Effective charges in QCD: Where has all the ambiguity gone?

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The method of effective charges has been proposed by Grunberg as a natural solution to the problem of renormalization-scheme dependence of finite-order perturbation-theory results. In this paper I shall argue that this solution relies in a crucial way on certain rather *ad hoc* assumptions. Some statements contained in his paper are furthermore commented upon and the peculiarities of the renormalization-scheme ambiguity for perturbation expansions which are divergent when considered in a fixed renormalization scheme briefly mentioned.

Perturbation expansions in QCD are burdened with two problems. First, because of the unavoidable truncation of these expansions, we face the ambiguity connected with the renormalization-scheme (RS) dependence of the resulting finite-order approximants. Second, since in any fixed RS the perturbation expansions most likely diverge (for a recent discussion of this problem see Ref. 1), we cannot use these finite-order approximants to construct the "full" results of perturbation theory directly order by order. Information from outside perturbation theory is necessary to give the full sums good mathematical meaning.

While most recent papers related to these problems have paid attention to the phenomenologically important first question, there have been attempts to address the second one as well.[2] Although I consider the problem of constructing physically well-motivated and mathematically satisfactorily defined full summations of perturbation expansions to be of primary importance even for the solution of the first problem, I return in this Comment to the paper of Grunberg,[3] where the method based on the concept of "effective charge" is suggested as the best and most natural solution to the mentioned finite-order ambiguity.

As all such attempts (for a closely related one see Ref. 4) this approach also suffers, however, from certain more or less arbitrary assumptions which must be adopted in order to arrive at unique results. This in itself is not principally wrong, as all other remedies such as those of Refs. 4-7 also contain some element of arbitrariness. As stressed some time ago by Politzer,[8] we have to put in a lot of guesswork if we want to get unique results. General criteria such as the principle of mimimal sensitivity (PMS),[5] or that of Ref. 3 can help in detail, where intuition is of no help, but most of the problem must be "solved" by some ansatz based on previous experience or physical insight. For the PMS criterion the merits and shortcomings have been discussed by a number of authors.[8-10] As Ref. 3 contains no such critical discussion the impression might arise that there is nothing arbitrary in the method of effective charges and consequently that this method is better than the others. The purpose of this paper is to demonstrate that this is not the case and that the RS ambiguity reappears even there, in somewhat disguised form.

Let me recall, in the notation of Ref. 3, that we are interested (within the framework of massless QCD) in the perturbation expansion of a given physical quantity $\sigma(Q)$, depending for the sake of simplicity on a single external momentum Q. In the fixed RS this expansion takes the form

$$\sigma(Q)=\alpha_s(1+\sigma_1\alpha_s+\sigma_2\alpha_s^2+\cdots),\quad \alpha_s=\alpha_s(\mu)\ , \qquad (1)$$

where μ is the arbitrary renormalization point. Equation (1) is a special case ($A=0$, $B=d=1$) of the general expression (2.1) in Ref. 3. The renormalized charge (coupling constant) α_s depends, in addition to μ, also on the choice of the so-called renormalization convention[5] (RC) and obeys the equation ($\rho\equiv\alpha_s/4\pi=g^2/16\pi^2$)

$$\frac{d\rho}{d\ln\mu^2}\equiv\beta(\rho)=-\beta_1\rho^2-\beta_2\rho^3-\beta_3\rho^4-\cdots\ . \qquad (2)$$

The first two coefficients β_1,β_2 are fixed once the number n_f of quark favors is given, but all higher-order coefficients β_i, $i\geq 3$, are free and define just the mentioned renormalization convention RC$=\{\beta_i\}$. Fixing the RC thus requires fixing all these β_i, $i\geq 3$ but leaving μ still free, while fixing the RS means that beside them also the value of μ was specified. Employing now μ and β_i to define the RS we work in, (1) should actually read

$$\sigma(Q)=\alpha_s(\mu,\beta_i)[1+\sigma_1(Q/\mu)\alpha_s(\mu,\beta_i)+\sigma_2(Q/\mu,\beta_3)\alpha_s^2(\mu,\beta_i)+\cdots] \qquad (3)$$

as the internal consistency of the perturbation theory determines the dependence of the coefficients σ_i on μ and β_i through relations

$$\pi\sigma_1=(\beta_1/2)\ln(\mu/\Lambda)-\rho_1\ , \qquad (4)$$

$$\pi^2\sigma_2=\pi^2\sigma_1^2+\pi\sigma_1\beta_2/(4\beta_1)+\rho_2-\beta_3/(16\beta_1)\ , \qquad (5)$$

etc., where all the ρ_i's are RS invariants introduced in Ref. 5 and Λ is the only free dimensionful parameter of the theory. In addition to setting the scale it also specifies [Eq. (2.7) in Ref. 3] which of the solutions to (2) we have in mind. To make my following arguments as simple as possible let me for technical reasons furthermore assume $\beta_2=0$. This would be the situation in the imaginary world with a noninteger number $n_f=306/38 \doteq 8.3$ of quark flavors where, however, the asymptotic freedom would still hold as $\beta_1=552/114>0$.

In the notation of Ref. 5 different RS's correspond to different μ and β_i for the same, fixed Λ, but as μ enters α_s always in the ratio μ/Λ it is alternatively possible to fix μ once and for all the considerations (by setting it equal to, say, the external momentum Q) and let Λ change instead. In this notation[11,12] different RS's correspond to different Λ and so RS$=\{\Lambda,\beta_i\}$.

Although the full renormalization group of Stuckelberg and Petermann[13] expresses the invariance with respect to the variation of all the parameters μ,β_i, some part of it may in practice be more relevant to our problem than the other. Nevertheless, any result we may arrive at must be a special case of (3). For instance, there is a claim in Ref. 12, repeated at the beginning of Ref 3, that in fact only the freedom connected with the change of the scale, i.e., either of μ or Λ, is relevant to the problem of RS dependence of finite-order approximants to (3). According to Ref. 12 one should not consider expressions such as (3), but should formally reexpand them in powers of $1/L$, where $L(Q/\Lambda)\equiv\beta_1\ln(Q^2/\Lambda^2)$ is an obvious RC invariant. After doing so we get

$$\sigma(Q)=\frac{1}{L}\left[1+\frac{\gamma_1(c)}{L}+\frac{\gamma_2(c)}{L^2}+\cdots\right], \tag{6}$$

where both the expansion parameter $1/L$ and the coefficients $\gamma_i(c)$ depend on a single dimensionless parameter c, describing the rescaling of Λ: $\Lambda(c)\equiv\exp(-c\beta_1/2)\Lambda(0)$. From (4) we then get

$$\gamma_1(c)=\sigma_1(\mu=Q)+c\beta_1^2 \tag{7}$$

and similarly for higher-order parameters γ_i, $i>1$. Thus if (6) is taken as the result of perturbation theory, the RS ambiguity reduces to the freedom in the choice of c and all the complications connected with coefficients β_i, $i\geq 3$ seem to disappear. When (6) is so simple, why bother with more complicated expressions such as (3)?

The catch is, of course, that the solution of the finite-order RS ambiguities does not reside in expressing physical quantities in terms of explicit RC invariants (and then somehow fixing Λ as well) but in the proof that this can be done *reasonably unambiguously.* As a matter of fact (6) can (in our simplified world with $\beta_2=0$) be obtained from (3) simply by adopting the 't Hooft renormalization convention RC$=\{\beta_i=0, i\geq 3\}$ (Ref. 14), where $\alpha_s=1/L$ identically and so (3) reduces directly to (6).

But instead of $1/L$ why not reexpand in another "manifest" RC invariant such as $1/L'\equiv 1/L+1/L^2$? Both $1/L'$ and $1/L$ depend quite smoothly on the ratio Q/Λ, and the fact that the latter when expressed in terms of elementary functions looks slightly more complicated is certainly no serious argument against it. But if $1/L'$ is acceptable, then so is, obviously,

$$\frac{1}{L(b)}\equiv\frac{1}{L}\left[\frac{1}{L}+\frac{b}{L}\right] \tag{8}$$

for *any b*. If, however, we now calculate the derivative of the quantity $\rho(Q/\Lambda)\equiv 1/[4\pi L(b)]$ with respect to lnQ^2 and recall that

$$\frac{d\dfrac{1}{4\pi L(Q/\Lambda)}}{d\ln Q^2}=-\beta_1\left[\frac{1}{4\pi L}\right]^2, \tag{9}$$

we easily find

$$\frac{d\rho(Q/\Lambda,b)}{d\ln Q^2}=-\beta_1\rho^2(Q/\Lambda,b)+(16\pi^2\beta_1 b)\rho^4(Q/\Lambda,b)+\cdots . \tag{10}$$

Thus the freedom in the choice of b in the definition (8) of the expansion parameter is in fact equivalent to the freedom in the choice of $\beta_3=-16\pi^2\beta_1 b$ in (2). The ambiguity related to β_3 thus reappears in somewhat disguised form, but it does so inevitably. The same holds for other higher-order parameters β_i, $i\geq 3$, as well. The arbitrariness of β_3 in (2) expresses in other words the fact that we have no reason to prefer $b=0$ in (8) to any $b\neq 0$.

Starting with (6) Grunberg then goes on the claim that "The solution to the RS problem proposed in (8) takes the care in the simplest possible way of this remaining difficulty" [15] that is of the choice of c in (6). The bulk of Ref. 3 represents then a detailed elaboration of the above statement, supplemented with a number of applications.

However, as will become clear in a moment the results of Ref. 3 are in fact not of the form (6) and thus cannot directly serve to fix the value of c therein. It is unclear why the expression (6) had been mentioned at all, as it has only remote relation to the method expounded in Ref. 3. That one boils down to a particular choice of the RS$=\{\overline{\Lambda},\overline{\beta}_i\}$, namely, such a choice in which the associated coupling α_s, called the "effective charge" appropriate to the physical quantity $\sigma(Q)$, is equal to $\sigma(Q)$ itself:

$$\sigma(Q)=\alpha_s(Q/\overline{\Lambda},\overline{\beta}_i)\equiv\overline{\alpha} . \tag{11}$$

The coefficients β_i, $i\geq 3$, which are process and RC dependent, are uniquely fixed by this requirement as is the value of the "effective" $\overline{\Lambda}$. Knowing the coefficient σ_1 this $\overline{\Lambda}$ can easily be related to Λ in any fixed RS, say the modified minimal subtraction scheme ($\overline{\text{MS}}$). To find $\overline{\Lambda},\overline{\beta}_i$ we solve the equations $\sigma_j(Q/\Lambda,\beta_i)=0$ for all $j\geq 1$. To second order we get $2\rho_1=\beta_1\ln(Q/\overline{\Lambda})$ which then determines $\overline{\Lambda}$ in terms of ρ_1; to third order $\overline{\beta}_3=16\rho_2\beta_1$ is fixed, etc. As the invariants ρ_i are in general nonzero, we immediately see why the results of this method are not of the form (6), which corresponds to $\beta_i=0$ for all $i\geq 3$.

Now the only reason mentioned in Ref. 3 for the above choice of $\overline{\Lambda}$, $\overline{\beta}_i$ is that it leads to the "simplest" possible form of (3). This , however, is certainly a very vague and subjective criterion, similar in essence to the choice $b=0$ in (8) (in this and only in this aspect is there some common point in Refs. 12 and 13). True, there are in Ref. 3 considerations leading its author to formulate his criterion for the "well-behaved" RC$=\{\beta_i, i\geq 3\}$ (I shall return to it later),

$$|\beta_{i+1}/\beta_i|\approx O|\beta_2/\beta_1| , \tag{12}$$

but they do not prevent us from defining another "effective charge" α_s^1 by means of the relation

$$\sigma(Q)=\alpha_s^1(1+\alpha_s^1) \tag{13}$$

or generally

$$\sigma(Q)=\alpha_s^r(1+r\alpha_s^r) \ . \tag{14}$$

The freedom connected with r is roughly equivalent to the choice of the renormalization point μ or equivalently Λ in (2). Although the β function associated with α_s^r as defined in (14) (denoted β^r) depends on r and does not, for large r satisfy the condition (12), there still remains, even if we accept the criterion (12), the problem of specifying which of the (loosely defined) "small" values of r to choose. The choice advocated in Ref. 3, namely, $r=0$, has no physical justification apart from its alleged "simplicity." The situation is analogous to that of choosing μ in (3). We know that in order to avoid large coefficients we have to set $\mu=\kappa Q$ with κ of the order of unity, but exactly which κ to choose is just the essence of the RS ambiguity. Of course, should we consider Eq. (2) (for all r) exactly to all orders the resulting α_s^r would, upon substituting into (14), yield results independent of r and equal to those of Ref. 3. If, however, (2) is truncated, the results will inevitably depend on the value of r, which we repeat is basically equivalent to the arbitrariness of μ (or equivalently Λ) in (3). And thus again, as in the discussion related to Eq. (6), the original freedom in the choice of the RS$=\{\Lambda,\beta_i\}$ has been traded for something else, this time the arbitrariness in the definition of the "effective charge" [we could obviously add to the right-hand side of (14) also the terms that would simulate the effects of still higher-order parameters β_i, $i\geq 3$].

Let me now return to the criterion (12) of "well-behaved" RC. Denoting $\beta_2/\beta_1\simeq w$ we have, due to (12),

$$\beta_i/\beta_1=(\beta_i/\beta_{i-1})(\beta_{i-1}/\beta_{i-2})\cdots(\beta_2/\beta_1)\simeq w^{i-1} \ , \tag{15}$$

and so if (12) should hold for all $i\geq 3$ the associated β function must in fact be convergent, behaving roughly like the geometric series. This might be a reasonable restriction of the allowed RC, but it rules out such commonly used RC's such as the MS, where the coefficients β_k are expected to grow factorially like k. Moreover, it also excludes the "effective" β function $\bar{\beta}$ of Ref. 3 itself, as it will inevitably diverge, too. If we assume a practical viewpoint and consider (12), as well as all other expansions, to only a finite (and low) order (hoping that the factorial growth of the coefficients β_k will make itself felt only at sufficiently higher orders to allow sensible phenomenology to be based on the first few explicitly known orders), then the MS RC passes, to the known order, the test as $\beta_3/\beta_2=7.92$, while $\beta_2/\beta_1=6.16$ (for $n_f=4$). Until very recently no such test could be done for the "effective" RC of Ref. 3 as it requires a full three-loop calculation of some physical quantity. However, in Ref. 16 a full three-loop calculation of the familiar ratio

$$R(Q)=\frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} \tag{16}$$

has been first reported. The results show that in the $\overline{\text{MS}}$ RS the third-order correction to (16) is very large. In terms of RS invariants the authors of Ref. 16 found (for $n_f=4$) $\rho_2=64.36$, which implies

$$\bar{\beta}_3/\bar{\beta}_2=167.73\doteq 27(\beta_2/\beta_1) \ . \tag{17}$$

So at least for the quantity (16) the "effective" β function $\bar{\beta}$ itself is not "well behaved" according to (12) not only for asymptotic orders but even for the lowest nonuniversal one. It may be a coincidence and other quantities may be better behaved but (17) is a warning that the divergence of perturbation expansions may be of more than academic interest [for (16) the third-order correction represents about 12% of the leading term and is, in the $\overline{\text{MS}}$ RS, about two and a half times bigger than the second-order one].

The method of Ref. 3 might be appropriate, though not unique, for convergent series, but for divergent ones encountered in QCD there are several problems. Besides the one connected with the behavior of lowest orders there is a fundamental question of the limiting value of finite-order approximants defined by means of (11). The algorithm suggested in Ref. 3 can of course be applied at any order but, as has been shown in Ref. 17, the resulting finite-order "effective charge" approximants do, for the case where the associated "effective" β function $\bar{\beta}$ of Ref. 3 is, as expected, divergent, vanish as the number of terms into account in (2) goes to infinity. In order to avoid this unwelcome fact the author of Ref. 3 must either specify at which order his procedure is to be terminated (because at that order we are supposed to be closest to the full sum) or his procedure must otherwise be modified.

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