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Renormalization-group flows as gradient flows in coupling-constant space for D-dimensional systems

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Under certain conditions we argue about the existence of a renormalization scheme in which the renormalization-group flow, defined by the β function of the theory, is expressed as a gradient flow in the neighborhood of a (fixed) point in theory space.

The renormalization group (RG) proved to be a very effective method for the study of statistical or field-theoretic systems near their critical points.[1] The Wilsonian description of the RG is a very powerful tool for a better understanding of the scaling behavior of physical systems in terms of the geometry of the underlying "coupling-constant space." Recently, attempts to understand this space globally (topology) have been undertaken by a number of authors.[2] The pioneering idea goes back to Zamolodchikov,[3] who proved that in the context of two-dimensional (2D) statistical mechanics, an "entropy theorem" for the RG flow can be shown to be valid under broad assumptions. What Zamolodchikov has shown in this c theorem was that if the space of running coupling constants $g^i(t)$ (t being the RG parameter) of a generic 2D theory admits a positive-definite metric $G_{ij}(g(t))$ then there exists a scalar function of the couplings $C(g(t))$ which decreases monotonically under the influence of the RG operator:

$$dC/dt = -\beta^i G_{ij}\beta^j \le 0 \ , \qquad (1)$$

where $\beta^i(g(t)) = dg^i(t)/dt$ is the β function.

When applied to σ models[4] this theorem suggests a geometrical understanding of the equivalence of the conformal invariance conditions and the background-field equations of motion. In the renormalization scheme (RS) where the β functions have an exact quadratic form, these conditions may be interpreted as string-field-theory equations of motion.[5]

It is natural to ask whether a corresponding statement to Eq. (1) holds in higher-dimensional theories. It is the purpose of this paper to answer positively this question, at least under some broad assumptions which encompass a large class of (scale-invariant) physical theories. In particular it will be shown that there exists a renormalization scheme (Wilson's scheme) in which the β function is expressed as a gradient flow in coupling-constant space (CCS). Moreover, arguments will be given for an interpretation, under certain restrictions to be stated below, of this scheme as a Riemann normal-coordinate scheme in CCS. It would be instructive first to review briefly Wilson's approach to the study of critical systems.[1,6]

Consider a d-dimensional theory described by an action $I(g^i,\alpha)$, where g^i is a possibly infinite set of couplings and α is an ultraviolet (UV) cutoff (we ignore possible infrared infinites, assuming that the correlation functions of the theory are infrared soft). Wilson assumed the existence of a complete set of *local* operators $\{O_i\}$ in theory space, and expanded I in terms of g^i around a point S_0 in that space as

$$I = I_0(S_0,\alpha) + g^i \int d^d r\, O_i(r) \ . \qquad (2)$$

From this point of view $\{g^i\}$ may be understood as a set of coordinates in CCS with the origin taken (for convenience) at S_0.

The existence of the cutoff in the propagator is assumed in the free part of the action I_0. Under this assumption and the scale independence of the partition function, Wilson has proven the existence of a RS such that the β function was quadratic (exact relation):

$$\beta^i(g(t)) = \beta^i(S_0) + \gamma^i{}_j(S_0) g^j(t) + \tfrac{1}{2} a^i{}_{jk}(S_0) g^j(t) g^k(t) \ , \qquad (3)$$

where $t = \ln\alpha$. If S_0 is a "critical point" then $\beta^i(S_0{}^*)=0$. In this case $\gamma^i{}_j(S_0{}^*)$ is defined to be the anomalous-dimension matrix, whose eigenvalues define the so-called critical exponents of the theory, y_i, which are assumed to be real for our purposes. The β^i function defined in (3) exhibits a highly nontrivial dependence on the expansion point S_0, as becomes clear from (3). From now on we shall abbreviate it by $\beta^i(g(t),S_0)$. Let us expand first around a critical point $S_0{}^* \equiv S^*$.

The nonlinearities in (3) are removed by the introduction of the so-called "scaling fields,"[6] λ^i, defined by $d\lambda^i/dt = y^i\lambda^i$. In this sense we may use exact scaling relations for the correlation functions of the bare operators $\overline{O}_i \equiv \partial I/\partial\lambda^i$. [In the case of real y_i we are considering, the couplings may be classified as irrelevant, marginal, and relevant depending on whether $y_i<0$, $y_i=0$, or $y_i>0$, respectively.[1,6] The usual formalism of perturbative renormalization (Gell-Mann–Low–type β functions) is achieved by solving the equations for the vanishing of the β functions of irrelevant couplings in terms of the relevant and marginal couplings. In this sense, rescaling λ^i by $\exp(-y^i \ln\alpha)$, to define dimensionless couplings $\lambda_0{}^i$, with $d\lambda_0{}^i/dt=0$, would define the conventional bare couplings λ_0 expressed in terms of renormalized couplings.]

The couplings g^i can be expressed in terms of the scaling fields λ^i through

$$g^i=\lambda^i+\tfrac{1}{2}B^i{}_{jk}\lambda^j\lambda^k+O(\lambda^3) \tag{4a}$$

with $B^i{}_{jk}=(y_j+y_k-y_i)^{-1}a^i{}_{jk}$ (Ref. 6). For the purposes of this work we shall assume that $y_j+y_k-y_i\neq 0$. The case of marginal operators (or in general cases where this quantity vanishes) can be dealt with by slightly perturbing the zero value and this will be assumed from now on (cf. also Refs. 2 and 3; for some discussion on incorporating strictly marginal fields, $y_i=0$, cf. Ref. 6). The operators O_i appearing in Eq. (2) are related to $\bar{O}_i$ by [using (4a)]

$$\int d^d r\, O_j \equiv O^*{}_j=\bar{O}^*{}_j-B^i{}_{jk}g^k\bar{O}^*{}_i+O(g^2)\ , \tag{4b}$$

where $\bar{O}^*{}_i\equiv\int d^d r\,\bar{O}_i$.

As we shall argue below, although there is no symmetry between upper and lower indices in $a^i{}_{jk}(S^*)$, it is possible to define a simple rule for lowering and raising indices (metric in theory space at S^*), which makes the "covariant" $a_{ijk}(S^*)$ totally symmetric. It is instructive, for this purpose, to discuss briefly some properties of the Kadanoff-Wilson operator-product expansion[7] (OPE).

Consider the two- and three-point functions in a d-dimensional field theory (for concreteness we assume that $\bar{O}_i$ are rotational and translational invariant):

$$\begin{aligned} H_{ij}(1,2)&=\langle\bar{O}_i(r_1)\bar{O}_j(r_2)\rangle_0\ ,\\ D_{ijk}(1,2,3)&=\langle\bar{O}_i(r_1)\bar{O}_j(r_2)\bar{O}_k(r_3)\rangle_0\ , \end{aligned} \tag{5a}$$

where $\langle\cdots\rangle_0$ indicates correlators taken with respect to $I_0(S^*,\alpha)$.

Scaling arguments[6] and symmetry considerations imply that (5a) depend only on the distances for isotropic systems (caution: no sum over indices implied)

$$H_{ij}(1,2)=G_{ij}|r_{12}|^{y_i+y_j-2d}\ , \tag{5b}$$

$$D_{ijk}(1,2,3)=C_{ijk}|r_{12}|^{\delta_{ij}-d}|r_{13}|^{\delta_{ik}-d}|r_{23}|^{\delta_{jk}-d}\ , \tag{5c}$$

where $\delta_{ij}=y_i+y_j-y_k$, etc., and $|r_{ij}|=|r_i-r_j|$ and G_{ij}, C_{ijk} are totally symmetric; the anomalous dimensions y_i (scaling exponents) for $\int d^d r\,\bar{O}_i(r)$, are defined by the action of the *linear* RG operator $l\bar{O}^*{}_i=y_i\bar{O}^*{}_i$ (in the notation of Ref. 6; from now on we draw the reader's attention to the fact that in the parts of the formulas which contain explicitly scaling exponents no sum over repeated indices will be implied).

Given the locality of $\bar{O}_i(r)$, conservation of free energy implies[6] that

$$G_{im}=0 \quad \text{if } y_i-y_m\neq 0\ . \tag{5d}$$

[Even in the case where $\bar{O}_i$ are not strictly local operators (i.e., when internal deformations are taken into account[8]) $G_{im}=0$ if y_i-y_m do not differ by an integer. In the case[3] $|y_i|\ll 1$, this does not make any difference relative to the strictly local case.]

Consider, now, the case where $\alpha\leq|r_1-r_2|<$ *any other scale in the theory*. In this limit the product $\bar{O}_i(r_2)\bar{O}_j(r_1)$ can be expanded around the "midpoint," since their arguments come close to each other. An OPE analysis[7] (assuming its infrared softness) can be used to recompute (5a). One has

$$\bar{O}_j(r_1)\bar{O}_k(r_2)\propto\tilde{U}^m{}_{jk}(r_1-r_2)\bar{O}_m(\tfrac{1}{2}(r_1+r_2))\ . \tag{6}$$

Scale invariance implies

$$\tilde{U}^m{}_{jk}(r)=u^m{}_{jk}|r|^{y_j+y_k-y_m-d}\ , \tag{7}$$

where $u^m{}_{jk}$ are constants for isotropic theories.

In the limit $|r_{23}|\approx r_{12}|\gg|r_{23}|$ (always within the framework of a finite-cutoff theory) one can write, after taking into account (5b), (5c), (6), and (7),

$$C_{ijk}=u^m{}_{ij}G_{mk}\ . \tag{8}$$

To relate $a^i{}_{jk}(S^*)$ with $u^i{}_{jk}$ we apply the RG program of Wilson and Wegner[6] (the linearized theory is sufficient as long as we are dealing with $\bar{O}_i$). The nonlinear RG equation implies

$$a^i{}_{mn}\bar{O}^*{}_i=-2\int_q h(q)(\delta\bar{O}^*{}_m/\delta S_q)(\delta\bar{O}^*{}_n/\delta S_{-q})\ , \tag{9}$$

where S_q are fields (e.g., "spins" etc.). $h(q)$ is a "short-range" function in momentum space used by Wilson to cut off the high-momentum components of the fields. His choice, which we can make generic use of, is $h(q)=\text{const}+2q^2$, or in configuration space $h(r)=(\text{const}-\partial^2/\partial r^2)\delta^{(d)}(r)$, where the δ function has to be understood as a regularized one (in terms of the cutoff α).

Using $\delta/\delta S_q=\int_r \exp(iqr)\delta/\delta S(r)$, it is straightforward to arrive at

$$a^i{}_{mn}\bar{O}^*{}_i=-2\int d^d r_1\int d^d r_2\int d^d r'\int d^d r''[\delta\bar{O}_m(r_1)/\delta S(r')][\delta\bar{O}_n(r_2)/\delta S(r'')]h(r'-r'')\ . \tag{10a}$$

Given the "short-range" of $h(r'-r'')$ and the *locality* of $\bar{O}_i(r)$ it is sufficient to use OPE analysis to compute the integrand in (10a). Noticing that $a^i{}_{jk}(S^*)$ is computed at $t\equiv\ln\alpha=0$ (i.e., $\alpha=1$) we can act on both sides of the OPE relations with the linear part of RG operator[6] and integrate over the configuration space to obtain, after some straightforward manipulations (due to the completeness of $\bar{O}_i$),

$$\begin{aligned} a^m{}_{jk}(S^*)&=\Omega_d\partial_t\left[\int_\alpha^\infty d(|r|)u^m{}_{jk}|r|^{y_j+y_k-y_i-1}\right]\Big|_{\alpha=1}\\ &=-\Omega_d u^m{}_{jk}\ , \end{aligned} \tag{10b}$$

where Ω_d is the d-dimensional solid angle, $\Omega_d=2(\pi^{d/2})/\Gamma(d/2)$. Infrared softness has been assumed.

Moreover Eqs. (8) and (10b) show that there is a natural metric $G_{ij}(S^*)$ in theory space such that the "covariant" $a_{ijk}(S^*)$ are totally symmetric in their indices:

$$a_{ijk}(S^*)\equiv G_{il}(S^*)a^l{}_{jk}(S^*)=-\Omega_d C_{ijk} \ . \qquad (11)$$

It is now clear that there exists at least a class of RS (Wilson scheme) in which "semi-off-shell" relations of the form (assuming that Wilson's scheme is compatible with a symmetric (or equivalently diagonalized) anomalous-dimension matrix[1,6]

$$\partial_i\Phi(g,S^*)=G_{ij}(S^*)\beta^j(g,S^*) \qquad (12)$$

hold. In (12), $G_{ij}(S^*)$ is the "value" of the metric at S^*, independent of g^i. The "effective action" $\Phi(g,S^*)$ is, thus, at most cubic in powers of the couplings g^i, and this is why, when applied to 2D σ models, this approach may be related to string field theory.[5] Note that Eq. (12) has been derived under very broad assumptions about y_i, namely, the reality of y_i and infrared softness, but not necessarily that $|y_i|\ll 1$, as in Refs. 1 and 3 (we remind the reader that in Ref. 1 this condition has been used for a proof of the existence of other fixed points within a sufficiently small neighborhood of the starting point). What we shall argue about, however, in this paper is that, under the condition $|y_i|\ll 1$, Eq. (12) may be understood as a tensorial relation in CCS through the identification of the Wilsonian scheme with a normal-coordinate system[9] in coupling-constant space (CCS), which by assumption has zero torsion. This can be justified by noting the vanishing of the (symmetric) Christoffel symbol of CCS at g^* in this scheme, as becomes evident from a straightforward three-point function analysis, which we sketch below. To establish the vanishing of the symmetrized higher derivatives of the connection it is necessary to consider higher-point function calculations, which depend on the details of the theory. Our conjecture is that this is true, i.e., that the g's in Wilson's formula (3) are normal coordinates [vectors in CCS, so that the right-hand side (RHS) of (3) is covariant]. We note in passing that the requirement of being in the neighborhood of a point S^*, is explained naturally in the geometric context of normal coordinate expansion, given that the two points must be connected by a *unique* geodesic, which is achieved, in general, if the points are close enough to each other.

A suitable metric $G_{ij}(g)$ for CCS, expressed in terms of the renormalized (finite) couplings, can be defined by the two-point function

$$G_{ij}\propto |r|^{2d-y_i-y_j}\langle O_i(r)O_j(0)\rangle|_{r=1} \ , \qquad (13)$$

where $\langle \cdots \rangle$ is taken with respect to the full action $I=I_0+I_{\text{int}}$, and is assumed to be well defined. From the expression for the metric at a point r and using (2) we have $\partial_k G_{ij}|_{g=0,r}=-|r|^{2d-y_i-y_j}[B^m{}_{ik}\langle \bar{O}_m(r)\bar{O}_j(0)\rangle_0 + B^m{}_{jk}\langle \bar{O}_i(r)\bar{O}_m(0)\rangle_0+|r|^{y_k+y_i+y_j-d}I_{ij}{}^k]$ where

$$I_{ij}{}^k=\pi^{d/2}C_{ijk}[\Gamma(\tfrac{1}{2}(d-\delta_{ik}-\delta_{jk}))\Gamma(\tfrac{1}{2}\delta_{ik})\Gamma(\tfrac{1}{2}\delta_{jk})] \times[\Gamma(\tfrac{1}{2}(d-\delta_{ik}))\Gamma(\tfrac{1}{2}(d-\delta_{jk}))\Gamma(\tfrac{1}{2}(\delta_{jk}+\delta_{ik}))]^{-1} \ .$$

In the limit $|y_i|\ll 1, I_{ij}{}^k\approx C_{ijk}\Omega_d 2y_k(y_i+y_k-y_j)^{-1}(y_j+y_k-y_i)^{-1}$. Using (5a)–(5c), (10b), and (11), we immediately obtain that $\partial_k G_{ij}|_{g=0,r=1}=0$ and thus the connection vanishes at $g^*(=0)$ (lowest-order condition for normal coordinate scheme;[9] we remind the reader that this is the only theory-independent condition as well).

A useful result is that for any vector A, when expanded in terms of (normal coordinates) $g^N=g^i$,

$$\partial^N{}_i\Phi^N[g^{iN},S^*]=G_{ij}(g^N)A^j(g^N)|_{\text{symm}} =G_{ij}(S^*)A^j(g^N)|_{\text{symm}} \ , \qquad (14)$$

where symm denotes that only the totally symmetric part of the coefficients in the expansion in powers of g^{iN} has been retained. Relation (14) is a consequence of the Riemann normal coordinate expansion of the metric tensor *(in any scheme)* as well as of symmetry properties of the curvature tensor in CCS. In the Wilson-normal coordinate scheme the RHS of (14) with $A^i=\beta^i$ yields Eq. (12), since due to the symmetry properties of the relevant quantities discussed above it can be represented as $G_{ij}(S^*)\beta^j(g,S^*)$. Thus we observe that in this RS the flow function Φ is cubic in powers of the couplings. This completes our geometric interpretation of (12).

Relation (12) has been conjectured for 2D models in Ref. 5, but here a proof has been given (under the above-stated broad assumptions) independently of the dimensionality of configuration space-time. It was sufficient for our purposes that we proved (12) in a given RS (in a general scheme the β function will not be exact and we do not know whether the expansion coefficients in powers of the couplings satisfy integrability conditions for the β function to be represented as a gradient flow). To arrive at (12) the assumption for the reality of y_i has been made. The converse is also true if the metric $G_{ij}(S^*)>0$ (positive definite), namely, (12) implies the reality of the critical exponents,[10] as can be seen by differentiating (12) with respect to g^k and going to the fixed point,

$$\partial_i\partial_k\bar{\Phi}|_{S^*}=\bar{G}_{ij}(S^*)\partial_k\bar{\beta}^j|_{S^*} \ . \qquad (15)$$

Inverting (15) we observe that $\partial_i\beta^j|_{S^*}$ is a product of a positive-definite matrix and a symmetric matrix in CCS and therefore it has real eigenvalues, i.e., real critical exponents. Notice that the critical exponents are invariant under RG transformations as can be seen by noticing the "vector" character of β^i under a RS change $\beta'^i(g')=(\partial g'^i/\partial g^k)\beta^k(g)$. The anomalous dimension matrix is defined as $\gamma_i{}^j(g^*)=\partial_i\beta^j|_{g=g^*}$, where g^* are the coordinates of the fixed point S^*; the eigenvalue problem of $\gamma_i{}^j$, which essentially defines the critical exponents, remains invariant under this transformation, as is directly seen.

Of course not all the systems admit real critical exponents. Apart from cases where OPE is not well defined, if G_{ij} is complex (allowing for non-Hermitian theories, say) or not positive definite, then this argument fails, although (12) may still be valid. Moreover two-point functions of the form (13), etc., are not always well defined.[11]

Although above we carried out the analysis in the neighborhood of a critical point in theory space, one can imagine repeating the above procedure in the vicinity of any other point S_0 [where $\beta^j(S_0)\neq 0$]. As long as the assumptions (about the existence of complete set of $\bar{O}_i$ around S_0, two-point functions of the theory, smallness of y_i, normal coordinate scheme conditions, etc.), are still valid one can perform Wilson's expansion around S_0 in the manner of Eq. (2), and in this way an effective action, $\Phi(S_0,g^i)$, which interpolates between fixed points will be established (this would be Zamolodchikov's version in d dimensions). Under the assumption about the positive definiteness of $G_{ij}(S_0)$, the fixed points will be the only stationary points of the "flow" Φ, which otherwise behaves monotonically under the influence of the RG transformation, $d\Phi(g(t))/dt=\beta^j\partial_j\Phi(g(t))$. Of course the question whether relations of the form (12) hold globally is theory dependent and we shall not attempt to answer it here. There may be topological obstructions in CCS to this, and certainly the subject is worth further investigation (noncompact CCS is also another possibility).

Off-shell relations of the form (12), are useful in making a connection between Morse theory and flow functions in CCS. In Ref. 2 arguments have been given for the identification of the flow function of Zamolodchikov with a perfect Morse function in CCS, which might lead to a better understanding of the topological (global) features of the theory space. We only mention that off-shell relations of the form (12), which express the β function as a curl, are necessary and sufficient for an interpretation of the RG flow problem in terms of a supersymmetric quantum-mechanical particle moving in a curved space with metric $G_{ij}(S_0)$. It is straightforward to show[2(b)] in such a case, that the supersymmetry charge $Q=\psi^i[\nabla_i+G_{ij}(S_0)\beta^j(g,S_0)]$ (∇_i being the covariant derivative in CCS and ψ^i spinor coordinates satisfying canonical anticommutation relations) is nilpotent $Q^2=0$ if and only if equations of the form (12) hold.

Another remark we would like to make concerns attempts to find closed expression of the flow Φ. In 2D σ models[4] Zamolodchikov's c theorem[3] expresses this function as an appropriate combination of two-point functions of components of the stress tensor of the theory. It can be shown (by a simple scheme change[4(d)] that in that case the flow function becomes the central charge coefficient.[4(e)] This can be relevant to attempts of finding closed expressions for the flow function in higher dimensions $d>2$. In the recent Ref. 12, suggestions have been made and perturbative arguments have been given for the integrated (over space-time) trace of the stress tensor (pertinent to rigid scale invariance) to play the role of the flow.

In any case, irrespective of the possibility of representing the flow Φ in a neat form in terms of some physical quantities of the theory, other than the β functions, our work provides the proof to all orders of the fact that under the above-stated broad assumptions the β function can be expressed (in a given scheme) as a gradient in CCS, which allows eventually for the construction of an exact flow. From this point of view, basic properties of systems exhibiting an initial scale invariance, e.g., asymptotic freedom in QCD, etc., can probably be understood in a geometric context this way. We hope we shall come to these issues in the near future.

As far as the string σ model is concerned, we mention that the construction of a flow seems possible in higher-genus Riemann surfaces.[4(f),13] Identification of this function with the (quantum) string effective action generating the correct scattering amplitudes (including loops) is not yet clear. We note in passing that, although some encouraging indications exist at the tree level,[4] however, the situation is quite complicated in summing up higher-genus surfaces. Problems associated with infinite genus surfaces, convergence of string perturbation theory, gauge symmetries in strings, etc., are far from being understood in this context. We also mention that in the case of supersymmetric strings the normal coordinate choice may not be allowed. As argued in Ref. 14 the connection terms in CCS may be identified with contact terms in OPE of vertex operators corresponding to "truly marginal operators" (i.e., marginal operators whose exact β function vanishes), which are known to be essential for a world-sheet supersymmetry in a Neveu-Schwarz-Ramond formulation or space-time supersymmetry in the light-cone approach to superstrings.[15] The trick of slightly perturbing these operators in order to define a normal coordinate system and identify it with that of Wilson, may, thus, not be allowed, although Eq. (12) is still valid.

We should also point out that in the case of d-dimensional perturbations with strictly marginal operators ($y_i=0$) or operators with $y_i+y_i-y_k=0$, etc., although one can define scaling fields,[6] so that Eq. (12) still holds, the scaling fields are related to g^i of Wilson's scheme via an expansion in powers of both g^i and $t=\ln\alpha$ since the B's in (4a) are made to depend explicitly on t in this case. Hence, the possibility of defining a normal coordinate scheme in CCS in this case is not clear. These are interesting issues the study of which, however, falls beyond the scope of this paper.

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