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Quark fragmentation function in the Schwinger model

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We derive an analytic expression for the N-boson distribution function in quark string fragmentation within the Schwinger model with massless fermions [$(1+1)$-dimensional Abelian gauge theory] employing the light-cone quantization.

I. INTRODUCTION

Quarks are confined inside hadrons when they have "low" relative momentum, and hadrons are created through the fragmentation of the strings between quarks in the case of "high" relative energy. For example, in the reactions $e^+ + e^- \rightarrow$hadrons, a quark and an antiquark are first created by a virtual photon. Then the field between the quark pair (we call this field the "string") decays into many hadrons.

There exists a vast amount of literature in which the fragmentation process is treated, see, for example, Refs. 1–8. These approaches are more or less phenomenological. For instance, Field and Feynman[3] treat fragmentation as a statistical process and derive an integral equation based on Kolmogorov's scaling hypothesis.[9] More in the direction of QCD, successful models have been proposed by Andersson *et al.*[5] and Webber.[6]

In this paper we wish to evaluate the fragmentation matrix elements within an exactly solvable model, the $(1+1)$-dimensional Abelian gauge theory (massless Schwinger model[10]). Obviously, a gauge theory in $1+1$ dimensions is a toy model, and its significance to the real world is not clear. For example, QED in $1+1$ dimensions confines fermions, in contrast with $(3+1)$-dimensional QED. However, QCD in $3+1$ dimensions confines fermions and therefore the string field may possibly be modeled by $(1+1)$-dimensional QED. Furthermore, phenomenological analyses indicate that the string formed between a q-$\bar{q}$ pair has essentially a one-dimensional space structure, and, therefore, a model field theory with one space and one time dimension may reproduce some significant features of the fragmentation physics.

The Schwinger model has often been studied and recently solved numerically for bound-state problems.[11] We are only aware of the interesting paper by Casher, Kogut, and Susskind[12] who treat the fragmentation process within this model. In fact, by comparing the classical field with the second-quantized expression, they infer the momentum-space number density $\langle a_p^\dagger a_p \rangle = 1/\omega_p$, which corresponds to one-particle distribution dx/x. We give a mathematical proof for their result and, at the same time, obtain a closed expression for the N-body distribution function. Our proof is in the context of the light-cone quantization of the Schwinger model. In this scheme, a closed analytical expression can be obtained for the boson operator in terms of the fermion creation and annihilation operators. Here, bosons play the role of the hadrons in the QCD-fragmentation process. The matrix element of the fermion-antifermion (f-$\bar{f}$) fragmentation into N-boson states is calculated exactly.

We organize this paper in the following way. In the next section we briefly explain the Schwinger model and the light-cone quantization procedure. Then the matrix element for the fragmentation is derived in Sec. III. The last section summarizes the results.

II. THE SCHWINGER MODEL

The Schwinger model is an Abelian gauge theory in $1+1$ dimensions and has been extensively and repeatedly discussed in many different contexts.[10–16] It is a super-renormalizable gauge theory and, most importantly, can be solved analytically for massless fermions.

The Lagrangian density is written as

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - g\gamma_\mu A^\mu)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} , \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The fermion described by the spinor ψ is coupled to the vector potential A_μ via the charge g. The mass of the fermion has been set to zero. The Dirac and Maxwell equations become

$$i\gamma_\mu \partial^\mu \psi = g\gamma_\mu A^\mu \psi , \tag{2.2}$$

$$\partial_\mu F^{\mu\nu} = j^\nu , \tag{2.3}$$

where the fermion current is defined as

$$j_\mu = g\bar{\psi}\gamma_\mu \psi .$$

We follow the notation used in Eller, Pauli, and Brodsky[11] and define the light-cone coordinates $x^\pm = x^0 \pm x^1$. The energy-momentum tensor is written as

$$T^{\mu\nu} = i(\bar{\psi}\gamma^\mu \partial^\nu \psi) + F^{\lambda\mu}\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L} , \tag{2.4}$$

where $g^{\mu\nu}$ is the metric tensor. The momentum P^μ can be written as

$$P^\mu = \tfrac{1}{2}\int dx^- T^{+\mu} . \tag{2.5}$$

Here we shall call P^+ the light-cone momentum and P^- the light-cone Hamiltonian. If we define the projection operator $\Lambda^{(+)}$, which only acts on spinor indices, as

$$\Lambda^{(+)}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \tag{2.6}$$

then we may write $\psi_+=\Lambda^{(+)}\psi$. The quantities P^+ and P^- take the form

$$P^+=\int dx^-\psi_+^\dagger i\partial^+\psi_+ \ , \tag{2.7a}$$

$$P^-=2g^2\int dx^-(\psi_+^\dagger\psi_+)\frac{1}{(i\partial^+)^2}(\psi_+^\dagger\psi_+) \ . \tag{2.7b}$$

Note that the above momenta can be expressed in terms of the field ψ_+ alone if we choose the light-cone gauge condition $A^+=0$.

To quantize the fields we employ the light-cone quantization scheme[17-19] because of its simplicity. We do not care to discuss the equivalence of this scheme with equal-time quantization.

We follow Eller, Pauli, and Brodsky[11] and quantize the field ψ_+ in a box of length L:

$$\psi_+=\frac{u}{\sqrt{2L}}\left[b_0+\sum_{n=1}^{\infty}(b_n e^{-in\xi}+d_n^\dagger e^{in\xi})\right] , \tag{2.8}$$

where $\xi=\pi x^-/L$ and u denotes the spinor with $u^+u=1$. The operator b_n (d_n) annihilates a fermion (an antifermion) with the light-cone momentum $k^+=2\pi n/L$. We have set $d_0=0$ as discussed in Ref. 11. The creation and annihilation operators obey the usual anticommutation relations:

$$\{b_n,b_m^\dagger\}=\{d_n,d_m^\dagger\}=\delta_{nm} \ . \tag{2.9}$$

Now we introduce the boson creation operators $a_n^\dagger$ ($n\geq 1$):

$$a_n^\dagger=\frac{1}{\sqrt{n}}\left[\sum_{m=0}^{n-1}b_m^\dagger d_{n-m}^\dagger+\sum_{m=0}^{\infty}b_{n+m}^\dagger b_m-\sum_{m=1}^{\infty}d_{n+m}^\dagger d_m\right] . \tag{2.10}$$

The a_n and $a_n^\dagger$ satisfy the Bose commutation relation if they operate on physical states which do not contain an infinite number of fermions and/or antifermions:

$$[a_n,a_m^\dagger]|\Psi\rangle=\delta_{nm}|\Psi\rangle \ . \tag{2.11}$$

The expression for the operator $a_n^\dagger$ in Eq. (2.10) has been constructed in order to diagonalize the light-cone Hamiltonian. In fact, the light-cone Hamiltonian H defined as $H=(2\pi/L)P^-$ can be expressed in terms of the boson operators as

$$H=\frac{g^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}a_n^\dagger a_n \tag{2.12}$$

for the charge zero sector where $g/\sqrt{\pi}=\mu$ is the mass of the boson. This is all we need in order to evaluate the fermion fragmentation matrix element.

III. FRAGMENTATION OF THE FIELDS BETWEEN FERMION-ANTIFERMION PAIRS

In the previous section we have given the light-cone Hamiltonian for the Schwinger model in the charge zero sector. The terms with finite charge diverge. Thus, QED in $1+1$ dimensions confines fermions, a well-known fact. A f-$\bar{f}$ pair (a "string") which is moving apart creates bosons. This process is treated in this section.

We start with an initial state $|\Psi_i\rangle$, which contains a f-$\bar{f}$ pair with light-cone momenta M_+ and M_- (in units of $2\pi/L$), respectively,

$$|\Psi_i\rangle=d_{M_+}^\dagger b_{M_-}^\dagger|0\rangle \ . \tag{3.1}$$

This state then evolves into a final state $|\Psi_f\rangle$ which contains N bosons with light-cone momenta $n_1,\ldots,n_N$,

$$|\Psi_f\rangle=a_{n_N}^\dagger\cdots a_{n_1}^\dagger|0\rangle \ . \tag{3.2}$$

The light-cone time development of $|\Psi_i\rangle\rightarrow|\Psi_f\rangle$ is governed by an operator $U(H)$ which depends on the light-cone Hamiltonian. The fragmentation matrix element is then

$$W_N=\langle\Psi_f|U(H)|\Psi_i\rangle \ . \tag{3.3}$$

Since the $|\Psi_f\rangle$ is an eigenstate of H [Eq. (2.12)] (namely, free boson states become eigenstates of the light-cone Hamiltonian, which is a particular feature of the Schwinger model with massless fermions), the operator $U(H)$ reduces to a c number U_0. Therefore, we obtain

$$W_N=U_0\langle 0|a_{n_1}\cdots a_{n_N}d_{M_+}^\dagger b_{M_-}^\dagger|0\rangle \ . \tag{3.4}$$

This matrix element can be evaluated by using the following recursion formula:

$$[a_n,d_{M_+}^\dagger b_{M_-}^\dagger]=-\frac{1}{\sqrt{n}}(\delta_{M_+,n-M_-}-d_{M_+}^\dagger b_{M_--n}^\dagger+d_{M_+-n}^\dagger b_{M_-}^\dagger) \ . \tag{3.5}$$

For massless fermions we may choose $M_-=0$ and then the antifermion $d_{M_+}^\dagger$ carries the momentum $M_+=(L/2\pi)\sqrt{s}$. In this case, Eq. (3.5) reduces to

$$[a_n,d_M^\dagger b_0^\dagger]=-\frac{1}{\sqrt{n}}(\delta_{n,M}+\delta_{M-n}^\dagger b_0^\dagger) \ . \tag{3.6}$$

Although Eq. (3.6) is easily derived using the explicit expression for the a_n, this equation contains important physics and is the central step in our derivation. If we denote the "string" $d_M^\dagger b_0^\dagger$ by $S(M,0)$ and if we introduce the symbol $B(n)$ for the boson related to a_n, Eq. (3.6) can be written symbolically:

$$S(M,0)\Longrightarrow B(M) \tag{3.7a}$$

or

$$S(M,0)\Longrightarrow B(n)+S(M-n,0) \ . \tag{3.7b}$$

The string either decays into one boson which carries the whole momentum M of the string, or it decays into a boson with momentum n and a string with the momentum $M-n$ remains. In the latter case, the boson is "chopped" off one end of the string. The two processes Eqs. (3.7a) and (3.7b) can be related to the different components of the wave function equation (2.10) of the bosons, Eq. (3.7a) to the component $d_{n-m}^\dagger b_m^\dagger$ in $a_n^\dagger$ and the process equation (3.7b) to the "particle-hole"-type admixtures $d_{n+m}^{\dagger} d_m$.

The probability amplitudes for the two processes, Eqs. (3.7), are equal and take the value $(-1)/\sqrt{n}$. This value is interesting, since the amplitude only depends on the quantum number of the created boson and not at all on the properties of the string, such as its energy. For the same reason, the amplitude is also independent of the history of the fragmentation sequence. These properties enter as assumptions into the approaches of Field and Feynman and of Andersson *et al.* for QCD string fragmentation. The iteration of the commutator Eq. (3.6) or the iteration of the symbolic equation (3.7) corresponds to an "outside-inside cascade." One finally obtains

$$W_N = U_0 \frac{(-)^N}{\sqrt{n_1 \cdots n_N}} \tag{3.8}$$

for the fragmentation matrix element.

The partial decay width for the f-$\bar{f}$ state (or "string") with total energy $\sqrt{s}$ into N bosons with light-cone momenta $n_1, \ldots, n_N$ is proportional to the square of the matrix element times the δ functions for energy and momentum conservation:

$$\Gamma_N(\sqrt{s}, n_1, \ldots, n_N) = \frac{2\pi |U_0|^2}{n_1 \cdots n_N} \delta\left[\sqrt{s} - \sum_{i=1}^{N} p_i^+\right] \delta\left[\sqrt{s} - \sum_{i=1}^{N} p_i^-\right] . \tag{3.9}$$

If one introduces the Feynman variable for the boson,

$$x_i = \frac{p_i^+}{\sqrt{s}} = \frac{n_i}{M_+} , \tag{3.10}$$

and uses $p_i^+ \cdot p_i^- = \mu^2$, one obtains the partial decay width

$$\Gamma_N(\sqrt{s}; x_1, \ldots, x_N) dx_1 \cdots dx_N = \frac{2\pi |U_0(s)|^2}{s} \left[\frac{dx_1}{x_1} \cdots \frac{dx_N}{x_N}\right] \delta\left[1 - \sum_{i=1}^{N} x_i\right] \delta\left[1 - \sum_{i=1}^{N} \frac{\mu^2}{s x_i}\right] . \tag{3.11}$$

This is the final result of our derivation. A boson with x_i is created in the fragmentation process with a probability $1/x_i$. In the rapidity variable y_i, the distribution of the bosons is uniform since $dy_i = dx_i / x_i$. The N-boson distribution equation (3.11) shows no correlations except those introduced by energy and momentum conservation. This follows as the light-cone Hamiltonian H contains no interactions among the bosons. This property is intrinsic to the *massless* Schwinger model. The strength $2\pi |U_0|^2/s$ of the width Γ_N is independent of N, a fact which is important for the calculation of the multiplicity distributions.

The contents of Eq. (3.11) may be summarized by saying that a $(1+1)$-dimensional massless QED string decays into all open channels with probabilities governed solely by "longitudinal phase space." This holds for the rapidity and the multiplicity distributions of the created bosons.

IV. SUMMARY

The results derived mathematically in this paper within the Schwinger model with massless fermions are known and often quoted. The references usually go back to the paper by Casher, Kogut, and Susskind where the result for the *one-body* distribution function is obtained. The explicit form, Eq. (3.11), for N-boson decay is stated in the paper by Field and Feynman who use it as a basis for a phenomenological treatment of QCD-string fragmentation. It forms the basis for many "longitudinal phase-space" phenomenologies, which have demonstrated success. However, as already stated in the Introduction, the Schwinger model is a toy world, albeit a beautiful and exactly solvable one.

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