Comments

Comments are short papers which comment on papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Conformally flat solution with heat flux

A. Banerjee, S. B. Dutta Choudhury, and Bidyut K. Bhui
Department of Physics, Jadavpur University, Calcutta 700 032, India
(Received 4 November 1988)

It is shown that the spherically symmetric solution previously given by Maiti is not the most general conformally flat solution for a shear-free and rotation-free fluid with heat flux. We have presented a more general solution for such a distribution and have considered the conditions of fit at the boundary of a simple spherically symmetric model with heat flux across the boundary with the exterior Vaidya metric.

I. INTRODUCTION

It was shown earlier by Maiti[1] that if the space-time of a shear-free, rotation-free fluid distribution with nonzero heat flux is conformally flat, it can then be shown using comoving coordinates that the curvature of the three-space is independent of the space coordinates. In the next step, however, Maiti incorrectly used for the three-space the form given by Eisenhart[2] who in fact considered the entire space and not the subspace. Thus the metric form wrongly claimed by Maiti to be general for the three-space was the standard form

$$dl^2=\frac{R^2(t)}{(1+kr^2/4)^2}(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2)\ ,$$

where $k=0,\pm1$. The metric of the space-time was further claimed to be spherically symmetric. In the present paper we have derived the more general form of the metric for the space-time having the properties mentioned above. It is not in general spherically symmetric but only reduces to it in certain special cases.

At the end of the paper we have considered a very simple model of a sphere with heat flux in the radial direction and have shown how one can obtain a suitable solution for the function $R(t)$ if the system is to match with the exterior radiating metric of Vaidya.[3]

II. SOLUTION WITH HEAT FLOW

The energy-momentum tensor of a fluid with heat flux is given by

$$T^\mu_\nu=(\rho+p)v^\mu v_\nu-p\delta^\mu_\nu+q^\mu v_\nu+v^\mu q_\nu\ . \tag{2.1}$$

The heat flux vector q^μ is orthogonal to the velocity vector v^μ which in a comoving coordinate system is $v^\mu=(g_{44})^{-1/2}\delta^\mu_4$. Since it is assumed that the fluid distribution is shear-free, rotation-free, and conformally flat it can be shown following Maiti that the curvature of the three-space is independent of the spatial coordinates and is a function of time alone. This follows from the expression

$$R^*_{ij}=\tfrac{2}{3}g_{ij}(\rho-\tfrac{1}{3}\theta^2)\ , \tag{2.2}$$

where $(\rho-\frac{1}{3}\theta^2)$ is a function of time alone. The above consideration leads us to write the metric in a chosen coordinate system as

$$ds^2=V^2(x^\mu)dt^2-U^{-2}(x^\mu)(dx^2+dy^2+dz^2)\ . \tag{2.3}$$

In the above the greek indices stand for 1,2,3,4 and the latin indices for 1,2,3. The relevant coordinate transformation, however, does not disturb the comoving character of the coordinate system. The special form of the metric for the three-space obtained in (2.3) follows from Eisenhart's result. Einstein's field equations from (2.1) and (2.3) yield

$$\begin{aligned}8\pi p&=U^2[-2U^{-1}(U_{,22}+U_{,33})+W^{-1}(W_{,22}+W_{,33})]+X\\&=U^2[-2U^{-1}(U_{,33}+U_{,11})+W^{-1}(W_{,33}+W_{,11})]+X\\&=U^2[-2U^{-1}(U_{,11}+U_{,22})+W^{-1}(W_{,11}+W_{,22})]+X\ ,\end{aligned} \tag{2.4}$$

$$8\pi\rho=2U\sum_i U_{,ii}-3\sum_i(U_{,i})^2+3W^{-2}\dot U^2-\Lambda\ , \tag{2.5}$$

$$UW_{,ij}=2WU_{,ij} \quad (i\neq j)\ , \tag{2.6}$$

$$8\pi q^i=2U^2(\dot{U}W^{-1})_{,i}\ , \tag{2.7}$$

where $W=UV$ and

$$X=3\sum(U_{,i})^2-2UW^{-1}\left[\sum_i U_{,i}W_{,i}\right]+(2U\ddot{U}-3\dot{U}^2-2U^2W^{-1}\dot{U}\dot{W})W^{-2}\ \Lambda\ . \tag{2.8}$$

The cosmological parameter Λ is also included for generality. The subscript indicates the derivative with respect to the relevant space coordinates and the overdot stands for the time derivative.

Using a result of Banerjee and Som[4] for a conformally flat space-time one can write

$$W=UV=A(t)(x^2+y^2+z^2)+A_1(t)x+A_2(t)y+A_3(t)z+A_4(t)\ , \tag{2.9}$$

where A, A_1, A_2, A_3, A_4 are arbitrary functions of time. This form of the metric when used in (2.6) yields

$$U_{,12}=U_{,23}=U_{,31}=0 \tag{2.10}$$

and

$$U_{,11}=U_{,22}=U_{,33}\ . \tag{2.11}$$

The relations (2.10) and (2.11) immediately on integration yield the explicit form of U as

$$U=B(t)(x^2+y^2+z^2)+B_1(t)x+B_2(t)y+B_3(t)z+B_4(t)\ , \tag{2.12}$$

where B, B_1, B_2, B_3, B_4 are arbitrary functions of time.

The Ricci scalar R^* of the three-space orthogonal to v^μ is given by (Ellis[5])

$$R^*=2(8\pi\rho+\Lambda-\theta^2/3) \tag{2.13}$$

which in view of the field equations can in the present case be written as

$$R^*=24BB_4-6(B_1^2+B_2^2+B_3^2)\ . \tag{2.14}$$

The Ricci scalar is clearly a function of time alone as is expected. The above solutions (2.9) and (2.12) represent a fluid distribution, which is shear-free, rotation-free, conformally flat, and has constant spatial curvature. It is thus contradicting the conclusion of Maiti that the only such solution must be spherically symmetric.

III. SPECIAL CASES

$A_i/A=B_i/B=C_i$ $(i=1,2,3)$, where C_i are constants. It is now possible to reduce the metric (2.9) and (2.12) into a spherically symmetric metric by a suitable time transformation and the following transformations of the space coordinates:

$$\bar{x}=x+C_1/2,\quad \bar{y}=y+C_2/2,\ \text{and}\ \bar{z}=z+C_3/2 \tag{3.1}$$

with C_1, C_2, C_3 being constants. Another simple transformation from $\bar{x},\bar{y},\bar{z}$ to the spherical coordinates r,θ,ϕ yields the metric in this special case to the form

$$ds^2=\left[1+\frac{a(t)}{1+k(t)r^2/4}\right]^2dt^2-\frac{R^2(t)}{[1+k(t)r^2/4]^2}(dr^2+r^2d\Omega^2)\ . \tag{3.2}$$

In (3.2) we have written

$$R(t)=\frac{K(t)}{4B},\quad K(t)=\frac{R^*}{6}R^2(t)\ ,\qquad a(t)=\frac{k}{4}\left[\frac{A_4}{A}-\frac{B_4}{B}\right]. \tag{3.3}$$

This form of metric was given by Modak,[6] which reduces to that of Maiti in a more special case of $k=$const that is $\dot{k}=0$. In another special case when $a(t)=0$, $\dot{k}\neq0$ the metric (3.2) represents Bergman's solution.[7] It may be noted, however, that in the metric (3.2) when both $a(t)=0$ and $\dot{k}=0$ the heat flux vanishes and the metric reduces to the well-known Robertson-Walker form.

IV. BOUNDARY CONDITIONS AND A SIMPLE MODEL WITH RADIAL HEAT FLUX

It was shown earlier (Santos[8]) that at the boundary of a fluid sphere with heat flux in the radial direction, the condition of fit for the interior with the exterior Vaidya metric demands

$$p-(q_\mu q^\mu)^{1/2}=0\ . \tag{4.1}$$

A very simple model of a fluid sphere with radial heat flux is given by

$$ds^2=\left[1+\frac{a}{1+\xi r^2}\right]^2dt^2-\frac{R^2(t)}{(1+\xi r^2)^2}(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2)\ , \tag{4.2}$$

where a and ξ are both constants. If, however, a is zero, the heat flux vanishes and the space-time is that of Robertson and Walker.

The conditions of fit at the boundary given in (4.1) is equivalent to

$$T_1^1-e^{(\mu-\nu)/2}T_4^1=0 \tag{4.3}$$

which in view of the field equations can be expressed in the form

$$-\frac{a+z}{R^2}\left[\frac{3z'^2}{z}-\frac{2z'^2}{a+z}+\frac{2zz'}{r(a+z)}-\frac{4z'}{r}\right]-\frac{z}{a+z}\left[\frac{2\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}\right]+\frac{2\dot{R}}{R^2}\left[z'-\frac{zz'}{a+z}\right]=0\ , \tag{4.4}$$

where a is a constant and $z=1+\xi r^2$, ξ being a constant quantity written for $k/4$. The condition (4.4) when considered at the boundary $r=r_0$ in fact yields a solution for $R(t)$ as a function of time. $R(t)$ can be obtained from the differential equation

$$\alpha-\beta(2R\ddot{R}+\dot{R}^2)+\gamma\dot{R}=0\ , \tag{4.5}$$

where α, β, and γ are constants. The solution of $R=R(t)$ obtained from (4.5) determines completely the metric (4.2). A very simple solution obtained by assuming $\ddot{R}=0$, which leads to

$$R=at+b\ . \tag{4.6}$$

Here a and b are constants. The constant a can, however, be expressed in terms of α,β,γ.

ACKNOWLEDGMENTS

B.K.B. thanks the Council for Scientific and Industrial Research, India, for financial support.

[1]S. R. Maiti, Phys. Rev. D **25**, 2518 (1982).
[2]L. P. Eisenhart, *Riemannian Geometry* (Princeton University, Princeton, NJ, 1925).
[3]P. C. Vaidya, Proc. Ind. Acad. Sci. **A33**, 264 (1951).
[4]A. Banerjee and M. M. Som, Int. J. Theor. Phys. **20**, 349 (1981).
[5]G. F. R. Ellis, in *Relativistic Cosmology,* proceedings of the International School of Physics, Enrico Fermi Course XLVII, edited by R. K. Sachs (Academic, New York, 1971).
[6]B. Modak, J. Astrophys. Astron. (India) **5**, 317 (1984).
[7]O. Bergman, Phys. Lett. **82A**, 383 (1981).
[8]N. O. Santos, Mon. Not. R. Astron. Soc. **216**, 403 (1985).