Can the electroweak vacuum be unstable?

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The electroweak vacuum need not be absolutely stable. For certain top-quark and Higgs-boson masses in the minimal standard model, our vacuum is instead metastable with a lifetime exceeding the present age of the Universe. It has been suggested that a metastable vacuum is generally ruled out because high-energy cosmic-ray collisions would have long ago induced its decay. I argue that the reasoning for this conclusion is erroneous. As a consequence, upper bounds on the top-quark mass derived from stability arguments are relaxed. Also presented is an analytic method for accurately approximating the lifetime of the vacuum from the effective potential without solving for the O(4) bounce solution numerically.

I. INTRODUCTION

In Weinberg-Salam theory, the weak gauge group is broken by a Higgs sector whose renormalizable potential is of the form

$$V(\phi)=-\tfrac{1}{2}\mu^2\phi^2+\tfrac{1}{4}\lambda\phi^4 . \tag{1.1}$$

This potential receives radiative corrections and the vacuum expectation of ϕ is determined by the effective potential which includes these corrections. One-loop corrections from bosons, such as the Higgs boson, give contributions of the form $\lambda^2\phi^4\ln\phi$ times numerical factors. These corrections dominate the usual $\lambda\phi^4$ at large ϕ. One-loop corrections from fermions give contributions of the form $-g_y^4\phi^4\ln\phi$ where the minus sign is due to Fermi statistics. If the Yukawa couplings are large enough, the fermion contributions will dominate over the bosonic ones at large ϕ with the result that our vacuum is only metastable.[1] The effective potential in such a case is depicted schematically in Fig. 1. Generally, however, the scale B at which the potential becomes unstable is very much larger than the scale A of the false vacuum.

Flores and Sher[2] have noted that our vacuum need not be absolutely stable; a metastable vacuum is acceptable if its lifetime exceeds that of the Universe. It is also necessary that the Universe can be trapped in the false vacuum in the first place, and they argue that this is plausible. In particular, the case at hand is different from the case of the Linde-Weinberg bound.[3] Below the Linde-Weinberg bound, there is a metastable vacuum at zero temperature which disappears at high temperature. For the cases examined in this paper, however, the metastable vacuum does not destabilize at high temperature.

The vacuum decays by quantum tunneling to form bubbles of the unstable phase which then expand classically to absorb all of the metastable phase. There are two types of forces acting on a bubble: the potential-energy advantage of the interior over the false vacuum, and its surface tension. The potential energy favors expansion of the bubble and grows with the volume; the surface tension favors contraction and grows with the surface area (or as the radius if the bubble has thick walls). Thus, small bubbles are dominated by surface tension and collapse. Large bubbles are dominated by the potential energy and expand. The quantum tunneling must create a bubble large enough that the bubble will continue to expand.

In general, the larger the top-quark mass or smaller the Higgs-boson mass, the more unstable the potential and the shorter the lifetime of our vacuum. Flores and Sher translated the constraint on the lifetime into a constraint on the top-quark and Higgs-boson masses.[2,4] Figure 2 shows my results for these constraints. Below the lower solid curve, the vacuum is absolutely stable. Between the two solid curves it is metastable with a lifetime exceeding the age of the Universe. These curves apply only to the minimal standard model with a single Higgs doublet that is valid up to $\Lambda=10^{19}$ GeV. The upper curve is also shown for different choices of the cutoff scale Λ, whereas the dependence of the lower solid curve on cutoff scale has been examined in Ref. 5. The lifetime has been computed at zero temperature. The curve corresponding to the lifetime constraint is significantly different from that of Ref. 2, perhaps due to the use of more modern results for the effective potential.

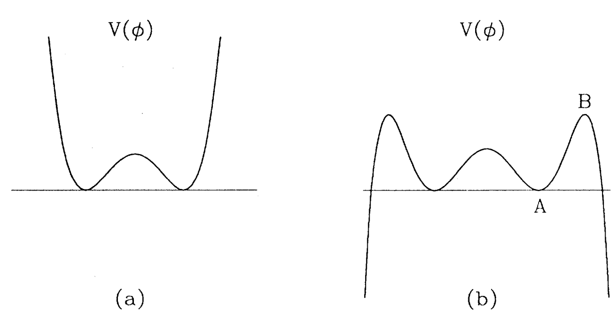

FIG. 1. The effective potential (a) when our vacuum is absolutely stable and (b) when fermion masses are large enough that it is not.

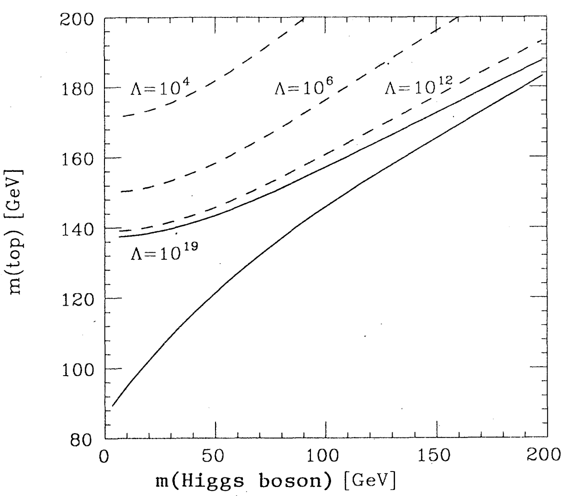

FIG. 2. Below the lower solid curve, our vacuum is absolutely stable. Between the solid curves it is unstable but with a lifetime greater than the age of the Universe. Both solid curves correspond to a cutoff of $\Lambda = 10^{19}$ GeV. The dashed lines show how the upper curve varies as Λ is taken to be 10^{12}, 10^{6}, and 10^{4} GeV.

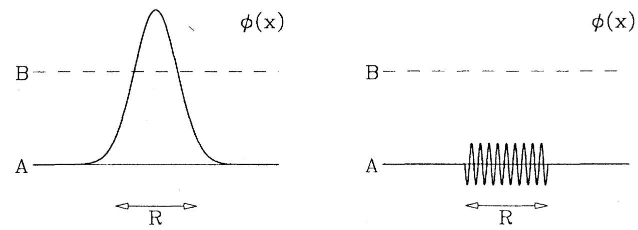

FIG. 3. Qualitative drawings of two configurations with equal energy density. The line A is the metastable vacuum; the line B is where the potential turns over and becomes unstable.

There is an energy barrier which separates the metastable vacuum from the region of instability. So far, I have discussed decay by quantum tunneling through that barrier. The lifetime can be long because quantum tunneling is an *exponentially* damped process parametrized by

$$\text{amplitude} \sim \exp[-(\Delta E)(\Delta t)] \sim \exp(-S_E) \;, \qquad (1.2)$$

where Δt is the time that energy conservation is violated and ΔE is how much it is violated by. The product is equivalent to the Euclidean action S_E (Ref. 6). There is another way to make the transition, however. If one could concentrate enough energy to go over the barrier *classically,* then there need be no exponential damping. In particular, Sher and Zaglauer[7] have argued that high-energy cosmic rays induce vacuum decay and that, except for a very thin sliver of parameter space, the possibility of a metastable vacuum is ruled out. One may then obtain stricter bounds on the top-quark and Higgs-boson masses in the single-Higgs-boson model.[5,7] The thesis of this paper is the refutation of this conclusion. I shall now synopsize their argument.

The object is to make a bubble of some radius R that is large enough to grow and absorb the metastable vacuum. In the process, the system must pass over the energy density barrier $\epsilon = V_{\max}$ of the potential. To make a bubble of size R classically then takes energy $E \sim \epsilon R^3$, but to get the proper energy density, ϵ, is not sufficient. For instance, Fig. 3 shows qualitatively two configurations with equal energy density. One is smoothly varying, has a large amplitude, and probes the region where the potential is unstable. The other has all of its energy in high frequencies and, as a result, does not achieve a large amplitude and does not know about the instability. To make the bubble, the energy therefore needs to be in frequencies k of order $1/R$. The typical number of quanta in such a classical configuration will be of order RE since each quanta carries energy $E_q \sim k \sim 1/R$. (I am assuming $k \gtrsim m_H$ here.) If $RE \gg 1$, one must address the problem of how likely it is for a cosmic-ray collision to create a large number of Higgs particles in a small region of space. However, Sher and Zaglauer argue that RE is less than one. Therefore, producing even a single Higgs boson of the right momentum seems likely to induce a transition. To come to this conclusion, they take for R the size of the "critical" bubble preferred when the false vacuum decays by quantum tunneling. Their small value of RE is a consequence of the fact that this R is very small compared to the weak scale and almost always much smaller than the scale $V_{\max}^{-1/4}$ fixed by the energy barrier. Finally, I should note that the rough nature of these approximations is mitigated by the fact that the space-time volume of the past light cone of our Universe is order e^{404} in units of the weak scale.[2] Since even a single bubble would have destroyed the false vacuum, only a gross estimate of the order of magnitude of the bubble creation rate is needed.

I shall argue that the bubble preferred for decay by quantum tunneling involves passing through configurations with energy density much greater than $\epsilon = V_{\max}$, resulting in $RE \gg 1$. Alternatively, if one passes over the barrier through configurations with energy density $\epsilon = V_{\max}$, I shall argue that R must be much larger than the value preferred by quantum tunneling and again $RE \gg 1$. The problem arises because of a confusion of the term "critical" bubble and because of a great difference between minimum-energy and minimum-action paths for crossing the barrier.

In the next section I briefly review the renormalization-group-improved effective potential for the Higgs boson. I then elaborate on a simple approximation to the potential which will highlight the qualitative arguments and about which much can be said analytically. The conclusions are then checked against various quantities derived numerically with the full potential. In Sec. III, I briefly consider alternative mechanisms for cosmic rays to induce vacuum decay. Without attempting a serious analysis, I indicate that the rate for producing *many* Higgs quanta to make a bubble is likely exponentially small. Since there is no longer a strong argument that the electroweak vacuum must be absolutely stable, I reexamine the constraints on top-quark and Higgs-boson masses in Sec. IV. There is a good analytic approximation to the tunneling rate in terms of the effective potential, and it agrees with my numerical work.

II. MINIMUM ENERGY VERSUS MINIMUM ACTION

The one-loop effective potential may be improved using the renormalization group,[8] and this procedure has been carried out in detail in the case of the Weinberg-Salam model.[9] The effective potential has the form

$$V(\phi)_{\rm eff}=-\tfrac{1}{2}\mu^2(t)G^2(t)\phi^2+\tfrac{1}{4}\lambda(t)G^4(t)\phi^4\ , \qquad (2.1)$$

where μ^2 and λ run logarithmically with ϕ through

$$d\mu^2(t)/dt=\mu^2(t)\beta_{\mu^2}(g(t),\lambda(t))\ ,$$
$$d\lambda(t)/dt=\beta_\lambda(g(t),\lambda(t))\ , \qquad (2.2)$$
$$t=\ln(\phi/\sigma)\ ,$$

and σ is the weak scale $\sigma=247$ GeV. $G(t)$ is defined in terms of the anomalous dimension γ of ϕ by

$$G(t)=\exp\left[-\int_0^t\gamma(t')dt'\right]\ . \qquad (2.3)$$

Explicit formulas may be found in Ref. 9.

At large ϕ, the potential is dominated by the quartic term in Eq. (2.1). The instability appears if $\lambda(t)$ runs to negative values as shown in Fig. 4. It is interesting that the potential described by Eq. (2.1) always turns upward again, leaving a new vacuum at very large ϕ. This new scale is exponentially larger than the weak scale, and in many cases it lies beyond the Planck scale and hence beyond any scale where the Higgs model might apply. The eventual stabilization of the potential is not, however, essential to the matter at hand.

A. $-\kappa\phi^4$ theory

I shall now focus on a simple toy model that approximates the true effective potential. For large ϕ, the potential can be approximated by the quartic term. The coefficient $\lambda(t)G^4(t)$ is slowly varying, so I shall approximate it to be a constant. As I am interested in unstable potentials, the constant is negative. The model is

$$V(\phi)=-\tfrac{1}{4}\kappa\phi^4\ , \qquad (2.4)$$

$$S_E=\int d^4x\left[\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{4}\kappa\phi^4\right]\ . \qquad (2.5)$$

This model has been considered by Lee and Weinberg.[10] I shall study it purely semiclassically, ignoring, in particular, any radiative corrections to the potential.

At first sight, this system may look completely unstable because there is no energy barrier in $V(\phi)$ to trap the system at $\phi=0$. The false vacuum $\phi=0$ is, nonetheless, stable against local perturbations. The stability arises from the surface energy required to make a bubble. More specifically, consider the creation of an initially static bubble of size R and amplitude ϕ_0. Its energy is

$$E=\int d^3x\left[\tfrac{1}{2}(\nabla\phi)^2-\tfrac{1}{4}\kappa\phi^4\right]$$
$$\sim R^3\left[\left[\frac{\phi_0}{R}\right]^2-\kappa\phi_0^4\right]$$
$$\sim\phi_0^2R-\kappa\phi_0^4R^3\ , \qquad (2.6)$$

where I have ignored factors of 2 and approximated the gradients by assuming the bubble has thick walls rather than thin ones. Bubbles are thick walled when the separation of the true- and false-vacuum energies is large compared to the potential barrier between them or, in the extreme case, when the potential is unbounded below.

For small R or ϕ_0, the first term of Eq. (2.6) dominates, creating an energy barrier to the creation of bubbles. This is the surface tension term, but notice that it grows as R in the thick-wall case rather than as the surface area. The dependence of bubble energy on R and ϕ_0 is sketched in Fig. 5. Notice that there is a ridge of "critical" configurations. Bubbles started on one side of this ridge will collapse back to the false vacuum; bubbles started on the other side will grow, destabilizing the false vacuum. The decay of the false vacuum occurs by quantum tunneling under the ridge.

All of the parameters of this system follow from the classical scale invariance of Eq. (2.5). By rewriting

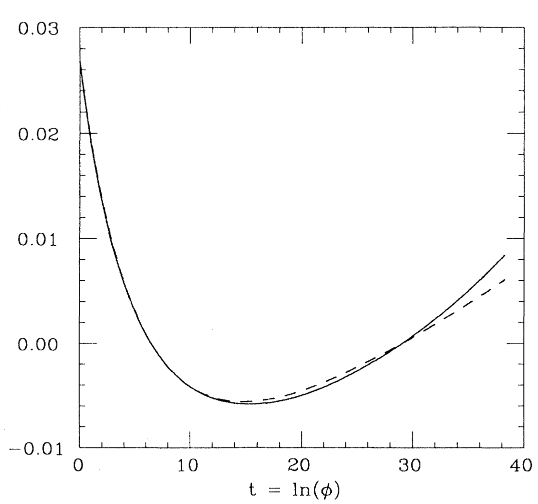

FIG. 4. λ (dashed line) and λG^4 (solid line) as a function of $t=\ln(\phi/\sigma)$ for $m_H=50$ GeV and $m_t=125$ GeV.

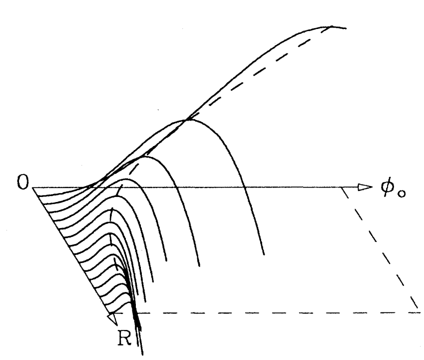

FIG. 5. The qualitative dependence of bubble energy on size and amplitude for $V(\phi)=-\kappa\phi^4/4$.

$$S_E=\frac{1}{\kappa}\int d^4x\,[\tfrac{1}{2}(\partial\bar{\phi})^2-\tfrac{1}{4}\bar{\phi}^4],\quad \phi=\frac{\bar{\phi}}{\sqrt{\kappa}}\ , \tag{2.7}$$

it follows on dimensional grounds that the ridge is described by

$$\phi_0\sim\frac{1}{\sqrt{\kappa}R},\quad E\sim\frac{1}{\kappa R}\ , \tag{2.8}$$

that the Euclidean action for tunneling under it is independent of R, and that

$$S_E\sim\frac{1}{\kappa}\ . \tag{2.9}$$

It will be useful to express the relation between the barrier and the tunneling probability more graphically as follows. Consider tunneling at fixed R and write the energy, including time derivatives:

$$E\sim R^3\left[\tfrac{1}{2}\dot{\phi}_0^2+\left[\frac{\phi_0}{R}\right]^2-\kappa\phi_0^4\right]\ . \tag{2.10}$$

For a simple quantum-mechanics problem of the form

$$E=\tfrac{1}{2}m\dot{x}^2+V(x)\ , \tag{2.11}$$

the tunneling probability grows with both the height and width of the barrier as $\int dx\sqrt{2mV(x)}\sim\Delta x\sqrt{mV_{\max}}$. Rescale Eq. (2.10) so that the kinetic term is normalized to one:

$$E\sim\tfrac{1}{2}\dot{\xi}^2+\left[\frac{\xi}{R}\right]^2-\kappa\frac{\xi^4}{R^3}\ ,\quad \xi\equiv R^{3/2}\phi_0\ . \tag{2.12}$$

The static energy is plotted versus ξ and R in Fig. 6. [In general, I shall use the term potential energy for contributions from $V(\phi)$ and static energy for contributions from $(\nabla\phi)^2+V(\phi)$.] One can see now that the energy barrier gets lower as $R\to\infty$ but also gets wider. It does so in such a way that the tunneling probability does not change. If one wishes to cross the barrier classically, using as little energy as possible, then $R\to\infty$ is preferred; if one crosses by tunneling, then there is no preference of scale. Because tunneling depends on the *width* as well as the height of the energy barrier, there is no reason why the minimum-energy and minimum-action paths for crossing the barrier must be characterized by the same scale. This observation is the key to my argument.

Note that, in the case at hand, the number of quanta in one of the critical configurations along the ridge is

$$N_q\sim RE\sim\frac{1}{\kappa}\sim S_E\ . \tag{2.13}$$

Thus, if the vacuum is very long lived, then many quanta are required to pass over the barrier classically.

Before extending the toy model, let me note that the minimum-action solution can be found analytically. With a radial ansatz, the Euclidean equations of motion are

$$\phi''=-\frac{3}{r}\phi'-\kappa\phi^3,\quad r^2=\tau^2+\bar{x}^2 \tag{2.14}$$

and are solved by[10]

$$\phi(r)=\left[\frac{2}{\kappa}\right]^{1/2}\left[\frac{2R}{r^2+R^2}\right]\ , \tag{2.15}$$

$$S_E=\frac{8\pi^2}{3\kappa}\ , \tag{2.16}$$

where R is arbitrary. This is the O(4) bounce solution. The static energy of the configuration is plotted versus Euclidean time in Fig. 7. One can see that the solution passes through the energy barrier of size $E\sim 1/\kappa R$. The turning point on the other side of the barrier is the bubble corresponding to the $\tau=0$ slice of the solution. Once this configuration is reached, it will expand classically. Note that the bounce solution double counts the transition, and so $\exp(-S_E)=|\exp(-S_E/2)|^2$ gives the rate rather than the amplitude. I shall show later how Eq. (2.16) gives an excellent approximation to the action in the real Weinberg-Salam theory.

B. $m^2\phi^2-\kappa\phi^4$ and other variations

Now consider what happens if a potential-energy barrier is included in $V(\phi)$. For instance, consider

$$V(\phi)=\tfrac{1}{2}m^2\phi^2-\tfrac{1}{4}\kappa\phi^4\ . \tag{2.17}$$

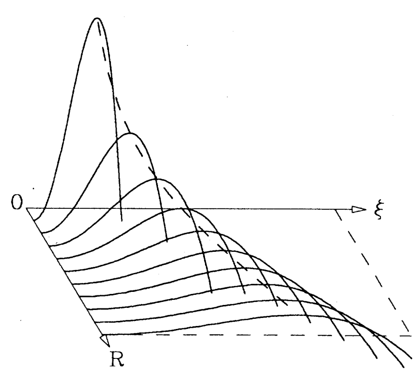

FIG. 6. The qualitative dependence of bubble energy on size and ξ for $V(\phi)=-\kappa\phi^4/4$.

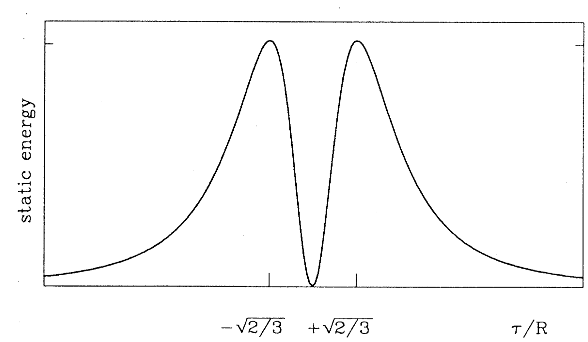

FIG. 7. The static energy of the O(4) bounce solution vs Euclidean time τ for $V(\phi)=-\kappa\phi^4/4$. The maximum is $E_{\max}=\sqrt{3/5}(12\pi^2)/(25\kappa R)$.

For large ϕ, the new term can be ignored and the ridge of critical bubbles is again given by Eq. (2.8). Large bubbles with small amplitude, however, no longer expand because the potential energy inside is positive rather than negative. This cuts off the vanishing of the static energy barrier at large R, and Fig. 6 is replaced by Fig. 8. There is now a well-defined minimum-energy barrier with size and energy fixed by the scale m:

$$R(E_{\min}) \sim \frac{1}{m}, \quad E \sim \frac{m}{\kappa} . \tag{2.18}$$

Note that there is a static, unstable solution to the classical equations of motion that corresponds to sitting on the minimum-energy barrier. Making a radial ansatz, this solution may be found numerically by solving

$$\phi'' + \frac{2}{r}\phi' = \frac{dV}{d\phi} . \tag{2.19}$$

This is the O(3) bounce solution.

By increasing the energy, the new term also increases the tunneling action. Tunneling is no longer indifferent to scale. Since the new term becomes negligible for large ϕ, tunneling prefers small-R, large-ϕ configurations over large-R ones, and so

$$R(S_{\min}) \to 0, \quad S_E \to \frac{8\pi^2}{3\kappa} . \tag{2.20}$$

Similarly, the number of quanta needed to pass over the barrier classically is still bounded below by $\sim 1/\kappa$. For $R \ll 1/m$, the situation is the same as before. For $R \gg 1/m$, the barrier energy is order $R^3 m^4/\kappa$ and the energy per quanta roughly m, giving order $R^3 m^3/\kappa \gg 1/\kappa$ quanta.

In the effective potential for the Weinberg-Salam theory, the radius preferred for tunneling is eventually cut off because the unstable potential eventually turns around to stabilize at a new, deeper vacuum. The radius and action can be approximated very easily. Using Eq. (2.16), the tunneling probability is maximized when the effective κ is maximized. So, from Eq. (2.1),

$$S_E \approx \frac{8\pi^2}{3\kappa_{\text{eff}}}, \quad \kappa_{\text{eff}} = \max[-\lambda(t) G^4(t)] > 0 . \tag{2.21}$$

The validity of this approximation can be seen by a simple variational argument. Consider the configuration of Eq. (2.15) but cut it off with an exponential once ϕ drops a decade or so from its central value. κ is roughly constant over this range and the action is still given by Eq. (2.16) to good approximation. This action provides an upper bound to the minimum tunneling action. Alternatively, consider the exact minimum-action solution for the real problem. Until ϕ drops a decade or so, the equations of motion are well approximated by the $-\kappa\phi^4$ theory and so the solution must be close to Eq. (2.15). The contribution to the action from this region shows that Eq. (2.16) must, to good approximation, also be a lower bound to the minimum tunneling action. For the applications of interest here, Eq. (2.21) is good to within 1%.

The minimum-energy and minimum-action paths for crossing the barrier have manifested at two very different scales. The minimum-energy path is determined by the scale at which the potential first becomes unstable, the minimum-action path at the exponentially larger scale where $-\lambda(t) G^4(t)$ reaches its maximum. The energy of the former times the radius of the latter is very small but does not give the number of quanta required to cross the barrier classically. Instead, the number of quanta is of order $1/\kappa_{\text{eff}}$ and is large. To verify this numerically, I have plotted RE in Fig. 9 for the configuration corresponding to the minimum-energy barrier. The parameters correspond to the upper solid curve of Fig. 2; the number of quanta increases for lower values of m_t. R has been taken as the radius where the solution reaches half of its central value. I also consider another characteristic path for crossing the barrier: the minimum-action solu-

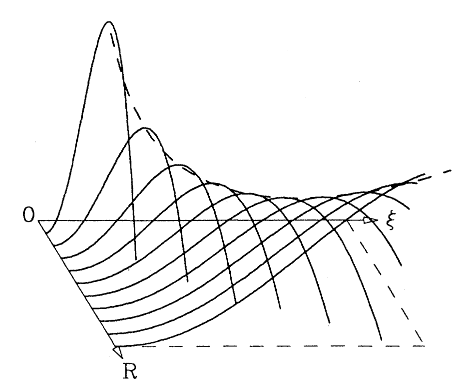

FIG. 8. The qualitative dependence of bubble energy on size and ξ for $V(\phi) = m^2\phi^2/2 - \kappa\phi^4/4$.

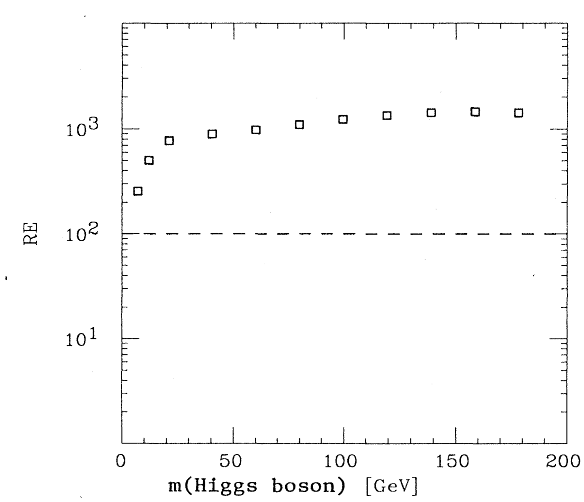

FIG. 9. The values of RE along the upper solid curve of Fig. 2. The points are computed numerically for the minimum-energy solution. The dashed curve is for the time slice of the minimum-action solution as it crosses the static-energy barrier.

tion. In this case, RE is taken for the time slice where the solution crosses the barrier, i.e., where the static energy is maximum, and I have plotted an analytic result derived from the $-\kappa\phi^4$ approximation:

$$RE = \tfrac{9}{50} S_E \ . \tag{2.22}$$

The reader might worry that the estimate of the number of quanta as RE could leave out important factors such as $1/(2\pi)^3$ which would bring the number down dramatically. In the perturbative regime, a more precise definition of the average number of quanta in a classical state would be

$$N_q = \int \frac{d^3k}{(2\pi^2)} \frac{\omega_k}{2} |\tilde{\phi}(k)|^2 \ . \tag{2.23}$$

As an example, reconsider the $-\kappa\phi^4$ model. Taking $\omega_k = k$, one finds with the above definitions that

$$N_q = \tfrac{9}{20} S_E = \tfrac{5}{2} RE \ . \tag{2.24}$$

There are no large numerical factors. As to the exact number, however, one should keep in mind that the number of quanta is only a rough concept outside of perturbation theory.

Based on the numerical and analytic work, I conclude that critical bubbles contain at least ~ 100 quanta for a sufficiently long-lived vacuum. The production of a single quanta in a high-energy collision will not induce vacuum decay.

III. PRODUCING MANY HIGGS BOSONS

Since the formation of critical bubbles requires a large number of quanta, it is worth considering whether a high-energy collision could produce a large number of Higgs bosons inside the relevant volume. This is not a question I shall address in depth, but I have a few observations. Remember that the Higgs bosons must all be produced inside a small volume. Therefore, it is not enough to produce 100 Higgs bosons in the course of a shower of primary and secondary collisions; they must all be produced in a single collision. At first sight, the probability of producing a large number N of Higgs bosons would seem proportional to α_y^N and hence exponentially small. One can do slightly better if the Higgs bosons are soft. As an example, the probability for radiating N consecutive Higgs bosons from a single fermion line is approximately

$$P \approx \frac{1}{N!} \left[\frac{\alpha_y}{4\pi} \ln(E/m) \right]^N \tag{3.1}$$

in leading order. E is the c.m. energy of the collision. This result is still exponentially small, even from the statistical factor $1/N!$ alone. Of course, I have considered only one diagram for making Higgs bosons; the radiated Higgs bosons are free to split into top-quark pairs which themselves radiate Higgs bosons which split into top-quark pairs and so forth. The sum of such "fan" diagrams still gives a statistical suppression factor similar to $1/N!$. I should note, however, that the perturbative expansion is controlled by $(\alpha_y/\pi)\ln(E/m)$. For a top-quark mass of 200 GeV and a rare collision between two 10^{12}-GeV cosmic rays, $(\alpha_y/\pi)\ln(E/m)$ is roughly 1. Thus, the formula for P has started to break down.

Regardless of the production probabilities, it is difficult to produce critical bubbles with soft Higgs radiation. The radiated Higgs bosons are produced with a uniform distribution in $\ln p_T$ between $\ln m$ and $\ln E$. The difficulty arises because the median transverse momentum is many orders of magnitude smaller than the longitudinal momentum. If the radiated Higgs bosons form a coherent classical configuration, its transverse size R_T will be much larger than its longitudinal size R_L. In the $-\kappa\phi^4$ model, one can estimate that using such pancake configurations increases the number of quanta needed to cross the barrier by a factor of $(R_T/R_L)^2$, i.e., by many orders of magnitude. Moreover, even such a pancake configuration would require the Fourier modes to be peaked around a particular $p_T \sim 1/R_T$. The radiated Higgs bosons are instead distributed across many orders of magnitude. Requiring otherwise will eliminate the logarithmic enhancement.

The above arguments are not, of course, a formal proof that critical bubbles cannot be formed by high-energy cosmic rays, though I hope my arguments make the contention plausible. In any case, in the absence of a convincing argument that cosmic rays *do* induce vacuum decay, one should not assume that our vacuum must be absolutely stable.

IV. COMPUTING THE DECAY RATE

Since the possibility that we live in a metastable vacuum remains open, limits on top-quark and Higgs-boson masses should be taken from the upper curve of Fig. 2 rather than the lower. The curve is determined by the condition that the bubble nucleation rate per unit volume, times the space-time volume of our past light cone, is order one. The space-time volume of the past light cone is e^{404} in units of the weak scale, which determines the curve to first approximation by

$$S_E \sim 404 \ . \tag{4.1}$$

Sometimes, $\kappa_{\rm eff}$ does not reach its minimum at a sensible scale. In such cases, I apply Eq. (2.21) with ϕ restricted to lie between the vacuum expectation values (VEV) σ and the Planck scale at 10^{19} GeV. For parameters where the tunneling scale is below the Planck scale, I solved for the O(4) bounce solution numerically and checked that the tunneling action agrees with the approximation to within 1%.

It is possible to do slightly better than Eq. (4.1). As discussed earlier, the scale ϕ relevant to tunneling is exponentially larger than the weak scale, and it is often pegged at the cutoff scale of 10^{19} GeV. On dimensional grounds, the bubble nucleation rate is approximately

$$\frac{d\Gamma}{dV} \approx \phi_0^4 \exp(-S_E) \ , \tag{4.2}$$

yielding

$$S_E - 4\ln(\phi_0/\sigma) \approx 404 \ . \tag{4.3}$$

Figure 2 was obtained using this improvement, which lowers the allowed top-quark masses by about 5 GeV.

I emphasize that these bounds apply only to a Higgs sector with a single Higgs doublet. As the Higgs sector may only be an effective description of nature, not valid up to the Planck scale, I indicate how the curve changes for different choices of the cutoff scale. The top-quark mass in this graph is measured at the scale $2m_t$. A recent two-loop calculation of the lower solid curve of Fig. 2, where the vacuum changes from stable to metastable, has been carried out in Ref. 5. I have not attempted such a careful calculation here.

I shall conclude this section with a brief discussion of the numerical issues involved in solving for the O(n) bounce solution in this situation. The equation to solve is

$$\phi'' = -\frac{n-1}{r}\phi' + \frac{dV}{d\phi} \ . \tag{4.4}$$

The solution must satisfy the boundary conditions

$$\phi'(0)=0, \quad \phi(\infty)=\sigma \ . \tag{4.5}$$

The general technique is to try different guesses $\phi_0 > \sigma$ for $\phi(0)$ and integrate the equations until either ϕ crosses σ or ϕ stops decreasing. The dividing line between these two behaviors, when ϕ comes to rest at σ, is the desired solution. One subtlety in solving the equation numerically is that the friction term is singular at $r=0$. This singularity may be avoided by solving analytically in the limit that r is very small. If r is small, then ϕ is not far from ϕ_0 and $V(\phi)$ may be approximated by a linear potential around $V(\phi_0)$:

$$\phi'' \approx -\frac{n-1}{r}\phi' + \partial V(\phi_0) \quad \text{as } r \to 0 \ , \tag{4.6}$$

where ∂V is short for $dV/d\phi$. The solution is

$$\phi(r) \approx \phi_0 + \frac{r^2}{2n}\partial V(\phi_0) \tag{4.7}$$

and is valid for

$$r \ll |\partial^2 V(\phi_0)|^{-1/2} \ . \tag{4.8}$$

This approximation may be used for the very first step in integrating ϕ, and the rest of the evolution is handled numerically.

Because of the large discrepancy of scale between ϕ_0 and σ, the integration is done with an adaptive stepsize. For the O(4) solution, one must also be careful how the action is computed. I integrate to the value of r at which a numerical solution either undershoots or overshoots as described above. The solutions which overshoot approach the correct action rather quickly. The solutions which undershoot, however, approach the correct action very slowly: a difference of ϕ_0 by 1% from the bounce solution can raise the computed action by a factor of 10. This results from the fact that solutions which undershoot fall much slower with r in the region $\sigma \ll \phi \ll \phi_0$ than do those which overshoot.

Finally, let me emphasize that the analytic approximation Eq. (2.21) is the best check of the algorithm. Another useful check is the virial theorem of Ref. 11.

V. CONCLUSION

I have argued that critical bubbles contain many quanta and so the decay of the false vacuum cannot be induced by a cosmic-ray collision that produces a single Higgs boson. Unless a strong argument is put forward that a single high-energy collision could generate the ~ 100 or more Higgs quanta needed, one should not assume that our electroweak vacuum must be absolutely stable. This relaxes the upper bound on the top-quark mass.

As a consequence, I note an important loophole in the conclusion of Lindner, Sher, and Zaglauer[5] concerning the minimal standard model. Assuming vacuum stability, they conclude that a future limit of $m_t > 110$ GeV set at the Fermilab Tevatron would imply that the Higgs boson could not be discovered at the Cornell Electron Storage Ring, the SLAC Linear Collider (SLC), or CERN LEP I. This conclusion, however, follows only if one demands that the vacuum be absolutely stable.

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