Some exact solutions of (2+1)-dimensional Yang-Mills equations with the Chern-Simons term

C. H. Oh and L. C. Sia
Department of Physics, Faculty of Science, National University of Singapore, Lower Kent Ridge, Singapore 0511, Republic of Singapore

Rosy Teh
Institut Teknologi Majlis Amanah Rakyat, 01700 Kangar, Perlis, Malaysia
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Two *Ansätze* for the gauge field potential are given so that the (2+1)-dimensional Yang-Mills equations with the Chern-Simons term can be solved in terms of the modified Bessel functions and the elliptic function respectively.

I. INTRODUCTION

Recently there has been some interest in (2+1)-dimensional gauge field theories, not merely because they can be regarded as a high-temperature limit of the four-dimensional theories or because a topological mass term can be introduced without violating the principle of local gauge invariance,[1] but also because they may be relevant in elucidating the high-T_c superconductivity physics.[2] Classical solutions play a preliminary and important role in our understanding of the quantized theories[3] and the purpose of this paper is to present some exact solutions of the (2+1)-dimensional SU(2) Yang-Mills (YM) equations with a Chern-Simons term. We note that numerical solutions have been obtained before in Euclidean as well as Minkowski spacetime[4] with the Euclidean one being complex. However our analytical solutions are different from these numerical solutions although our Euclidean solutions are also complex as expected, since the coefficient ξ of the Chern-Simons term becomes imaginary in the Euclidean version. In the presence of a Higgs field, vortex solutions were also found in Ref. 5 but none of them are exact and analytical.

As in the (3+1)-dimensional case, to obtain analytical solutions of the YM equations, the choice of *Ansatz* is of utmost importance. With the right choice of *Ansatz*, the reduced YM equations become simple and solvable. We shall present two *Ansätze* here. The first *Ansatz* yields solutions in both Minkowski and Euclidean spacetimes. The reduced equations are linear although the nonlinear terms $[A_\mu, A_\nu]$ and $[A_\mu, F^{\mu\nu}]$ are in general nonvanishing. An interesting feature of this class of solutions is that the action vanishes in the Euclidean case. The second *Ansatz* applies only in the Euclidean spacetime. The reduced equation is nonlinear with Jacobi's elliptic functions as solutions.

II. THE SOLUTIONS

The YM action with the Chern-Simons term is

$$S=\int d^3x\,(\mathcal{L}_{\rm YM}+\mathcal{L}_{\rm CS}) \,, \tag{1a}$$

$$\mathcal{L}_{\rm YM}=-\tfrac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu} \,, \tag{1b}$$

$$\mathcal{L}_{\rm CS}=\frac{\xi}{2}\epsilon^{\mu\nu\alpha}(\partial_\mu A^a_\nu A^a_\alpha+\tfrac{1}{3}\epsilon^{abc}A^a_\mu A^b_\nu A^c_\alpha) \,, \tag{1c}$$

where, for convenience, we set the gauge field coupling constant $g=1$ and the metric is $g_{\mu\nu}=(-++)$. The energy-momentum tensor is given by

$$\theta_{\mu\nu}=F^a_{\mu\alpha}F^{a\alpha}_\nu+g_{\mu\nu}\mathcal{L}_{\rm YM} \,, \tag{2}$$

while the angular momentum is

$$J=\int d^2x\,\epsilon^{ij}x_i\theta_{0j} \,. \tag{3}$$

With the action given by expression (1), the equation of motion is

$$\partial_\mu F^{a\mu\nu}+\epsilon^{abc}A^b_\mu F^{c\mu\nu}+\frac{\xi}{2}\epsilon^{\nu\alpha\beta}F^a_{\alpha\beta}=0 \,. \tag{4}$$

In Euclidean spacetime, the coefficient ξ is replaced by $-i\xi$.

We now introduce the first *Ansatz* in Minkowski spacetime:

$$A^a_\mu(x)=\hat{\boldsymbol\phi}^a[\delta^0_\mu\psi_1(x)-\hat{\boldsymbol\phi}^\mu\psi_2(x)]+\delta^a_3\hat{\boldsymbol\phi}_\mu\psi_3(x) \,, \tag{5}$$

where ψ's are assumed to depend on $\rho=(x_1^2+x_2^2)^{1/2}$ only and $\hat{\boldsymbol\phi}^a$ and $\hat{\boldsymbol\phi}^\mu$ denote unit vectors,

$$\hat{\boldsymbol\phi}^a=\epsilon^{ai}x^i/\rho\equiv\epsilon^{ai}\hat\rho^i, \quad i=1,2 \,, \tag{6}$$

with $\hat{\boldsymbol\phi}^\mu$ being similarly defined. Clearly $\hat{\boldsymbol\phi}^3=\hat{\boldsymbol\phi}^0=0$. Substituting the above *Ansatz* into Eq. (4), the reduced equations are

$$\psi_2'+\psi_2/\rho-\xi\psi_1=0 \,, \tag{7a}$$

$$\psi_2B'-B\psi_2'+\xi B\psi_1=0 \,, \tag{7b}$$

$$B''+(B/\rho)'+\psi_1^2B=0 \,, \tag{7c}$$

$$\psi_1''+\psi_1'/\rho-\psi_1B^2-\xi(\psi_2'+\psi_2/\rho)=0 \,, \tag{7d}$$

$$\psi_2\psi_1B-\xi(B'+B/\rho)=0 \,, \tag{7e}$$

and

$$B=\psi_3-\frac{1}{\rho} \ . \tag{7f}$$

Here prime denotes differentiation with respect to the variable ρ. Equations (7) simplify tremendously if we set $B=0$ so that the reduced equations become

$$\psi_2'+\psi_2/\rho-\xi\psi_1=0 \ , \tag{8a}$$

$$\psi_1''+\psi_1'/\rho-\xi(\psi_2'+\psi_2/\rho)=0 \ . \tag{8b}$$

Solutions of Eqs. (8) are the modified Bessel functions and in order for the gauge field to vanish fast enough at large distances we choose $K_\nu(\rho)$ and write the potential as

$$A_\mu^a=\hat{\phi}^a(\delta_\mu^0 K_0-\hat{\phi}_\mu K_1)d+\delta_3^a\hat{\phi}_\mu\frac{1}{\rho} \ , \tag{9}$$

where d is a constant. Note that Eqs. (8) yield

$$\nabla^2\psi_1-\xi^2\psi_1=0 \ , \tag{10}$$

indicating the massive nature of the field configuration. We now proceed to compute some useful quantities from solution (9). The action becomes simple,

$$S=(\xi d)^2\int d^3x\,(K_1^2-K_0^2) \ , \tag{11}$$

and the total energy is just

$$H=(\xi d)^2\int d^2x\,(K_1^2+K_0^2)/2 \ . \tag{12}$$

The electric and magnetic field strengths are, respectively, given by

$$E_i^a=F_{0i}^a=\xi\, d\hat{\phi}^a\hat{\rho}_i K_1(z) \ , \tag{13}$$

$$B^a=\tfrac{1}{2}\epsilon^{ij}F_{ij}^a=-\xi\, d\hat{\phi}^a K_0(z) \ , \tag{14}$$

with $z=\xi\rho$. Near the origin the fields behave as

$$E_i^a\simeq d\hat{\phi}^a\hat{\rho}_i\frac{1}{\rho} \ , \tag{15a}$$

$$B^a\simeq\xi\, d\hat{\phi}^a\ln(z/2) \ , \tag{15b}$$

while, at large distances,

$$E_i^a\simeq d\xi\,\hat{\phi}^a\hat{\rho}_i(\pi/2z)^{1/2}e^{-z} \ , \tag{16a}$$

$$B_i^a\simeq -d\xi\,\hat{\phi}^a(\pi/2z)^{1/2}e^{-z} \ . \tag{16b}$$

For the *Ansatz* (5) with $\psi_3=1/\rho$, the time component of Eq. (4) reduces to

$$\partial^i E_i^a=\xi B^a \ . \tag{17}$$

If we regard ξB^a on the right-hand side (RHS) of Eq. (17) as charge density then the total non-Abelian charge carried by our solution is

$$Q^a=\xi\int d^2x\, B^a=\int d^2x\,\partial^i E_i^a \ . \tag{18}$$

A gauge-invariant characterization of this charge can be written as

$$Q=\xi\int d^2x\, B^a\hat{\phi}^a=-2\pi d \ .$$

The angular momentum as defined by Eq. (3) can be computed:

$$J=\pi d^2/\xi=-(Qd)/(2\xi) \ . \tag{19}$$

In passing we note that from the asymptotic behavior (16), our solutions are not vortexlike.

The *Ansatz* (5) can also be used for Euclidean field equation with $\mu=1,2,3$. The reduced equations are the same as Eqs. (7) with slight modification. Again simple solution can be found and the gauge potential is given by

$$A_\mu^a=\hat{\phi}^a(\delta_\mu^3 K_0-i\hat{\phi}_\mu K_1)+\delta_3^a\hat{\phi}_\mu\left[\frac{1}{\rho}\right] \tag{20}$$

which is inevitably being complex because of the Chern-Simons term. The action vanishes identically since

$$\mathcal{L}_{\rm YM}=-\mathcal{L}_{\rm CS}=(\xi d)^2(K_0^2-K_1^2) \ . \tag{21}$$

The field strengths, non-Abelian charge Q_a, and the angular momentum J are, respectively, as given by Eqs. (13), (14), (18), and (19) with appropriate modification by the imaginary number i.

The *Ansatz* (5) leads to reduced equations which can be linearized and hence solvable. By hard work we find it is possible to devise another *Ansatz* which renders the YM equation (4) to become a single nonlinear solvable reduced equation. The new *Ansatz* can be written as

$$A_\mu^a=(\alpha^a A_1+\gamma^a A_2)\alpha_\mu+(\alpha^a A_3+\gamma^a A_4)\gamma_\mu+\beta^a\beta_\mu A_5 \ , \tag{22}$$

where the A's are functions of the variable $u=\beta_\mu x^\mu$ and α,β,γ are three mutually perpendicular unit vectors. Substituting expression (22) into the Euclidean YM field equation we find five coupled nonlinear equations. Equating A_1 with A_4 and A_2 with $(-A_3)$, the reduced equations are consistent when $A_5=-i\xi/2$. By further setting $A_3=fA$, $A_4=eA$, we finally obtain

$$A''-\xi^2A/4-(e^2+f^2)A^3=0 \ , \tag{23}$$

where e and f are constant. Clearly the solutions of Eq. (23) are Jacobi's elliptic functions $E(u,k)$ (Ref. 6),

$$(E')^2+aE^2+\tfrac{1}{2}bE^4=c \ , \tag{24}$$

and the constants a,b,c depend on the parameter k. The action can be calculated and is infinite:

$$S=-\frac{e^2+f^2}{2}\int d^3x\{2[c+(e^2+f^2)E^4]+\xi^2E^2\} \ . \tag{25}$$

To simplify our calculations we now consider a special case of the *Ansatz* (22) and the solution is

$$A_\mu^a=[e(\delta_1^a\delta_\mu^1+\delta_3^a\delta_\mu^3)+f(\delta_1^a\delta_\mu^3-\delta_3^a\delta_\mu^1)]E(x_2,k) -i(\xi/2)\delta_2^a\delta_\mu^2 \ . \tag{26}$$

By replacing the constants e and f by ie and if, respectively, and setting $\xi=2$, $k^2=1$, one has

$$E(x_2,k)={\rm dn}(x_2) \ , \tag{27}$$

where ${\rm dn}(x_2)$ is the basic Jacobi elliptic function and $c=0$. The action (25) becomes integrable with respect to the variable x_2:

$$S=\tfrac{8}{3}\int dx_1 dx_3 \ . \tag{28}$$

The energy H can be evaluated and is finite per unit x_1 length:

$$H=\int d^2x\ \theta_{33}=\tfrac{8}{3}\int dx_1 \ . \tag{29}$$

One can also compute the total non-Abelian charge

$$Q^a=\xi\int d^2x\ B^a=(\pi\xi/2)(e\delta_3^a+f\delta_1^a)\int dx_1 \tag{30}$$

which is again finite per unit x_1 length. However in contrast with Eq. (18), Q^a is now not equal to the integral of $\partial^i E_i^a$ since, for the *Ansatz* (26), $\epsilon^{abc}A_i F^{ci0}\neq 0$. The angular momentum for solution (27) vanishes since the elliptic function E depends on x_2 only and $\theta_{3i}=0$.

III. COMMENTS

We end with some remarks.

(i) That an *Ansatz* can reduce the nonlinear YM equations with the Chern-Simons term to linear equations is not surprising since, for the self-dual instanton solution, the reduced equation is also linear.

(ii) Jacobi's elliptic functions have previously been used to construct time-dependent periodic solutions for the four-dimensional YM equations.[7]

(iii) The solution (26) is entirely imaginary when the constants e and f both become imaginary. An imaginary gauge field potential has also been discussed in Ref. 4.

(iv) Our solutions are not completely "sourceless" since they have singularities.

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