued in Ref. 2 that instead the magnitude of $[(\bar{\beta}_3 - \beta_3)/\beta_1]\alpha_s^2$ compared to unity gives a relevant indication for convergence of PT in the considered RC. This remark typically applies to $\bar{\alpha}_s^{e^+e^-}(Q)$, since $c_1\alpha_s$ turns out to be rather small in the usually considered modified minimal subtraction ($\overline{\text{MS}}$) scheme, with the standard choice $\mu = Q$; i.e., the latter is quite close to the FAC choice in this case. The possibility that $c_1\alpha_s$ is accidentally small was in fact considered in Ref. 6. The method of effective charges does support this point of view, and sees no sign of breakdown of PT in the finding[6] that the third-order correction is more than twice as large as the second-order one. Furthermore, from Ref. 6 one obtains that $[(\bar{\beta}_3 - \beta_3)/\beta_1]\alpha_s^2 \approx 0.11$ [using[6] $\alpha_s^{\overline{\text{MS}}}(34^2\ \text{GeV}^2) = 0.132$], which is essentially equal to the relative magnitude of the third-order correction $c_2\alpha_s^2$ as computed in Ref. 6, and again appears to be of reasonable size.

(3) Finally, let me comment on some remaining, more peripheral issues raised in Ref. 1.

(i) Concerning the problem of the nonsensical results eventually obtained through naive application of the method of effective charges to all orders, it is no different in principle from similar difficulties always encountered when dealing with (presumably) divergent perturbation series. It is obvious that a resummation prescription must be given when dealing with such series to all orders, which, in the present case, means a resummation prescription for the $\bar{\beta}$ functions. This clearly requires extra information, beyond PT (as recognized by Chýla himself)—consequently the method of Ref. 2, which deals exclusively with PT, has nothing particular to say on this subject. In fact, as stated in Ref. 2, this method is "compatible with any resummation procedure for $\bar{\beta}$ that might emerge from future progress in the field" (although it is not clear whether it will be more advantageous at this stage to deal with the series for the $\bar{\beta}$ functions, rather than with the original series in α_s themselves). Meanwhile, the $\bar{\beta}$-function series can presumably be used as respectable asymptotic series, provided the criterion discussed in point (2) above is satisfied.

(ii) Concerning the connection between the method of Ref. 2 and the standard $1/L$ expansion ($L \equiv \ln Q^2/\Lambda^2$), I first note that the former does give some suggestion to resolve the ambiguity pointed out by Chýla [$1/L$ vs $1/L(b) = (1/L)(1 + b/L)$ as expansion parameters]: namely, it favors the choice $b = 0$ (or small). Indeed, as noticed by Chýla himself, the use of $1/L$ corresponds essentially to the use of 't Hooft RC. Since the latter is, by construction, "well behaved" (no higher-order corrections to the β function), it follows[2] that the $1/L$ expansion (with the FAC choice of Λ; see below) will be reliable (provided, of course, PT is reliable for the $\bar{\beta}$ function associated with σ). On the other hand, using $1/L(b)$ implies using a RC whose β function is not well behaved for large b (since b is essentially the three-loop β-function coefficient), and would, therefore, not give a reliable perturbation expansion, even if $\bar{\beta}$ is well behaved. Furthermore, I note that, contrary to Chýla's statement, the method of Ref. 2 does provide a rationale for the choice of c in Eq. (7) of Ref. 1: namely, it suggests the FAC choice $\gamma_1(c) = 0$ (see Sec. III C in the first paper of Ref. 2). In addition, it gets rid of the need of actually specifying a value for c, since in its full-fledged form it yields results which are *independent* of c.

In summary, the connection between the $1/L$ expansion and the method of Ref. 2 is simply that both of them focus on the Q^2 dependence of σ. The additional virtue of the effective-charge method is that it leads directly to the asymptotic expansion of the inverse function $Q^2/\Lambda^2 = F^{-1}(\sigma)$, hence getting rid both of the ambiguity associated with the choice of parameter in the $1/L$ expansion, as well as of the ambiguity due to the choice of c (i.e., of Λ, and incidentally, of Q—a troublesome problem often considered in the literature as that of the "choice of the relevant momentum scale").

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[1] J. Chýla, preceding paper, Phys. Rev. D **39**, 676 (1989).

[2] G. Grunberg, Phys. Rev. D **29**, 2315 (1984); Phys. Lett. **95B**, 70 (1980); **110B**, 501(E) (1982); Ecole Polytechnique Report No. A510.0782 (unpublished).

[3] For further applications, see G. Grunberg, Phys. Lett. **114B**, 271 (1982); **135B**, 455 (1984); **153B**, 427 (1985); Acta Phys. Pol. B **16**, 491 (1985).

[4] W. Celmaster, in *Gauge Theories and Lepton-Hadron Interactions,* proceedings of the Symposium on Particle Physics, Visegrad, Yugoslavia, 1981, edited by Z. Horwath *et al.* (Centra Research Institute for Physics, Budapest, 1981).

[5] G. Grunberg (in preparation).

[6] S. G. Gorishny, A. L. Kataev, and S. A. Larin, Phys. Lett. B **212**, 238 (1988).