Covariant functional diffusion equation for Polyakov's bosonic string

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I write a covariant functional diffusion equation for Polyakov's bosonic string with the string's world-sheet area playing the role of proper time.

The attempt to formulate a covariant quantum theory of strings in terms of the line functional has as a basic object the string transition amplitude.[1-3] The main idea in this framework is to consider the string world-sheet area playing the role of a proper time. The string propagator, thus, should satisfy a kind of functional diffusion equation in the area space variable.[2]

In this paper we analyze the associated functional diffusion equation in Polyakov's quantum bosonic string theory by taking into account in an explicit way the theory's conformal anomaly.

The transition amplitude for an initial (Euclidean) string state

$$\{(x_\mu^{\text{in}}(\sigma),e^{\text{in}}(\sigma)),\quad 0\le\sigma\le 1\}$$

propagating to a final string

$$\{(x_\mu^{\text{out}}(\sigma),e^{\text{out}}(\sigma)),\quad 0\le\sigma\le 1\}$$

in Polyakov's theory is given by[3]

$$G[c^{\text{out}};c^{\text{in}}]=\int d\mu[g_{ab}]d\mu[\phi_\mu]\exp[-I_0(g_{ab},\phi_\mu,\mu^2,\lambda)]\ , \tag{1}$$

where the covariant string action with a cosmological term μ^2 and a "quark-mass" parameter λ is the Brink–Di Vecchia–Howe action[4]

$$I_0(g_{ab},\phi_\mu,\mu^2,\lambda)=\tfrac{1}{2}\int_D d\sigma\, d\xi(\sqrt{g}\; g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu+\mu_0^2)+\lambda_0\int_{\partial D}ds\ . \tag{2}$$

The string surface parameter domain is taken to be the rectangle $D=\{(\sigma,\xi),0\le\sigma\le 1,0\le\xi<T\}$. The covariant functional measures $d\mu[g_{ab}]d\mu[\phi_\mu]$ are defined over all cylindrical (random) surfaces without holes and handles with the string configurations as nontrivial boundaries: i.e., $\phi_\mu(\sigma,0)=x_\mu^{\text{in}}(\sigma)$; $\phi_\mu(\sigma,T)=X_\mu^{\text{out}}(\sigma)$.

In order to write an area functional diffusion equation for the string propagator, Eq. (1), we rewrite it in a form where the string's world-sheet area plays a role as a string proper time:

$$G[C^{\text{out}},C^{\text{in}}]=\exp\left[-\lambda_0\int_{C^{\text{in}}}ds-\lambda_0\int_{C^{\text{out}}}ds\right]\int_0^\infty dA\; e^{-\mu^2 A}\bar{G}[C^{\text{out}},C^{\text{in}},A]\ , \tag{3}$$

where $\bar{G}[C^{\text{out}},C^{\text{in}},A]$ is the fixed-area string propagator

$$\bar{G}[C^{\text{in}},C^{\text{out}},A]=\int d\mu[g_{ab}]d\mu[\phi_\mu]\delta\left[\left[\int_D d\sigma\, d\xi[\sqrt{g(\sigma,\xi)}]-A\right]\right]\exp[-I_0(g_{ab},\phi_\mu,\mu^2\equiv 0)]\ . \tag{4}$$

The δ-function constraint in Eq. (4) ensures that only the random surfaces with fixed area A contribute.

Let us evaluate the area partial derivative of the area-fixed string propagator: namely,

$$\frac{\partial}{\partial A}\bar{G}[C^{\text{in}},C^{\text{out}},A]=\int d\mu[g_{ab}]d\mu[\phi_\mu]\delta'\left[\int_D d\sigma\, d\xi\sqrt{g(\sigma,\xi)}-A\right] \tag{5}$$

with $\delta'(x)$ being the first derivative of the δ distribution.

At this point we consider the identity

$$-\delta'\left[\int_D d\sigma\, d\xi\sqrt{g(\sigma,\xi)}-A\right]=\lim_{\xi\to 0^+}\left[\frac{1}{2\sqrt{g}g^{00}}\frac{\delta}{\delta g_{00}}\right](\bar{\sigma},\xi)\delta\left[\int_D d\sigma\, d\xi\sqrt{g(\sigma,\xi)}-A\right] \tag{6}$$

which can be easily verified by using the Fourier integral representation for the δ functional and the relationship $\delta\sqrt{g}=\frac{1}{2}\sqrt{g}\; g^{00}\delta g_{00}$.

By substituting Eq. (6) into Eq. (5) we obtain the result

$$\frac{\partial}{\partial A}\bar{G}[C^{\text{out}},C^{\text{in}},A]=\lim_{\xi\to 0^+}\int d\mu[g_{ab}]\left[-\frac{1}{2\sqrt{g}\; g^{00}}\frac{\overleftrightarrow{\delta}}{\delta g_{00}}\right](\bar{\sigma},\xi)F(\phi_\mu,g_{ab})\ , \tag{7}$$

where $\delta/\delta g_{00}(\bar{\sigma},\zeta)$ acts on the measure $d\mu[g_{ab}]$ and on the string-field term

$$F(\phi_\mu,g_{ab})=\int d\mu[\phi_\mu]\exp[-I_0(\phi_\mu,g_{ab},\mu^2\equiv 0)] \ . \tag{8}$$

The $\delta/\delta g_{00}(\bar{\sigma},\zeta)$ functional derivative of the term $F(\phi_\mu,g_{ab})$ is subtle since the covariant functional measure $d\mu[\phi_\mu]$ depends in a nontrivial way on the metric $g_{ab}(\sigma,\zeta)$ as a consequence of its definition as the functional volume element associated with the covariant functional metric[5]

$$\|\delta\phi_\mu\|^2=\int_D(\sqrt{g}\ \delta\phi_\mu\delta\phi_\mu)(\sigma,\zeta)d\sigma\, d\zeta \ . \tag{9}$$

Its evaluation proceeds in the following way. The $g_{00}(\bar{\sigma},\zeta)$ functional derivative of the Brink–Di Vecchia–Howe action without the boundary term is trivially given by the (0,0) component of the stress-energy tensor:[3]

$$\frac{\delta}{\delta g_{00}(\bar{\sigma},\zeta)}I_0(g_{ab},\phi_\mu,\mu^2\equiv 0)=(\partial_0\phi^\mu\partial_0\phi^\mu-\tfrac{1}{2}g_{00}g^{cd}\partial_c\phi^\mu\partial_d\phi^\mu)(\bar{\sigma},\tau) \ . \tag{10}$$

In the conformal gauge $g_{ab}=e^\rho\delta_{ab}$ Eq. (10) takes the simple form below at the boundary limit $\zeta\to 0^+$ with $\pi_\mu^{\rm in}(\bar{\sigma})=\lim_{\zeta\to 0^+}\partial_0\phi^\mu(\bar{\sigma},\zeta)$ being the string canonical momentum and $x_\mu^{\prime\,\rm in}(\bar{\sigma})=\lim_{\zeta\to 0^+}\partial_1\phi_\mu(\bar{\sigma},\zeta)$:

$$\tfrac{1}{2}[\pi_\mu^{\rm in}(\bar{\sigma})^2-x_\mu^{\prime\,\rm in}(\sigma)^2] \ . \tag{11}$$

Let us evaluate the $\delta/\delta g_{00}(\bar{\sigma},\zeta)$ functional derivative of the functional measure $d\mu[\phi_\mu]$ in the conformal gauge where the results are given by local expressions.

The Frechet derivative of the functional measure is (by its definition) given by the relationship

$$e^{-\rho(\bar{\sigma},\bar{\zeta})}\frac{\delta}{\delta\rho(\bar{\sigma},\bar{\zeta})}(d\mu[\phi_\mu;e^\rho\delta_{ab}])=\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}(d\mu[\phi_\mu;e^{\rho+\delta h}\delta_{ab}]-d\mu[\phi_\mu,e^\rho\delta_{ab}]) \tag{12}$$

with $\delta h=\epsilon\delta(\sigma-\bar{\sigma})\delta(\zeta-\bar{\zeta})$.

Since we have, as a straightforward consequence of the theory's covariance [see Eq. (9)],

$$d\mu[\phi_\mu,e^{\rho+\delta h}\delta_{ab}]=d\mu[e^{\delta h/2}\phi_\mu,e^\rho\delta_{ab}] \tag{13}$$

and the effect of the functional measure $d\mu[\phi^\mu]$ under a conformal rescaling can be exactly evaluated,[6]

$$d\mu[\phi^\mu,e^{\rho+\delta h}\delta_{ab}]=d\mu[\phi^\mu,e^\rho\delta_{ab}]\exp\left[\frac{D}{24\pi}\left(\int_D\tfrac{1}{2}(\partial_a\rho)(\partial_a\delta h)+\mu^2(\epsilon)e^\rho\delta h+\lambda_0(\epsilon)\int_{\partial D}e^\rho\delta h\right)\right] , \tag{14}$$

we thus have the result

$$e^{-\rho(\bar{\sigma},\zeta)}\frac{\delta}{\delta\rho(\bar{\sigma},\zeta)}d\mu[\phi^\mu,e^\rho\delta_{ab}]=\frac{D}{24\pi}[R(\rho(\bar{\sigma},\zeta))+\mu_0^2(\epsilon)+\lambda_0(\epsilon)]d\mu[\phi^\mu,e^\rho\delta_{ab}] , \tag{15}$$

where $R(\rho(\bar{\sigma},\zeta))=e^{-\rho(\hat{\sigma},\zeta)}\Delta\rho(\bar{\sigma},\zeta)$ is the scalar of curvature associated with the intrinsic metric $e^{-\rho}\delta_{ab}$ and $\mu_0(\epsilon),\lambda_0(\epsilon)$ are infinite constants which depend on the regularization scheme used to evaluate the functional determinants of two-dimensional Beltrami-Laplace operators in Polyakov's effective action.[3]

It is instructive to remark that one can implement the above calculation without choosing the conformal gauge since the measure functional derivative may be alternatively defined by the ratio

$$\frac{\delta}{\delta g_{00}(\bar{\sigma},\bar{\zeta})}d\mu[\phi_\mu,g_{ab}]=\frac{\det^{-D/2}[\Delta g_{ab}+\delta g_{00}(\bar{\sigma},\bar{\zeta})]}{\det^{-D/2}(\Delta g_{ab})} \tag{16}$$

and we have the general covariant result

$$\ln\det(\Delta g_{ab})=\frac{1}{48\pi}\int_D d\sigma\, d\zeta\int_D d\sigma' d\zeta'(\sqrt{g}\ R)(\sigma,\zeta)\Delta_{g_{ab}}^{-1}(\sigma-\sigma',\zeta-\zeta')(\sqrt{g}\ R)(\sigma',\zeta') , \tag{17}$$

where $\Delta_{g_{ab}}^{-1}(\sigma-\sigma',\zeta-\zeta')$ denotes the Green's function of the Laplace Beltrami operator $\Delta_{g_{ab}}=(1/\sqrt{g})\partial_a(g^{ab}\partial_b)$ in the presence of the intrinsic metric $\{g_{ab}\}$.

However, it is important to note that only in the conformal gauge do our calculations take a local form as a functional of the intrinsic metric tensor. This is the technical reason that we use the conformal gauge at the end of our calculations.

Finally the $g_{00}(\bar{\sigma},\zeta)$ derivative of $d\mu[g_{ab}]$ in the conformal gauge is easily evaluated:[3,5]

$$e^{-\rho(\bar{\sigma},\zeta)}\frac{\delta}{\delta\rho(\bar{\sigma},\zeta)}d\mu[g_{ab}=e^\rho\delta_{ab}]=-\frac{26}{24\pi}[R(\rho(\bar{\sigma},\zeta))+\mu_0^2(\epsilon)+\lambda_0(\epsilon)]d\mu[g_{ab}=e^\rho\delta_{ab}] , \tag{18}$$

since we have explicitly

$$d\mu[g_{ab}=e^{\rho}\delta_{ab}]=D[\rho]\exp\left[-\frac{26}{48\pi}\int_D[\tfrac{1}{2}(\partial_a\rho)^2+\mu^2(\epsilon)e^{\rho}]+\lambda(\epsilon)\int_{\partial D}e^{\rho}ds\right]$$

$$\left[D[\rho]=\prod_{(0,\zeta)\in D}d\rho(\sigma,\zeta)\right] . \quad (19)$$

By grouping together Eqs. (11), (15), (18), and introducing the covariant string commutation relation[1]

$$[\pi^{\mu}_{\rm in}(\sigma),x^{\nu}(\sigma')]=\frac{i\delta(\sigma-\sigma')}{\hbar e_{\rm in}(\sigma)} \quad \{e_{\rm in}(\sigma)=\lim_{\zeta\to 0^+}\exp[+\rho(\sigma,\zeta)]\}$$

which produces the Schrödinger representation $\pi^{\mu}_{\rm in}(\sigma)=-i\hbar e^{-1}_{\rm in}(\sigma)\delta/\delta x^{\rm in}_{\mu}(\sigma)$, we can finally write Eq. (7) as a covariant diffusion equation for Polyakov's bosonic string which takes into account in a explicitly and local way the presence of the world-sheet intrinsic metric

$$\exp[\rho(\sigma,\zeta)]\left[-\frac{1}{2}\frac{\delta^2}{e_{\rm in}(\bar\sigma)^2\delta x^{\rm in}_{\mu}(\bar\sigma)\delta x^{\rm in}_{\mu}(\bar\sigma)}-\tfrac{1}{2}|x'^{\rm in}_{\mu}(\bar\sigma)|^2+\frac{26-D}{24\pi}\lim_{\zeta\to 0^+}[R(\rho(\bar\sigma,\zeta))+C_{\infty}]\right]\bar G[C^{\rm out},C^{\rm in},A]$$

$$=\frac{\partial}{\partial A}\bar G[C^{\rm out},c^{\rm in},A] . \quad (20)$$

The above-written string wave equation is the main result of this paper.

Let us comment that at $D=26$, where the invariance of Polyakov's string theory under the world-sheet diffeomorphism group is restored (otherwise it is partially broken to the quotient group of the complete diffeomorphism group by the Weyl diffeomorphism subgroup) we can fix $e_{\rm in}(\sigma)=1$ and the above area diffusion equation takes the simple form

$$\frac{\partial}{\partial A}\bar G[C^{\rm out},C^{\rm in},A]=\left[-\frac{1}{2}\frac{\delta^2}{\delta x^{\rm in}_{\mu}(\bar\sigma)\delta x^{\rm in}_{\mu}(\bar\sigma)}-\tfrac{1}{2}|x'^{\rm in}_{\mu}(\bar\sigma)|^2\right]\bar G[C^{\rm out},C^{\rm in},A] . \quad (21)$$

A simple functional solution of Eq. (21) is

$$\bar G[C^{\rm out},C^{\rm in},A]=e^{-EA}\Phi[C^{\rm in}]\Phi[C^{\rm out}] , \quad (22)$$

where the string functional $\Phi[C^{\rm in}]$ satisfies the string wave equation

$$\left[-\frac{1}{2}\frac{\delta^2}{\delta x^{\rm in}_{\mu}(\bar\sigma)\delta x^{\rm in}_{\mu}(\bar\sigma)}-|x'^{\rm in}_{\mu}(\bar\sigma)|^2\right]\Phi_E[C^{\rm in}]$$

$$=-E\Phi_E[C^{\rm in}] . \quad (23)$$

Here we can see that the possible values of E are exactly the eigenvalues of the "functional Klein-Gordon" operator on the left-hand side of Eq. (23) which can be identified with the $-L_0$ Virasoro constraint written in the Schrödinger representation.[7]

Finally, similar results have been obtained in Refs. 2, 8, and 9 for the case of the usual Nambu string at the critical dimension by using WKB and Hamilton-Jacobi techniques to solve directly Eq. (23) for $E=0$.

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