Gross-Neveu and Thirring models. Covariant Gaussian analysis

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Two-dimensional fermionic theories, the SU(N) Gross-Neveu and the massive Thirring models, are analyzed in the covariant Gaussian approximation. In the Gross-Neveu model we find three phases (renormalizations). In one of them the results coincide with the leading order in $1/N$ expansion. In the other two phases the gap equation has no solution and there are no fermionic excitations in the spectrum of the theory. It is argued that those renormalizations are relevant for $N=1,2$. The massive Thirring model is found to possess a line of ultraviolet fixed points. In the limit $m_b \to 0$ the axial symmetry is not broken. The $2\to 2$ S-matrix element for the nonasymptotically free phase is calculated and it qualitatively agrees with the exact expression. We also find an asymptotically free phase with vanishing bare coupling.

I. INTRODUCTION

Two-dimensional renormalizable quantum field theories (QFT's) have been very well studied by a variety of methods. The exact results for many of them are known.[1,2] For this reason they have served as convenient test cases for any new nonperturbative method developed to investigate more realistic renormalizable theories in four space-time dimensions.

Recently we formulated a covariant Gaussian approximation (CGA) (Ref. 3) and applied it to the scalar Φ^4 theories[4] and the Abelian Higgs model[5] in four dimensions. This is a generalization of the Gaussian effective potential approach[6] to the calculation of the effective action. With its help one can calculate the Green's functions of a theory at arbitrary momenta and the implementation of the full renormalization program, including wave-function renormalization, becomes possible.

The essence of the method is the following. The effective action is calculated in accordance with the Jackiw-Kerman formula:[7]

$$S_{\rm eff}[\phi(t)]=\int dt\,\langle i\partial_t - H\rangle \ . \tag{1.1}$$

Here, however, not all the states in Hilbert space are considered but only those having a form of a Gaussian (in field representation):

$$|\phi,\pi,G,I\rangle = N\exp\{-\tfrac{1}{2}[\Phi-\phi(t)][G^{-1}(t)+iI(t)]\times[\Phi-\phi(t)]+i\pi(t)[\Phi-\phi(t)]\} \ . \tag{1.2}$$

In field theory of several degrees of freedom $\phi(t),\pi(t)$ become $\phi^i(x,t),\pi^i(x,t)$ and $G(t),I(t)$ become matrices $G^{ij}(x,y,t)$ and $I^{ij}(x,y,t)$. Hence the approximate effective action is a functional of four real functions ϕ, π, G, I. It can be recast into a more convenient, explicitly Lorentz-covariant form.[13] Expressing π and I in terms of ϕ, $\dot\phi$, G, $\dot G$ by means of the equations of motion one obtains $S_{\rm eff}[\phi(t),G(t)]$. Further, instead of G one introduces a covariant object-truncated propagator

$$G_{\rm tr}^{-1}(t,t')=\frac{\partial S_{\rm eff}}{\partial\phi(t)\partial\phi(t')} \ , \tag{1.3}$$

where the derivatives are partial functional derivatives [$G(t)$ is not differentiated]. Then $S_{\rm eff}$ can be reexpressed in terms of $\phi(t)$ and $G_{\rm tr}(t,t')$. The result is that $S_{\rm eff}$ is an approximation to the Cornwall-Jackiw-Tomboulis composite operator effective action[8] in which only the nonoverlapping diagrams are kept. The minimization of this action with respect to $\phi(t)$ and $G_{\rm tr}(t,t')$ leads to shift and gap equations which are equivalent to truncated first and second Dyson-Schwinger equations (DSE's) (Ref. 9) with all terms containing three-point and higher proper Green's functions (PGF's) omitted. The approximate Green's functions of a theory are found by functional differentiation of the effective action (or equivalently the shift equation). It was shown in Ref. 4 that this is equivalent to summing all the nonoverlapping Feynman diagrams of a theory in which the propagator is $G_{\rm tr}$ the solution of the gap action.

In this work we apply the CGA to two well-studied two-dimensional theories: the SU(N) Gross-Neveu and the massive Thirring models. Our aim is twofold. First, we want to demonstrate the generalization of the method to fermionic theories. Second, we are interested in comparing the results of our approximation with rigorous results that are known for the S-matrix elements in these two cases (Ref. 2).

In Sec. II we formulate the CGA for a Lagrangian with general four-Fermi interaction. General expressions for the fermion propagator and four-point function are derived.

In Sec. III we consider the SU(N) Gross-Neveu model. This is a relativistic analog of the theory of superconductivity in which the Gaussian approximation originated. After performing the full renormalization program we calculate the renormalized propagator and four-point function. Our results are very similar to the $1/N$ leading-order results.[10] The theory is asymptotically free, fermion mass is dynamically generated, and the phenemenon of dimensional transmutation occurs. The β function for large N is very close to the perturbative one. The S-matrix elements of two-particle scattering at high energies are in good agreement with the exact results[2] when N is large enough. In addition to the $1/N$ renormalization two more possibilities to choose the bare coupling as a function of cutoff exist. We argue that they are relevant for the cases $N=1$ and 2. In both of these cases there is no consistent solution of the gap equation, which means that no fermionic excitations exist in the spectrum of the theory in this approximation.

In Sec. IV we consider the massive Thirring model along the same lines. Here we find a solution with finite bare coupling. The theory has a line of ultraviolet fixed points. This corresponds to the solution of the Thirring model presented in Ref. 2. The S-matrix elements qualitatively agree with exact results. In the limit $m_b \to 0$ axial symmetry is not broken and fermions disappear from the spectrum. There is an additional possibility to renormalize the theory with a vanishing bare coupling in the infinite-cutoff limit. This phase is asymptotically free and does not correspond to a known solution of the Thirring model. We also consider the SU(N) generalization of the Thirring model. It has the same qualitative structure for any N.

Results are discussed in Sec. V.

The Appendix contains some technical details.

II. THEORIES WITH GENERAL FOUR-FERMI INTERACTION

In this section we develop the CGA for arbitrary fermionic theory with quartic interaction of the type

$$L = \bar{\Psi} D^{-1} \Psi + \tfrac{1}{2} g (\bar{\Psi} \Gamma^i \Psi)^2 \,, \tag{2.1}$$

where Ψ is a bispinor that generally carries Lorentz and internal-symmetry indices and Γ^i is a general matrix in these indices. We work in Euclidean space and use the following representation of Dirac matrices:

$$\gamma_0 = i\sigma_3, \quad \gamma_1 = i\sigma_1, \quad \gamma_5 \equiv \gamma_0\gamma_1 = -\sigma_2 \,, \qquad \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu} \,. \tag{2.2}$$

The most natural and straightforward generalization of the CGA to fermions is to consider the Grassmannian shift ψ and a truncated propagator matrix

$$G = \begin{bmatrix} S & T \\ T^* & S^* \end{bmatrix} \tag{2.3}$$

as variational parameters. The matrix elements are defined by

$$S^{\alpha\beta}(x,y) = \langle T\Psi^\alpha(x)\bar{\Psi}^\beta(y)\rangle - \psi^\alpha(x)\bar{\psi}^\beta(y) \,, \qquad T^{\alpha\beta}(x,y) = \langle T\Psi^\alpha(x)\Psi^\beta(y)\rangle - \psi^\alpha(x)\psi^\beta(y) \,. \tag{2.4}$$

Here the index α stands for Lorentz and any internal index that is carried by the field Ψ. We introduced ψ^α—classical Grassmannian shift. This simplifies the procedure of calculation of Green's functions and retains likeness to the usual path-integral formulation of fermionic systems. Eventually we shall be interested in solutions for $\psi=0$.

The Gaussian effective action in analogy with the bosonic case is an approximation to the Cornwall-Jackiw-Tomboulis effective action[8] with "bubble" diagrams only retained:

$$S_{\text{eff}}[\psi, G] = S_{\text{cl}}[\psi] - \operatorname{tr} \ln D^{-1}S - \operatorname{tr} D^{-1}S - \text{bubble vacuum diagrams} \,. \tag{2.5}$$

The minimization of this effective action with respect to ψ and G results in the set of truncated first two DSE (Ref. 9). For the Lagrangian Eq. (2.1) these two equations (which we shall call the shift and gap equations correspondingly) are

$$(D^{-1}\psi)_\alpha + g(\Gamma^i\psi)_\alpha(\bar{\psi}\Gamma^i\psi) + g\left(-(\Gamma^i\psi)_\alpha\Gamma^i_{\beta\gamma}\bigcirc_{\beta\gamma} + \Gamma^i_{\alpha\beta}\bigcirc_{\beta\gamma}(\Gamma^i\psi)_\gamma + \Gamma^i_{\alpha\beta}\bigcirc_{\beta\gamma}(\bar{\psi}\Gamma^i)_\gamma\right) = 0 \,, \tag{2.6}$$

$$S^{-1}_{\alpha\beta} = D^{-1}_{\alpha\beta} + g\left(\Gamma^i_{\alpha\beta}(\bar{\psi}\Gamma^i\psi) + (\Gamma^i\psi)_\alpha(\bar{\psi}\Gamma^i)_\beta - \Gamma^i_{\alpha\beta}\Gamma^i_{\gamma\delta}\bigcirc_{\gamma\delta} + \Gamma^i_{\alpha\gamma}\Gamma^i_{\delta\beta}\bigcirc_{\gamma\delta}\right) . \tag{2.7}$$

Here and in what follows we use the notation

$$S^{\alpha\beta}(x,y) = \alpha x \,{-}\!\leftarrow\!{-}\!{-}\!\leftarrow\!{-}\, \beta y \,, \quad T^{\alpha\beta}(x,y) = \alpha x \,{-}\!\leftarrow\!{-}\!{-}\!\rightarrow\!{-}\, \beta y \,.$$

To obtain the effective action $S_{\text{eff}}[\psi(x)]$ one has to express the truncated propagators S and T via ψ using the gap equation and substitute into Eq. (2.5). The (functional) derivatives of $S_{\text{eff}}[\psi]$ are the time-dependent Green's functions of the theory in the CGA (Ref. 11).

The essential step in the calculation of effective action is renormalization. In two space-time dimensions the theory described by the Lagrangian Eq. (2.1) is just renormalizable. We therefore impose the following normalization conditions in order to establish the dependence of the bare parameters m_b and g on the ultraviolet cutoff:[13]

$$S^{-1}_{p=0}=m_{\text{ph}} , \tag{2.8}$$

$$\frac{\partial}{\partial p_\mu}S^{-1}_{p=0}=\gamma_\mu , \tag{2.9}$$

$$\Gamma^4(0,0,0,0,)=-g_r . \tag{2.10}$$

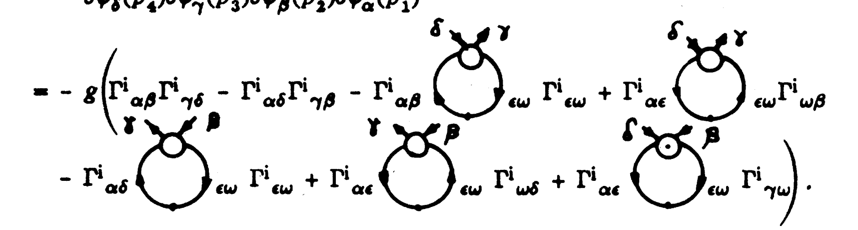
The condition equation (2.9) is necessary in order to establish the wave-function renormalization factor z which scales the field Ψ so that the Green's functions are finite for all momenta.

In order to do that we must find two- and four-point proper Green's functions (PGF). The shift equation always has a zero-shift solution $\psi=0$. For obvious reasons we shall be interested in the Green's functions only in this case (the vacuum must obey the fermion number selection rule). The truncated propagator can be easily found in every particular case from the gap equation.

In the no-shift case the full Gaussian propagator is equal to the truncated one as can be seen by differentiation of the shift equation.

The four-point PGF is the third derivative of the shift equation:

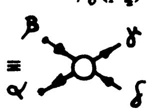
$$\Gamma^4_{\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1) \equiv -\frac{\delta^4 S_{\text{eff}}}{\delta\psi_\delta(p_4)\delta\bar\psi_\gamma(p_3)\delta\psi_\beta(p_2)\delta\bar\psi_\alpha(p_1)}$$

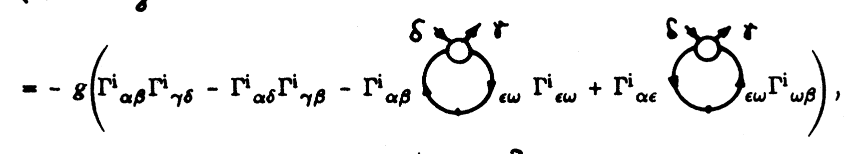


(2.11)

Here we took into account the fact that the first derivative of the truncated two-point PGF vanishes for zero shift. The auxiliary four-point functions that enter Eq. (2.11) are found by differentiation of the right-hand side (RHS) of the gap equations:

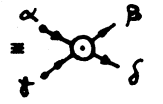
$$\Gamma^4_{a_1\,\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1) \equiv -\frac{\delta^2}{\delta\psi_\delta(p_4)\delta\bar\psi_\gamma(p_3)}S^{-1}_{\alpha\beta}(p_1,p_2)$$

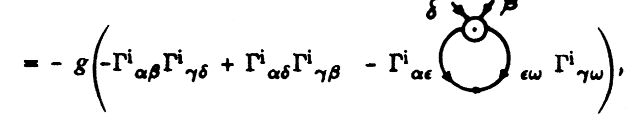




(2.12)

$$\Gamma^4_{a_2\,\delta\beta\gamma\alpha}(p_4,p_3,p_2,p_1) \equiv -\frac{\delta^2}{\delta\psi_\delta(p_4)\delta\psi_\beta(p_3)}T^{-1}_{\alpha\gamma}(p_1,p_2) \equiv$$





(2.13)

where the dotted lines are the ones that constitute loops in Eq. (2.11).

These equations are formally solved by

$$\Gamma^4_{a_1}=-g\Lambda[1+X(p_1+p_2)]^{-1} , \tag{2.14}$$

$$\Gamma^4_{a_2}=g\Lambda[1+Y(p_1+p_2)]^{-1} , \tag{2.15}$$

where

$$\Lambda_{[\delta\gamma][\beta\alpha]}=\Gamma^i_{\beta\alpha}\Gamma^i_{\gamma\delta}-\Gamma^i_{\beta\delta}\Gamma^i_{\gamma\alpha} , \tag{2.16}$$

$$X_{[\delta\gamma][\beta\alpha]}(p)=\int\frac{d^2k}{(2\pi)^2}\{-\Gamma^i_{\alpha\beta}[S(k+p)\Gamma^i S(k)]_{\gamma\delta}+[\Gamma^i S(k+p)]_{\alpha\gamma}[S(k)\Gamma^i]_{\delta\beta}\} , \tag{2.17}$$

$$Y_{[\delta\gamma][\beta\alpha]}(p)=\int\frac{d^2k}{(2\pi)^2}[\Gamma^i S(p-k)]_{\alpha\gamma}[\Gamma^i S(k)]_{\delta\beta}\ . \quad (2.18)$$

Using Eqs. (2.12) and (2.13) we can rewrite Eq. (2.11) in the simpler form

$$\Gamma^4_{\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1)=2g\Lambda_{\delta\gamma\beta\alpha}+\Gamma^4_{a_1\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1)-\Gamma^4_{a_1\beta\gamma\delta\alpha}(p_2,p_3,p_4,p_1)-\Gamma^4_{a_2\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1)\ . \quad (2.19)$$

The formulas (2.12)–(2.19) will be used for calculations in the particular cases of the Gross-Neveu and Thirring models in the following sections.

III. SU(N) GROSS-NEVEU MODEL

The model is described by the Lagrangian

$$L=zi\bar{\Psi}\partial\!\!\!/\Psi^i+z^2\frac{g}{2}(\bar{\Psi}^i\Psi^i)^2\ . \quad (3.1)$$

Here Ψ^i is a complex Dirac spinor transforming as the fundamental representation of SU(N) group. The wave-function renormalization factor z was explicitly introduced into the Lagrangian. Thus the Green's functions that we are calculating are already the renormalized ones.

It is known that the actual symmetry of the theory is not SU(N) but rather O($2N$) (Ref. 14). The transformations forming this group mix not only particles but also particles with antiparticles. The eigenstates of the theory are O($2N$) multiplets and thus it is one of the theories whose exact S matrix is found in Ref. 2.

In the notation of the previous section,

$$D^{-1ij}_{\alpha\beta}(p)=z\delta^{ij}p\!\!\!/_{ab}\ ,\quad \Gamma^{ij}_{\alpha\beta}=\delta^{ij}\delta_{\alpha\beta}\ . \quad (3.2)$$

The gap equation is

$$S^{-1}=zD^{-1}+z^2g\int\frac{d^2k}{(2\pi)^2}[S^{ij}_{\alpha\beta}(k)-\delta^{ij}\delta_{\alpha\beta}\mathrm{tr}S(k)]\ . \quad (3.3)$$

The general form of its solution is

$$S^{-1}_{\mathrm{tr}}=z\delta^{ij}(p\!\!\!/+M)_{\alpha\beta}\ , \quad (3.4)$$

where M satisfies the equation

$$M=-z(2N-1)gMI_0(M)\ . \quad (3.5)$$

Here I_0 is a divergent "bubble" integral which (introducing the momentum cutoff Λ) is equal to

$$I_0(M)=\frac{1}{(2\pi)^2}\int\frac{d^2p}{p^2+M^2}=\frac{1}{4\pi}\ln\left[\frac{\Lambda^2}{M^2}+1\right]\ . \quad (3.6)$$

A nontrivial solution of this equation exists if

$$gz=-\frac{4\pi}{2N-1}\left[\ln\frac{\Lambda^2}{\mu^2}\right]^{-1}\ . \quad (3.7)$$

We shall, however, have to calculate the four-point function to see whether this value of g renders it finite and whether other possibilities to renormalize the theory exist.

The wave-function renormalization can be established already at this stage. Since the full Gaussian propagator in the absence of shifts is equal to the truncated one, the normalization condition Eq. (2.9) becomes

$$z=1\ . \quad (3.8)$$

Thus there is no wave-function renormalization.

Using Eqs. (2.17) and (2.18) we calculate

$$\begin{aligned}X^{[lk][ji]}_{[\delta\gamma][\beta\alpha]}(p)=&-[Q^{klji}+(1-2N)O^{lkji}]2B(p)a_{1[\delta\gamma][\beta\alpha]}\\&-(Q^{klji}+O^{lkji})\{[4M^2A(p)-2B(p)]b_{1[\delta\gamma][\beta\alpha]}+4M^2A(p)b_{2[\delta\gamma][\beta\alpha]}\\&+2i|p|MA(p)(c_{1[\delta\gamma][\beta\alpha]}+c_{2[\delta\gamma][\beta\alpha]})\}\ ,\end{aligned} \quad (3.9)$$

$$\begin{aligned}Y^{[ik][ji]}_{[\delta\gamma][\beta\alpha]}(p)=&-\delta^{ik}\delta^{jl}\{2B(p)b_{1[\delta\gamma][\beta\alpha]}+[4M^2A(p)-2B(p)]a_{1[\delta\gamma][\beta\alpha]}+4M^2A(p)a_{2[\delta\gamma][\beta\alpha]}\\&-2i|p|MA(p)(d_{1[\delta\gamma][\beta\alpha]}+d_{2[\delta\gamma][\beta\alpha]})\}\ .\end{aligned} \quad (3.10)$$

Here we introduce the following notation:

$$\begin{aligned}&A(p)=-2\frac{g}{(2\pi)^2}\int\frac{d^2k}{(k^2+M^2)[(k+p)^2+M^2]}=\frac{g}{4\pi}\frac{1}{\sqrt{p^2}\sqrt{p^2+4M^2}}\ln\frac{\sqrt{1+4M^2/p^2}-1}{\sqrt{1+4M^2/p^2}+1}\ ,\\&B(p)=\frac{g}{2}I_0(M)+2\left[M^2+\frac{p^2}{4}\right]A(p),\quad O^{lkji}=\frac{1}{N}\delta^{ij}\delta^{kl},\quad Q^{lkji}=\delta^{ik}\delta^{jl}-O^{lkji}\ ,\\&a_{1[\delta\gamma][\beta\alpha]}(p)=\tfrac{1}{2}\delta_{\alpha\beta}\delta_{\gamma\delta},\quad a_{2[\delta\gamma][\beta\alpha]}(p)=-\frac{1}{2p^2}p\!\!\!/_{\alpha\beta}p\!\!\!/_{\gamma\delta}\ ,\\&b_{1[\delta\gamma][\beta\alpha]}(p)=\tfrac{1}{2}\gamma^5_{\alpha\beta}\gamma^5_{\gamma\delta},\quad b_{2[\delta\gamma][\beta\alpha]}(p)=-\frac{1}{2p^2}(p\!\!\!/\gamma^5)_{\alpha\beta}(p\!\!\!/\gamma^5)_{\gamma\delta},\quad c_{1[\delta\gamma][\beta\alpha]}(p)=-\frac{i}{2p}(p\!\!\!/\gamma^5)_{\alpha\beta}\gamma^5_{\gamma\delta}\ ,\\&c_{2[\delta\gamma][\beta\alpha]}(p)=-\frac{i}{2p}\gamma^5_{\alpha\beta}(p\!\!\!/\gamma^5)_{\gamma\delta},\quad d_{1[\delta\gamma][\beta\alpha]}(p)=-\frac{i}{2p}\gamma_{\alpha\beta}p\!\!\!/_{\gamma\delta},\quad d_{2[\delta\gamma][\beta\alpha]}(p)=-\frac{i}{2p}p\!\!\!/_{\alpha\beta}\delta_{\gamma\delta}\ .\end{aligned} \quad (3.11)$$

To invert the matrices $1+X$ and $1+Y$ we use the algebra of matrices a_i, b_i, c_i, and d_i. All the relevant formulas are given in the Appendix. The result is

$$\Gamma^{4lkji}_{a_1\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1)=-gO^{ijkl}\left[\left[\frac{2N-1}{1-(1-2N)B}+\frac{1}{1-2B}\right]a_1+\frac{4i|p_1+p_2|MA(c_1-c_2)}{(1-4M^2A)(1-4M^2A+2B)+4(p_1+p_2)^2M^2A^2}\right]$$
$$+g\delta^{il}\delta^{jk}\left[\frac{1}{1-2B}a_1+a_2+\frac{-(1-4M^2A)b_1+(1-4M^2A+2B)b_2+2i|p_1+p_2|MA(c_1-c_2)}{(1-4M^2A)(1-4M^2A+2B)+4(p_1+p_2)^2M^2A^2}\right], \quad (3.12)$$

$$\Gamma^{4lkji}_{a_2\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1)=g(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})\frac{1}{1+2B}b_1$$
$$+g(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk}\left[b_2+\frac{(1+4M^2A)a_1+(1+4M^2A-2B)a_2-2i|p_1+p_2|MA(d_1+d_2)}{(1+4M^2A)(1+4M^2A-2B)+4(p_1+p_2)^2M^2A^2}\right]. \quad (3.13)$$

In the above expressions a_i should be understood as $a_{i[\delta\gamma][\beta\alpha]}$, etc. The argument of the functions A and B and the tensors a_i , b_i, c_i, and d_i is (p_1+p_2).

There are four possible choices of g that make Γ^4 finite:

$$g=-\frac{4\pi}{2N-1}\ln^{-1}\left[\frac{\Lambda^2}{\mu^2}\right], \quad (3.14)$$

$$g=g_r , \quad (3.15)$$

$$g=4\pi\ln^{-1}\left[\frac{\Lambda^2}{\mu^2}\right], \quad (3.16)$$

$$g=-4\pi\ln^{-1}\left[\frac{\Lambda^2}{\mu^2}\right]. \quad (3.17)$$

Here μ and g_r are arbitrary finite constants.

Let us first consider the possibility Eq. (3.14). In this case the gap equation [Eq. (3.5)] has a nontrivial solution. The renormalized four-point PGF is

$$\Gamma^{4lkji}_{\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1)=\frac{1}{2N}[\delta^{\alpha\beta}\delta^{\gamma\delta}\delta_{ij}\delta_{kl}J^{-1}(p_1+p_2)$$
$$-(j\to l,\beta\to\delta,p_2\to p_4)] , \quad (3.18)$$

where

$$J(p)=2\left[M^2+\frac{p^2}{4}\right]\frac{1}{(2\pi)^2}\int\frac{d^2k}{(k^2+M^2)[(k+p)^2+M^2]}$$
$$=-\frac{1}{4\pi}\sqrt{1+4M^2/p^2}\ln\frac{\sqrt{1+4M^2/p^2}-1}{\sqrt{1+4M^2/p^2}+1}. \quad (3.19)$$

One notices that this expression does not depend on any dimensionless parameter. Once M is specified by the normalization condition Eq. (2.8) there are no additional parameters in the theory. This is the phenomenon of dimensional transmutation discussed in the framework of this model in Ref. 9. The theory is asymptotically free—Γ^4 vanishes at large momenta.

Let us define the effective coupling at the symmetric point as

$$\lambda(s)\equiv-\frac{1}{2N(2N-1)}\Gamma^{4iijj}_{\alpha\alpha\beta\beta}\ (s=t=u) . \quad (3.20)$$

Then for large momenta we have

$$\lambda(s)=-\frac{4\pi}{2N}\left[\ln\frac{s}{M^2}\right]^{-1} \quad (3.21)$$

and the β function is

$$\beta(\lambda)\equiv2s\frac{\partial\lambda(s)}{\partial s}=-2\frac{2N}{4\pi}\lambda^2 . \quad (3.22)$$

For large enough N this coincides with the perturbative one-loop β function

$$\beta_{\text{pert}}(\lambda)=-2\frac{2(N-1)}{4\pi}\lambda^2 . \quad (3.23)$$

In fact our expressions are very close to those obtained in the leading order in $1/N$ expansion. The renormalized four-point PGF coincides exactly and Eq. (3.14) becomes the $1/N$ result with the substitution $2N-1\to2N$. It is interesting to contrast them with the exact solution of the model given in Ref. 2. Figures 1(a) and 1(b) show the scattering amplitude corresponding to σ_2 of Ref. 2 calculated with our Γ^4 versus the rigorous solution for different values of N. For large enough N they coincide practically in the whole range of energies. For smaller N the agreement is poorer but the qualitative behavior of the approximate solution at high energies is still the same as the exact one.

The situation here is similar to that in the negative coupling Φ^4 scalar theory in four dimensions. There too for large N the β function was close to the perturbative one. For smaller N the discrepancy became larger but the qualitative features remained correct.[4]

There is another similarity with the Φ^4 theory. Let us consider the energy density of the Gaussian states Eq. (1.3):

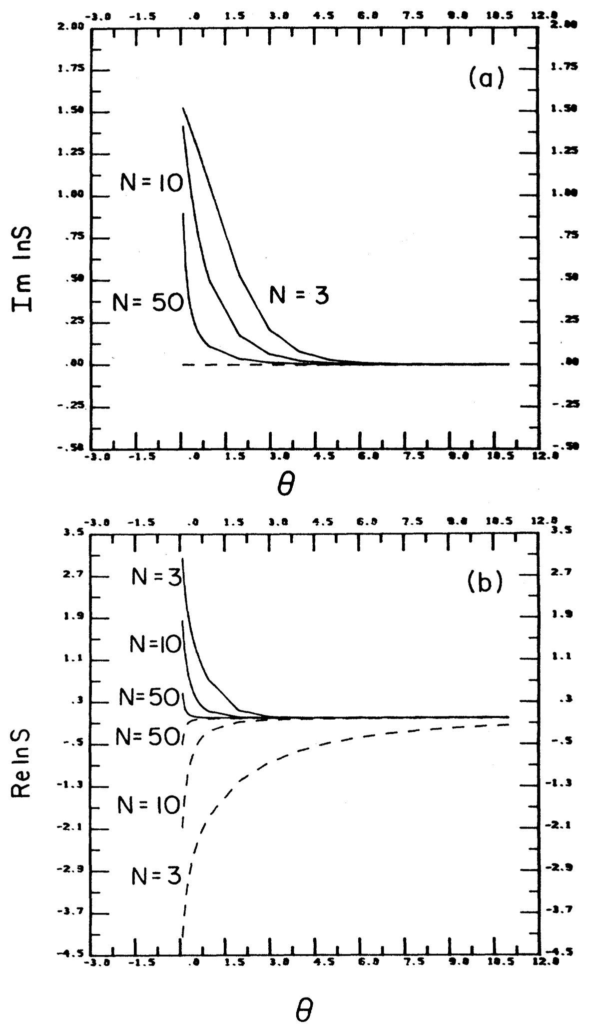


FIG. 1. The S-matrix element equal to σ_2 of Ref. 2 of the Gross-Neveu model as a function of relative rapidity of two particles for different values of N. For the exact definition of θ see Eq. (4.17). The exact solution is shown in dashed lines, the Gaussian approximation result in solid lines.

$$E_g = 2NM^2 \left\{ \frac{1}{4\pi} \left[\ln \left[\frac{\Lambda^2}{M^2} + 1 \right] - \frac{\Lambda^2}{M^2} \ln \left[1 + \frac{M^2}{\Lambda^2} \right] \right] + (2N-1) g \left[\frac{1}{4\pi} \ln \left[\frac{\Lambda^2}{M^2} + 1 \right] \right]^2 \right\} . \quad (3.24)$$

One notices that with the bare coupling given by Eq. (3.14) this is not bounded from below as $M^2 \to \infty$ for any finite Λ. However, if we take the limit $\Lambda \to \infty$ and leave out of Eq. (3.24) terms that vanish in this limit for finite M^2 we obtain

$$E_g = \frac{2N}{4\pi} M^2 \left[\ln \frac{M^2}{\mu^2} - 1 \right] . \quad (3.25)$$

This is finite and bounded from below for any value of M. The function (3.24) is schematically depicted on Fig. 2. The solution of the gap equation that we found corresponds to the local minimum of the energy at $M^2 = \mu^2$. It is a complete analog of the "metastable vacuum" of Ref. 15 in the Φ^4 theory. Here also as the cutoff is removed the lifetime of this metastable state becomes infinite since the potential barrier is proportional to the cutoff.

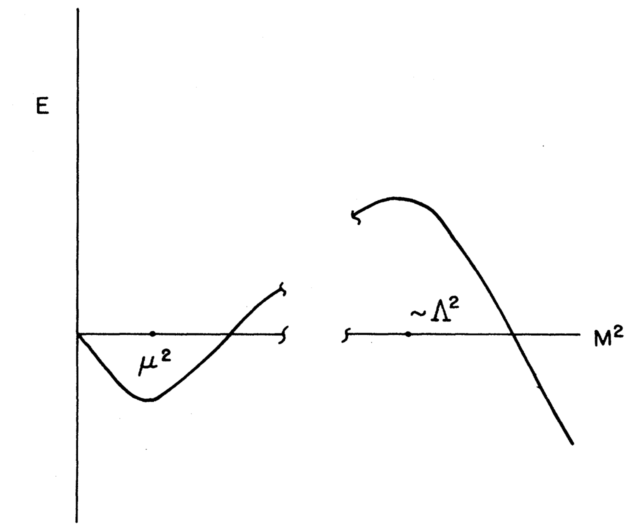


FIG. 2. The schematic dependence of energy density of Gaussian states on the mass parameter M^2 in the Gross-Neveu model for bare coupling of Eq. (3.14).

It is known that for $N > 2$ the $1/N$ leading order is a good approximation to the exact results. However, for $N = 1, 2$ it does not agree with them even qualitatively. For $N = 1$ the Lagrangian equation (2.1) coincides with the Lagrangian of massless Thirring model. It is known[1] that in this theory the coupling-constant renormalization is finite, mass is not generated, and there are no fermions in the spectrum of the theory. In the case $N = 2$ the situation is similar.[2] It is known that the S matrix found formally in Ref. 2 does not contain poles corresponding to massive fermions and the theory does not contain fermions in the spectrum.

Keeping this in mind we are now going to ask what is the meaning of additional solutions for the normalization condition Eqs. (3.15)–(3.17). First it can be seen that the solution (3.17) is unacceptable. Consider the energy density equation (3.24). For the bare coupling equation (3.17) and $N > 1$ it is unbounded from below as a function of M^2. In this case the energy does not have a local minimum for finite M^2 and thus no interpretation similar to Eq. (3.25) can be given to it. For $N = 1$ it coincides with the solution equation (3.14).

For other solutions, Eqs. (3.15) and (3.16), the energy is bounded from below. In these two cases there is no nontrivial solution of the gap equation (3.5). The status of the solution $M = 0$ is also rather shaky. The energy density [Eq. (3.24)] is a function of M^2 rather than M. If we minimized this expression with respect to M^2 as is usually done in bosonic theories [and not M which was done in order to arrive at Eq. (3.5)] no solution at all would have been found. In any case $M^2 = 0$ is an end point of our parameter range and our attitude towards it must be very cautious. It seems more correct to interpret it in the following way: for the bare couplings, Eqs. (3.15) and (3.16), there are no reliable solutions of the gap equation and therefore no real fermions exist in the spectrum of the theory in these phases.

Here one could argue that since $M=0$ is the end point of the parameter range of our variational calculation, bare couplings that lead to $M=0$ are senseless at least in the framework of the approximation. However this is not convincing. If we were to consider the same bare Lagrangian but with a nonzero bare mass (which also defines a renormaliable theory) the couplings, Eqs. (3.15) and (3.16), would lead to a nonzero solution of the gap equation which would depend on m_b. Thus for nonzero m_b those renormalizations certainly make sense. When the limit $m_b \to 0$ is taken this would mean not that the renormalizations no longer make sense but rather that the fermion mass is not dynamically generated in these phases and fermions disappear from the spectrum. Thus the renormalizations equations (3.15) and (3.16) reproduce correctly the gross feature of the theory for $N=1,2$.

Let us look in more detail on the Γ^4 for those two renormalizations.[16] For Eq. (3.15) we have

$$\begin{aligned}\Gamma^{4lkji}_{\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1)\\ &= -\frac{g}{(p_1+p_2)^2}\delta^{il}\delta^{jk}\left[(\not p_1+\not p_2)_{\alpha\beta}(\not p_1+\not p_2)_{\gamma\delta}+\frac{1}{1-4M^2A(p_1+p_2)}[(\not p_1+\not p_2)\gamma^5]_{\alpha\beta}[(\not p_1+\not p_2)\gamma^5]_{\gamma\delta}\right]\\ &+\frac{g}{(p_1+p_4)^2}\delta^{ij}\delta^{kl}\left[(\not p_1+\not p_4)_{\alpha\beta}(\not p_1+\not p_4)_{\beta\gamma}+\frac{1}{1-4M^2A(p_1+p_4)}[(\not p_1+\not p_4)\gamma^5]_{\alpha\beta}[(\not p_1+\not p_4)\gamma^5]_{\beta\gamma}\right]\\ &-\frac{g}{(p_1+p_3)^2}(\delta^{il}\delta^{jk}-\delta^{ij}\delta^{kl})\left[\frac{1}{1-4M^2A(p_1+p_3)}(\not p_1+\not p_3)_{\alpha\gamma}(\not p_1+\not p_3)_{\beta\delta}+[(\not p_1+\not p_3)\gamma^5]_{\alpha\gamma}[(\not p_1+\not p_3)\gamma^5]_{\beta\delta}\right]\\ &+2g(\delta^{ij}\delta^{kl}\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta^{il}\delta^{jk}\delta_{\alpha\delta}\delta_{\beta\gamma})\ .\end{aligned} \tag{3.26}$$

The effective coupling is

$$\lambda(s)=-g\left[\frac{3}{2}-\frac{1}{2(1-4M^2A(s))}\right]. \tag{3.27}$$

It has a finite limit as the scale s goes to infinity, $\lambda \to -g$, and is completely N independent. For Eq. (3.16) one obtains

$$\begin{aligned}\Gamma^{4lkji}_{\delta\gamma\beta\alpha}(p_4,p_3,p_2,p_1)=&\left[-\frac{1}{N}\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}\right]\delta_{\alpha\beta}\delta_{\gamma\delta}\frac{1}{\frac{1}{2\pi}\ln\frac{M^2}{\mu^2}+2J(p_1+p_2)}-(j\to l,\beta\to\delta,p_2\to p_4)\\ &-(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk})\gamma^5_{\alpha\gamma}\gamma^5_{\beta\delta}\frac{1}{\frac{1}{2\pi}\ln\frac{M^2}{\mu^2}+2J(p_1+p_3)}\ .\end{aligned} \tag{3.28}$$

The effective coupling is

$$\lambda(s)=\frac{N+1}{N}\frac{1}{\frac{1}{4\pi}\ln\frac{M^2}{\mu^2}+J(s)}\underset{s\to\infty}{\longrightarrow}\frac{N+1}{N}\frac{4\pi}{\ln\frac{s}{M^2}} \tag{3.29}$$

and the β function

$$\beta(\lambda)=-\frac{2N}{N+1}\frac{1}{4\pi}\lambda^2\ . \tag{3.30}$$

This phase is asymptotically free. Its N dependence is also very weak and differs strongly from the $1/N$ phase.

Thus in these two phases the effective coupling behaves in such a way that the addition of new fields to the theory does not change it. The effective coupling defined in Eq. (3.20) is essentially the coupling per degree of freedom. Thus the picture is more like that of N particles with self-interaction λ or particles interacting in pairs than N particles genuinely interacting with each other. This can indicate that these renormalizations are nontrivially realized only for some small values of N, while for large N they result in a direct sum of several systems rather than a system with a really different dynamics.[17] This conjecture is of course outside the range of the CGA and in the framework of the approximation we have all the three phases for any value of N. However it is consistent with the known results for $N=1,2$.

Hence we find that the $1/N$ renormalization correctly describes the theory for $N>2$. In this case the fermion mass is generalized dynamically and the phenomenon of dimensional transmutation takes place. For $N=1,2$ the other two renormalizations are probably relevant. In these cases there is no consistent solution of the gap equation and no fermions in the spectrum of the theory.

It is satisfying to discover that in addition to the $1/N$ renormalization the CGA provides possibilities for existence of the other phases whose major feature—nonexistence of real fermions—is confirmed by exact results for small N.

IV. THIRRING MODEL

In this section we analyze the massive Thirring model along the same lines. The theory is described by the Lagrangian

$$L=z\overline{\Psi}(i\partial\!\!\!/+m_b)\Psi+z^2\frac{g}{2}(\overline{\Psi}\gamma_\mu\Psi)^2 . \tag{4.1}$$

Our motivation here is again to test the method. It is known that this theory has a phase which is not asymptotically free and the coupling constant undergoes finite renormalization.[1,2] All the theories investigated up to now with the help of the CGA including the four-dimensional U(1) Higgs model were found to be asymptotically free. The impression could arise that this is an artifact of the method rather than the real feature of the theories. Hence it is interesting to test it on a theory which is known to have a nonzero ultraviolet fixed point.

The second truncated DSE in the absence of shifts is

$$S_{\alpha\beta}^{-1}(p)=D_{\alpha\beta}^{-1}(p)+g\int\frac{d^2k}{(2\pi)^2}\{[\gamma_\mu S(k)\gamma_\mu]_{\alpha\beta}-(\gamma_\mu)_{\alpha\beta}\mathrm{tr}[\gamma_\mu S(k)]\} , \tag{4.2}$$

where

$$D_{\alpha\beta}^{-1}(p)=z(p\!\!\!/+m_b)_{\alpha\beta} . \tag{4.3}$$

Its solution has a general form

$$S^{-1}=z(p\!\!\!/+M) , \tag{4.4}$$

where the mass M is the solution of the gap equation

$$M=m_b-z\frac{g}{2\pi}M\ln\frac{\Lambda^2}{M^2} . \tag{4.5}$$

Since the truncated propagator is equal to the full Gaussian one, the two normalization conditions Eqs. (2.8) and (2.9) can be already implemented. They lead to

$$M=m_{\mathrm{ph}} , \tag{4.6}$$

$$z=1 . \tag{4.7}$$

In the same way as in the previous section we calculate the auxiliary functions

$$\Gamma^4_{a_1\alpha\beta\gamma\delta}(p_4,p_3,p_2,p_1)=-2g\left[\frac{1}{1+4B}a_1-a_2+\frac{[(1-8m^2A)b_1-(1-8M^2A+4B)b_2+4i|p_1+p_2|MA(c_1-c_2)]}{(1-8M^2A)(1-8M^2A+4B)+16(p_1+p_2)^2M^2A^2}\right] , \tag{4.8}$$

$$\Gamma^4_{a_2\alpha\beta\gamma\delta}(p_4,p_3,p_2,p_1)=\frac{4g}{1-4B}b_1 . \tag{4.9}$$

The argument of A,B,a_1, etc., is (p_1+p_2) and a_i stands for $a_{i[\delta\gamma][\beta\alpha]}$, etc. Here we use the notation of Eq. (3.11).

There are three possible choices of g that make this function finite:

$$g=g_r , \tag{4.10}$$

$$g=2\pi\left[\ln\frac{\Lambda^2}{\mu^2}\right]^{-1} , \tag{4.11}$$

$$g=-2\pi\left[\ln\frac{\Lambda^2}{\mu^2}\right]^{-1} . \tag{4.12}$$

First let us consider the renormalization equation (4.10). The renormalized four-point PGF in this phase is

$$\Gamma^4_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4)=2g(\gamma_\mu^{\alpha\beta}\gamma_\mu^{\gamma\delta}-\gamma_\mu^{\alpha\beta}\gamma_\mu^{\gamma\beta})+g\left[\frac{1}{(p_1+p_2)^2}\left[-(p\!\!\!/_1+p\!\!\!/_2)_{\alpha\beta}(p\!\!\!/_1+p\!\!\!/_2)_{\gamma\delta}-\frac{1}{1-8M^2A}(p\!\!\!/_1\gamma^5+p\!\!\!/_2\gamma^5)_{\alpha\beta}(p\!\!\!/_1\gamma^5+p\!\!\!/_2\gamma^5)_{\gamma\delta}\right]-(\beta\to\delta,p_2\to p_4)\right] . \tag{4.13}$$

The effective coupling is

$$\lambda(s) \equiv -\tfrac{1}{4}\Gamma^4_{\alpha\alpha\beta\beta}(s=t=u)$$

$$= -\frac{g}{2}\left[3-\frac{1}{1-8M^2A(s)}\right] \underset{s\to\infty}{\longrightarrow} -g \tag{4.14}$$

and the β function

$$\beta(\lambda) = -2[\lambda(s)+g] \; . \tag{4.15}$$

Hence the theory is not asymptotically free but has an ultraviolet fixed point $\lambda=-g$. This coincides with the nonasymptotically free phase discussed in the previous section.

From the definition of A [Eq. (3.11)] we see that in order for Γ^4 to have no tachyon poles the following must hold: $g>-\pi$. It is known that the theory is well defined only for $g>-\pi/2$ (see, for example, Ref. 14). Thus although the restriction we obtain is milder it reflects the qualitatively correct feature. The effective two-particle interaction for positive g is that of attraction while for negative g repulsion. This is in agreement with exact results.[2]

It is interesting to compare our results with the rigorous formula of Ref. 2. To this end we calculate the S-matrix element corresponding to fermion-antifermion backward scattering:

$$S(\theta) = -\frac{g}{\sinh\theta}\left[\frac{1}{1+\dfrac{g}{\pi}\dfrac{\theta}{\sinh\theta}} + \frac{1}{1+\dfrac{g}{\pi}\dfrac{i\pi-\theta}{\sinh\theta}} + 2\right] . \tag{4.16}$$

Here θ is the relative rapidity of the two particles:

$$p_2^0 p_1^1 - p_1^0 p_2^1 = M^2 \sinh\theta \; ,$$
$$p_1^1 p_2^1 = \cosh\theta \; . \tag{4.17}$$

Figures 3(a)–3(c) show the comparison of our result to the exact one for different values of g. As in the case of the Gross-Neveu model there is a qualitative agreement for large energies and small couplings.

The next question to address is what is the status of other renormalization equations (4.11) and (4.12). First, consider the renormalization equation (4.11). It has a feature that in our view makes it unacceptable. Let us concentrate on the limit $m_b \to 0$. In this case the gap equation (4.4) has a nontrivial solution:

$$M = \mu \; . \tag{4.18}$$

The renormalized four-point PGF is

$$\Gamma^4_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4)$$
$$= -\left[\delta_{\alpha\beta}\delta_{\gamma\delta}\frac{g}{[(p_1+p_2)^2+4M^2]2A(p_1+p_2)} + \gamma^5_{\alpha\beta}\gamma^5_{\gamma\delta}\frac{g}{2(p_1+p_2)^2A(p_1+p_2)} - (\beta\to\delta, p_2\to p_4)\right] . \tag{4.19}$$

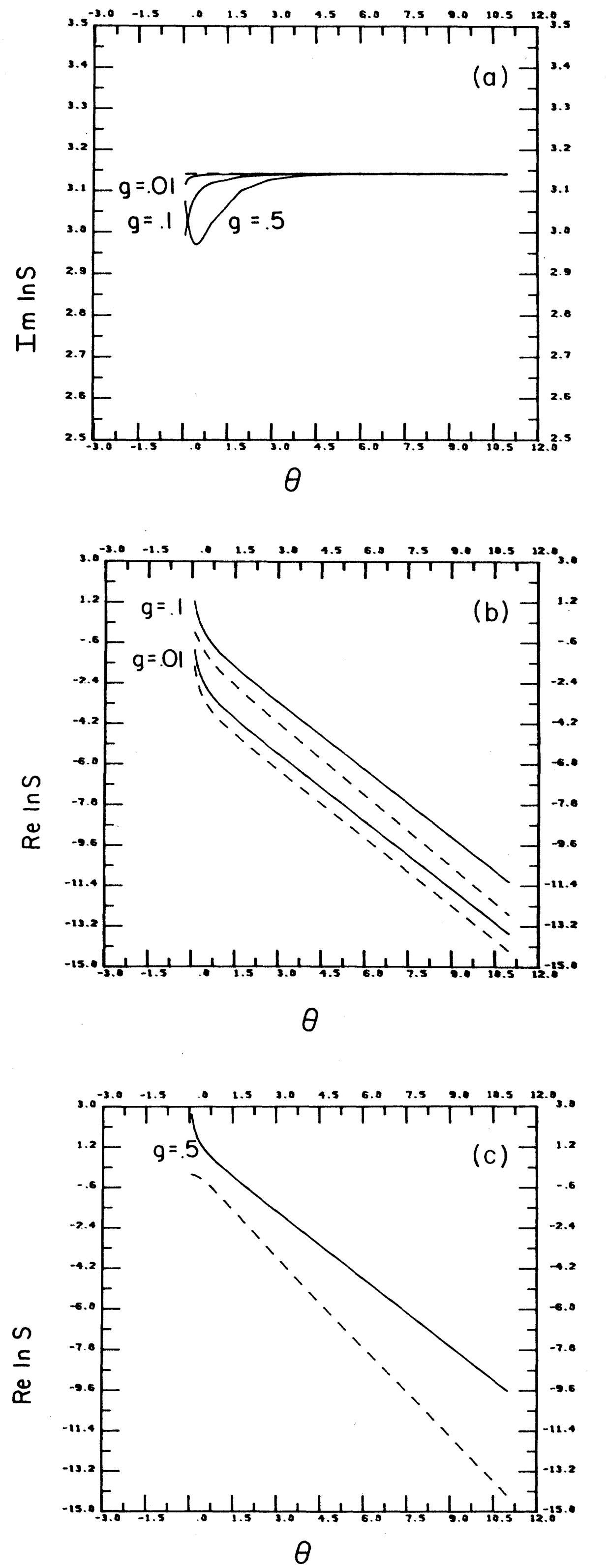


FIG. 3. The fermion-antifermion backward-scattering S-matrix element of the massive Thirring model for different values of coupling g. The exact solution of Ref. 2 is shown in dashed lines, the Gaussian approximation result in solid lines.

Here again the dimensional transmutation phenomenon occurs. However the theory of Eq. (4.1) with $m_b=0$ possesses a continuous U(1) axial symmetry. The generation of mass would mean that this symmetry is spontaneously broken, which is forbidden in two dimensions.[19] Indeed Γ^4 has a Goldstone-particle pole at zero momentum. If one would consider post-Gaussian corrections[20] this massless excitation is likely to lead to a severe infrared problem destabilizing this phase. A similar situation occurs in theories where the axial symmetry is broken in $1/N$ leading order when the next to leading corrections are taken into account.[21] Thus this phase must be unstable beyond Gaussian approximation.

Turning to the third solution of the normalization conditions Eq. (4.11) we find the renormalized four-point function to be

$$\Gamma^4_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4)=-\frac{1}{\frac{1}{2\pi}\ln\frac{M^2}{\mu^2}+2J(p_1+p_3)}\times(\gamma^\mu_{\alpha\beta}\gamma^\mu_{\gamma\beta}-\gamma^\mu_{\alpha\delta}\gamma^\mu_{\beta\delta})\ . \tag{4.20}$$

The effective coupling vanishes at large momenta

$$\lambda(s)\underset{s\to\infty}{\longrightarrow}-2\pi\left[\ln\frac{s}{M^2}\right]^{-1},\quad \beta(\lambda)=-\frac{1}{\pi}\lambda^2\ . \tag{4.21}$$

So we find an additional, asymmetrically free phase in the massive Thirring model. In this phase in the limit $m_b=0$ the only solution of the gap equation is the trivial one $M=0$, and the axial symmetry is not broken.

It is easy to generalize these results to the SU(N)-Abelian Thirring model:

$$L=\overline{\Psi}(i\partial\!\!\!/+m_b)\Psi^i+\frac{g}{2}(\overline{\Psi}\gamma_\mu\Psi^i)^2,\quad i=1,\ldots,N\ . \tag{4.22}$$

It is straightforward to repeat all the steps of our analysis in this case. The gap equation is

$$M=m_b-\frac{g}{2}M\ln\frac{\Lambda^2}{M^2}\ . \tag{4.23}$$

Again there are three possibilities to renormalize the theory:

$$g=g_r\ , \tag{4.24}$$

$$g=2\pi\left[\ln\frac{\Lambda^2}{\mu^2}\right]^{-1}, \tag{4.25}$$

$$g=-2\pi\left[\ln\frac{\Lambda^2}{\mu^2}\right]^{-1}. \tag{4.26}$$

In the first case one obtains the renormalized four-point PGF:

$$\Gamma^{4ijkl}_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4)=2(\delta^{ij}\delta^{kl}\gamma^\mu_{\alpha\beta}\gamma^\mu_{\gamma\delta}-\delta^{il}\delta^{jk}\gamma^\mu_{\alpha\beta}\gamma^\mu_{\gamma\beta})+g\left[\frac{1}{(p_1+p_2)^2}\delta^{ij}\delta^{kl}\left[-(p\!\!\!/_1+p\!\!\!/_2)_{\alpha\beta}(p\!\!\!/_1+p\!\!\!/_2)_{\gamma\delta}-\frac{1}{1-8NM^2A(p_1+p_2)}(p\!\!\!/_1+p\!\!\!/_2)\gamma^5_{\alpha\beta}(p\!\!\!/_1+p\!\!\!/_2)\gamma^5_{\gamma\delta}\right]-(j\to l,\beta\to\delta,p_2\to p_4)\right]\ . \tag{4.27}$$

This is a theory with finite ultraviolet fixed point. In the case of Eq. (4.26) one obtains

$$\Gamma^{4ijkl}_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4)=g\left[-\delta^{il}\delta^{jk}\delta_{\alpha\beta}\gamma_{\gamma\delta}\left[\frac{1}{[(p_1+p_2)^2+4M^2]2A(p_1+p_2)+\frac{g}{4\pi}\ln\frac{M^2}{\mu^2}}-\gamma^5_{\alpha\beta}\gamma^5_{\gamma\delta}\frac{1}{2(p_1+p_2)^2A(p_1+p_2)+\frac{g}{4\pi}\ln\frac{M^2}{\mu^2}}\right]-(j\to l,\beta\to\delta,p_2\to p_4)\right]\ . \tag{4.28}$$

This has the same infrared problem in the limit $m_b\to0$ as in the one-component case (since then $M=\mu$). The solution, Eq. (4.25), leads to an asymptotically free phase in analogy with the one component case.

Thus we conclude, that the CGA gives the following results for the Thirring model [and indeed the theory Eq. (4.22) for any N]. The theory has a line of ultraviolet fixed points $g<-\pi/N$. In the limit $m_b\to0$ the axial symmetry remains unbroken. The only solution of the gap equation in this case is $M=0$. As discussed in Sec. III this solution cannot be taken at its face value. The correct interpretation is that no consistent solution of the gap equation exists and thus there are no elementary fermions in the spectrum of the theory. This is in full accord with the rigorous reults.[1] In addition to that an asymptotically free solution with vanishing bare coupling is possible.

V. CONCLUSIONS

We have analyzed in covariant Gaussian approximation two solvable two-dimensional fermionic QFT: the

SU(N) Gross-Neveu and massive Thirring models.

In the Gross-Neveu theory we found that three renormalizations are possible. One is almost identical to the $1/N$ leading-order renormalization. In this case the theory is asymptotically free. Fermion mass is dynamically generated and a dimensional transmutation phenomenon takes place. There are two more possibilities to renormalize the theory. In these two cases mass is not generated and the spectrum of the theory does not contain fermions. The effective coupling in these cases does not depend on N which makes is plausible that they are realized for some specific values of N. This is in agreement with the known fact that for $N=1$ and 2 the theory does not have real fermions in the spectrum.

An important difference in this connection must be noted between the CGA and the leading order of $1/N$ expansion. In $1/N$ expansion only one renormalization is possible. Since all the expressions are analytic in N the same qualitative picture persists for all the values of N (if the results of the approximation are taken seriously). The situation for $N=1$ is rather strange. In this case there still is the dynamical mass generation and the fermion acquires mass. However for $N=1$ the Gross-Neveu model coincides with the massless Thirring model and possesses an additional U(1) axial symmetry. The appearance of a massive fermion signalizes spontaneous breakdown of this symmetry. However the Green's function equation (3.18) shows no sign of a Goldstone pole. This paradox is resolved in the CGA. For $N=1$ two renormalizations, Eqs. (3.14) and (3.17), coincide. Therefore, the limit $N\rightarrow 1$ must be taken not in the renormalized expression for the four-point GF but rather in the bare one [Eq. (3.13)]. When this is done carefully additional contributions to Eq. (3.18) appear and we indeed find the four-point function of Eq. (4.19) which exhibits a pole at zero momentum in the axial channel. In addition to solving this puzzle the CGA provides for the discovery of other phases which better represent the true behavior of the theory for small N. Thus although the CGA is in some cases similar to $1/N$ leading order it does not always give the same results and sometimes can have greater generality.

It is interesting to note that in the case of $1/N$ renormalization there is a strong similarity with the Φ^4 theory in four dimensions. The energy density of Gaussian states with different masses is not bounded from below as a function of mass as long as the ultraviolet cutoff is finite. In the limit $\Lambda\rightarrow\infty$ the solution of the gap equation is protected from decaying into a state with infinite mass by an infinite potential barrier and thus becomes stable. Exactly the same feature is observed in the Φ^4 theory where (for the case of infinitesimal negative bare coupling) the global minimum for finite cutoff is a state with infinite field shift ϕ (Ref. 15). Notwithstanding this instability at finite cutoff, the results of our approximation (which in this case coincide with the $1/N$ leading order) are in good agreement with exact results for the Gross-Neveu model, which is exactly solvable. This in our view is a strong indication that Φ^4_{3+1} theory cannot be discarded just because of the same instability (or equivalently because its bare coupling is negative). Thus the theory with negative coupling discussed in Refs. 4, 6, and 15 is a good candidate to be a nontrivial well-defined QFT.

In the case of the massive Thirring model we found two phases. In the first one the coupling constant undergoes finite renormalization. The theory has a line of ultraviolet fixed points. In the second phase the bare coupling is infinitesimally small and the theory is asymptotically free. The same is true for the SU(N) generalization of the Thirring model.

The comparison of S-matrix elements of both Gross-Neveu and Thirring models with exact expressions reveals that in both cases there is a qualitative agreement. In the former model it is better for large values of N and in the latter case—for small values of coupling.

We conclude that the CGA is capable of giving qualitatively good results for QFT. This justifies further attempts to apply the approximation to more complicated and phenomenologically relevant theories.

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APPENDIX

In this appendix we present the relevant formulas for the algebra of the "projector" matrices defined by Eq. (3.11). We use the same representation of Dirac matrices as throughout the paper:

$$\gamma_0=i\sigma_3,\quad \gamma_1=i\sigma_1,\quad \gamma_5\equiv\gamma_0\gamma_1=-\sigma_2\,, \qquad \{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu}\,. \tag{A1}$$

They obey the Fierz identity

$$\gamma^\mu_{\alpha\gamma}\gamma^\mu_{\beta\delta}=\delta_{\alpha\gamma}\delta_{\beta\delta}+\gamma^5_{\alpha\gamma}\gamma^5_{\beta\delta}-2\delta_{\alpha\beta}\delta_{\gamma\delta}\,. \tag{A2}$$

The algebra of the "projector" matrices is

$$\begin{aligned}
&a_1a_1=a_1,\quad a_2a_2=a_2,\quad b_1b_1=b_1,\quad b_2b_2=b_2,\\
&c_1c_1=c_2c_2=d_1d_1=d_2d_2=0\,,\\
&a_1a_2=b_1b_2=0,\quad a_ib_j=0,\quad a_ic_j=c_ia_j=0\,,\\
&b_id_j=d_ib_j=0\,,\\
&b_1c_2=b_2c_1=c_1b_1=c_2b_2=0,\\
&a_1d_1=a_2d_2=d_1a_2=d_2a_1=0\,,\\
&b_1c_1=c_1,\quad b_2c_2=c_2,\quad c_1b_2=c_1,\quad c_2b_1=c_2\,,\\
&a_1d_2=d_2,\quad a_2d_1=d_1,\quad d_1a_1=d_1,\quad d_2a_2=d_2\,.
\end{aligned} \tag{A3}$$

Using Eq. (A2) and the definition equations (3.11) one can obtain the following expressions that were used to arrive at formulas of Secs. III and IV:

$$\gamma^\mu_{\alpha\gamma}\gamma^\mu_{\beta\delta}=2(b_{1[\alpha\beta][\gamma\delta]}-a_{1[\alpha\beta][\gamma\delta]}),\quad \gamma^\mu_{\alpha\beta}\gamma^\mu_{\gamma\delta}=-2(b_{2[\alpha\beta][\gamma\delta]}+a_{2[\alpha\beta][\gamma\delta]}),\quad \gamma^\mu_{\alpha\beta}\gamma^\mu_{\beta\gamma}=-2(b_{1[\alpha\beta][\gamma\delta]}+a_{1[\alpha\beta][\gamma\delta]}),$$

$$\gamma^5_{\alpha\gamma}\gamma^5_{\beta\delta}=b_{1[\alpha\beta][\gamma\delta]}+a_{1[\alpha\beta][\gamma\delta]}-b_{2[\alpha\beta][\gamma\delta]}-a_{2[\alpha\beta][\gamma\delta]},\quad \gamma^5_{\alpha\delta}\gamma^5_{\beta\gamma}=a_{1[\alpha\beta][\gamma\delta]}-b_{1[\alpha\beta][\gamma\delta]}-b_{2[\alpha\beta][\gamma\delta]}-a_{2[\alpha\beta][\gamma\delta]}, \tag{A4}$$

$$\delta_{\alpha\delta}\delta_{\beta\gamma}=a_{1[\alpha\beta][\gamma\delta]}-b_{1[\alpha\beta][\gamma\delta]}+b_{2[\alpha\beta][\gamma\delta]}+a_{2[\alpha\beta][\gamma\delta]},\quad \delta_{\alpha\gamma}\delta_{\beta\delta}=a_{1[\alpha\beta][\gamma\delta]}+b_{1[\alpha\beta][\gamma\delta]}+b_{2[\alpha\beta][\gamma\delta]}+a_{2[\alpha\beta][\gamma\delta]}.$$

To invert the matrices $(1+X)$ and $(1+Y)$ we used

$$(1-\alpha_1a_1-\alpha_2a_2-\beta_1b_1-\beta_2b_2-\gamma_1c_1-\gamma_2c_2)^{-1}$$
$$=\frac{1}{1-\alpha_1}a_1+\frac{1}{1-\alpha_2}a_2+\frac{1}{(1-\beta_1)(1-\beta_2)-\gamma_1\gamma_2}[(1-\beta_2)b_1+(1-\beta_1)b_2+\gamma_1c_1+\gamma_2c_2], \tag{A5}$$

$$(1-\alpha_1a_1-\alpha_2a_2-\beta_1b_1-\beta_1b_2-\gamma_1d_1-\gamma_2d_2)^{-1}$$
$$=\frac{1}{1-\beta_1}b_1+\frac{1}{1-\beta_2}b_2+\frac{1}{(1-\alpha_1)(1-\alpha_2)-\gamma_1\gamma_2}[(1-\alpha_2)a_1+(1-\alpha_1)a_2+\gamma_1d_1+\gamma_2d_2)]. \tag{A6}$$

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[17]An example of phenomenon of this kind is furnished by the massless Abelian SU(N) Thirring model. By means of bosonisation techniques one can represent the theory as a theory of N decoupled bosonic degrees of freedom, where only one of the degrees of freedom is sensitive to the bare coupling g. See Ref. 21.

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