Loop expansion in a functional space

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As an alternative to the loop expansion of the effective potential, we suggest a functional expansion of the generating functional for an n-point Euclidean Green's function. The formulation of the scheme is independent of the space-time dimension of the model. The scheme yields standard perturbation theory in the regime of a small coupling constant and allows to extract information on the regime of strong coupling. As an explicit example we consider the scalar Φ^4 model and compute contributions up to the four-loop level.

I. INTRODUCTION

Coleman and Weinberg[1] have shown the usefulness of the loop expansion of the effective potential to study spontaneous symmetry breaking in massless scalar electrodynamics. In Ref. 2 Jackiw has discussed the two-loop approximation of the effective potential for the model of n self-interacting scalar fields. In particular he pointed out the need for computing higher-order loop contributions in field theory. The purpose of this paper is to propose a systematic functional approximation of the generating functional of the n-point Euclidean Green's function. The n-point Green's function plays an important role in many-body, condensed-matter, and particle physics. We choose as finite-dimensional functional space the functionals obtained by expanding the generating functional in the coupling constant, and then compute $\exp\{\int d^4x\,\mathcal{L}^{\rm int}[\partial/\partial J(x)]\}$ in this finite-dimensional functional space. This functional approximation has the following properties. It corresponds to summing all loop diagrams, generated by the finite-dimensional functional space, up to infinite order. Expansion of the functional approximation in the coupling constant reproduces standard perturbation theory. We want to underline two features of this scheme: (a) The dimension of the functional space is independent from the space-time dimension of the model (contrary to lattice field theory, where the number of lattice nodes is related to the space-time dimension); (b) the scheme is Lorentz invariant, like ordinary perturbation theory (but contrary to lattice field theory, which violates Lorentz symmetry on a finite lattice). In Sec. II we introduce the functional approximation. In Sec. III we define the finite-dimensional functional space and introduce a scalar product. Section IV explains how to compute $\exp\{\int d^4x\,\mathcal{L}^{\rm int}[\partial/\partial J(x)]\}$. In Sec. V we comment on mathematical properties of the approximation. In Sec. VI we give an explicit example applying it to the scalar Φ^4 model and compute contributions up to the four (six)-loop level. In Sec. VII we discuss renormalization and perform it explicitly up to the two-loop level. A conclusion is given in Sec. VIII.

II. FUNCTIONAL EXPANSION

We consider the scalar Φ^4 model in $3+1$ dimensions in its Euclidean form, given by the Lagrangian

$$\mathcal{L}=-\left[\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4\right]. \tag{1}$$

Let us recall some notations and properties of the generating functional $W[\mathcal{J}]$:

$$W[\mathcal{J}]=\mathcal{N}^{-1}\int\mathcal{D}\phi\,\exp\left[\int d^4x\,\mathcal{L}(\phi)+\mathcal{J}\phi\right],$$
$$\mathcal{N}{:}\,W[0]=1\,. \tag{2}$$

From $W[\mathcal{J}]$, one can obtain the n-point Green's function via

$$G(x_1,\ldots,x_n)=\frac{\delta^n}{\delta\mathcal{J}(x_1)\cdots\delta\mathcal{J}(x_n)}W[\mathcal{J}]\bigg|_{\mathcal{J}=0}. \tag{3}$$

As a starting point for perturbation theory, one usually defines a free generating functional $W^{\rm free}[\mathcal{J}]$, defined as in Eq. (2), but substituting the Lagrangian by the free Lagrangian $\mathcal{L}^{\rm free}$. The following well-known relation holds:[3,4]

$$W[\mathcal{J}]=\mathcal{N}^{-1}\exp\left[\int d^4x\,\mathcal{L}^{\rm int}\left[\frac{\delta}{\delta\mathcal{J}(x)}\right]\right]W^{\rm free}[\mathcal{J}]\,. \tag{4}$$

The free generating functional can be computed explicitly to give

$$W^{\rm free}[\mathcal{J}]=\exp\left[\tfrac{1}{2}\int d^4x\,d^4y\,\mathcal{J}(x)\Delta(x,y)\mathcal{J}(y)\right], \tag{5}$$

where Δ denotes the free propagator

$$\Delta(x,y)=\int\frac{d^4k}{(2\pi)^4}\frac{\exp[ik\cdot(x-y)]}{k^2+m^2}\,. \tag{6}$$

We write Eq. (4) in shorthand, suppressing the normalization $\mathcal{N}$,

$$W[\mathcal{J}]=\exp(A)W^{\rm free}[\mathcal{J}]\,, \tag{7}$$

where A is an operator in an infinite-dimensional functional space $\mathcal{F}$. Let us construct a finite-dimensional functional space

$$\mathcal{F}^{(N)}=\{f_n[\mathcal{J}]\,|\,n=0,\ldots,N\}\ . \tag{8}$$

In order to make this a meaningful definition, the functionals $f_n[\mathcal{J}]$ should have something to do with $W[\mathcal{J}]$. That will be specified below. Firstly, we suggest to approximate A by $A^{(N)}$, chosen such that

$$A^{(N)}\mathcal{F}^{(N)}\rightarrow\mathcal{F}^{(N)}\ . \tag{9}$$

However, $\mathcal{F}^{(N)}$ is not an invariant subspace of A, in general. We demand that the mapping of A agrees with the mapping of $A^{(N)}$ on a subspace of $\mathcal{F}^{(N)}$ as large as possible. Secondly, we suggest to approximate $W^{\text{free}}[\mathcal{J}]$ by $W^{(N)}_{\text{free}}[\mathcal{J}]$. Finally, we suggest to approximate $W[\mathcal{J}]$ by $W^{(N)}[\mathcal{J}]$, given by

$$W^{(N)}[\mathcal{J}]=\exp(A^{(N)})W^{(N)}_{\text{free}}[\mathcal{J}]\ . \tag{10}$$

The above construction has been introduced because the functional operator $A^{(N)}$ is represented by a finite-dimensional matrix, which can be diagonalized and hence $\exp(A^{(N)})$ can be computed. That means that we can compute $W^{(N)}[\mathcal{J}]$, i.e., the generating functional in a finite-dimensional "toy" space $\mathcal{F}^{(N)}$. The procedure can be interpreted as partial summing of perturbation theory, if we choose the functionals in $\mathcal{F}^{(N)}$ as those which would appear in perturbation theory. Then one hopes to learn something about the strong-coupling regime from the small eigenvalues of $A^{(N)}$. Below we will elaborate the idea, using the functional basis from perturbation theory; which will lead to a loop expansion. It might be interesting, however, to try finite-dimensional functional spaces constructed from other functionals also.

III. CHOOSING $\mathcal{F}^{(N)}$ FROM PERTURBATION THEORY

In ordinary perturbation theory, one would expand Eq. (7) in the coupling constant, i.e., in powers of A, to a given order K,

$$W^{\text{pert}}[\mathcal{J}]=\sum_{n=0}^{K}\frac{A^n}{n!}W^{\text{free}}[\mathcal{J}]\ . \tag{11}$$

We identify $W^{(N)}_{\text{free}}\equiv W^{\text{free}}$ and define $\mathcal{F}^{(N)}$ via the basis

$$f_n[\mathcal{J}]=A^nW^{\text{free}}[\mathcal{J}],\quad n=0,\ldots,N\ . \tag{12}$$

Clearly, A does not leave $\mathcal{F}^{(N)}$ invariant. In order to define $A^{(N)}$, we introduce on $\mathcal{F}^{(N)}$ (or more generally on $\mathcal{F}$) a scalar product. Because of the requirement that A and $A^{(N)}$ should agree on a maximally large subspace we define an orthogonal projector $P^{(N)}$ and put

$$A^{(N)}=P^{(N)}AP^{(N)}\ . \tag{13}$$

Thus $A^{(N)}$ leaves $\mathcal{F}^{(N)}$ invariant and agrees with A on $\mathcal{F}^{(N-1)}$.

In order to specify a suitable scalar product, we look at the functional dependence of f_n on $\mathcal{J}$. It has the form

$$A^nW^{\text{free}}[\mathcal{J}]=Q_n[\mathcal{J}]W^{\text{free}}[\mathcal{J}]\ , \tag{14}$$

where $Q_n[\mathcal{J}]$ is functional polynomial in $\mathcal{J}$ of degree $4n$, in the case of the Φ^4 model, and $W^{\text{free}}[\mathcal{J}]$, according to Eq. (5), is a Gaussian. Hence $A^nW^{\text{free}}[\mathcal{J}]$ is a functional like a Hermite function. However, there is a small but important difference. The exponent of the Hermite functions is negative, while here it is positive (the Euclidean propagator Δ being a positive operator). On the space of those Hermite-type functionals we define a scalar product by

$$\begin{aligned}\langle f|g\rangle&=\langle PW^{\text{free}}|QW^{\text{free}}\rangle=\int d[i\mathcal{J}]P[i\mathcal{J}]^*W^{\text{free}}[i\mathcal{J}]^*Q[i\mathcal{J}]W^{\text{free}}[i\mathcal{J}]\\&=\lim_{M\to\infty}i^M\int_{-\infty}^{\infty}d\mathcal{J}_1\cdots\int_{-\infty}^{\infty}d\mathcal{J}_MP[i\mathcal{J}]^*Q[i\mathcal{J}]\exp\left[-\sum_{m,n=1}^{M}\mathcal{J}_m\Delta_{mn}\mathcal{J}_n\right]\ .\end{aligned} \tag{15}$$

We have introduced the imaginary unit i into the definition, in order to obtain a minus sign in the exponent. Now we have a Wiener-type functional integral with an exponential fall-off behavior, which guarantees the existence of the integral. The scalar product has all the properties of an ordinary scalar product, except that it is not positive, $\langle f|f\rangle\not\geq 0$, due to the factor $[i]$ in the measure. This is no problem, because in the following we will have to compute ratios of scalar products; hence this factor will cancel out in the final expression for the generating functional.

With this definition of the scalar product, one can verify the following properties of the functional operator A, defined by Eqs. (1) and (4):

$$A=\frac{-\lambda}{4!}\int d^4x\left[\frac{\delta}{\delta\mathcal{J}(x)}\right]^4\ . \tag{16}$$

Firstly, A is Hermitian, i.e.,

$$\langle f|A|g\rangle=\langle Af|g\rangle\ , \tag{17}$$

and secondly, A is a negative operator for a positive λ: i.e.,

$$\frac{\langle f|A|f\rangle}{\langle f|f|\rangle}\leq 0\ . \tag{18}$$

IV. COMPUTATION OF $\exp(A^{(N)})$ IN THE EIGENREPRESENTATION OF $A^{(N)}$

Starting from the basis, given by Eq. (12), we define the matrices

$$\begin{aligned}A_{mn}&=\langle f_m|A|f_n\rangle\ ,\\B_{mn}&=\langle f_m|f_n\rangle\ .\end{aligned} \tag{19}$$

which both are real and symmetric. Then the projector $P^{(N)}$ and the functional operator $A^{(N)}$ can be expressed as

$$P^{(N)}=\sum_{m,n=0}^{N}|f_m[\mathcal{J}]\rangle(B^{-1})_{mn}\langle f_n[\mathcal{J}]| ,$$
$$A^{(N)}=\sum_{m,n=0}^{N}|f_m[\mathcal{J}]\rangle(B^{-1}AB^{-1})_{mn}\langle f_n[\mathcal{J}]| . \quad (20)$$

Now we can diagonalize $A^{(N)}$:

$$A^{(N)}|v_m[\mathcal{J}]\rangle=\alpha_m|v_m[\mathcal{J}]\rangle,\ \ \langle v_m|v_n\rangle=\delta_{m,n} . \quad (21)$$

Let us express the eigenvectors v_m in the basis of the functionals f_n:

$$|v_m[\mathcal{J}]\rangle=\sum_{n=0}^{N}\eta_{mn}|f_n[\mathcal{J}]\rangle . \quad (22)$$

Then the eigenvalues α are determined by

$$\det(A-\alpha_n B)=0 , \quad (23)$$

while the eigenvectors η are determined by

$$\eta AB^{-1}=\alpha\eta , \quad (24)$$

with the normalization

$$\eta B\eta^{\dagger}=1 . \quad (25)$$

Then we can write

$$\exp(A^{(N)})=\sum_{n=0}^{N}|v_n[\mathcal{J}]\rangle\exp(\alpha_n)\langle v_n[\mathcal{J}]|$$
$$=\sum_{m,n=0}^{N}|f_m[\mathcal{J}]\rangle[\eta^T\exp(\alpha)\eta^*]_{mn}\langle f_n[\mathcal{J}]| \quad (26)$$

and the generating functional, Eq. (10),

$$|W^{(N)}[\mathcal{J}]\rangle=\frac{1}{\mathcal{N}^{(N)}}\sum_{n=0}^{N}|f_n[\mathcal{J}]\rangle[\eta^T\exp(\alpha)\eta^*B]_{n,0} ,$$
$$\mathcal{N}^{(N)}:W^{(N)}[0]=1 . \quad (27)$$

Here we can already say something about the structure of the solution. All functional dependence on $\mathcal{J}$ is given by the functionals $f_n[\mathcal{J}]$. The coefficient matrix is independent of $\mathcal{J}$ (all $\mathcal{J}$ dependence has been integrated out in the scalar products) and hence it will be composed of loop diagrams only. This will be discussed in the next section in some detail.

Because the operator A is Hermitian and negative for a positive coupling constant λ, the behavior of the generating functional in the region of a large (positive) λ is dominated by the smallest eigenvalues of A. We can compute the eigenvalues of the $A^{(N)}$. However, the question arises, is A still negative after renormalization? This is an open question. But what we can do is compute the renormalized $A^{(N)}$ (see Sec. VII), determine its eigenvalues, and check if they all have the same sign.

We want to conclude this section with a comparison of $W^{(N)}[\mathcal{J}]$ and $W[\mathcal{J}]$ in the coupling-constant expansion. We claim that both agree in the coupling-constant expansion to order K if $K\le N$. Expansion of $W^{(N)}[\mathcal{J}]$ in the coupling constant to order K reads

$$W^{(N)}_{\text{pert}}[\mathcal{J}]=\sum_{n=0}^{K}\frac{A^{(N)^n}}{n!}W^{\text{free}}[\mathcal{J}] . \quad (28)$$

We find that this agrees with the expansion of $W[\mathcal{J}]$, Eq. (11), for $K\le N$. This is due to the construction of $A^{(N)}$, Eq. (13), which leaves $\mathcal{F}^{(N)}$ invariant and agrees with A on $\mathcal{F}^{(N-1)}$. This is true independent of the particular choice of the scalar product.

V. COMMENT ON CONVERGENCE PROPERTIES

The idea to approximate the function of an operator, $\exp(A)$, by $\exp(A^{(N)})$, where $A^{(N)}$ is an operator in a finite-dimensional subspace, has been found useful in time-dependent scattering calculations, where the time evolution $\exp(iHt)$ has been approximated in this way.[5] In nonrelativistic few-body systems one can show for a wide class of potentials that $\exp(iH^{(N)}t)$ converges to $\exp(iHt)$ in the strong sense.[6] Although this approximation has been shown to work in numerical calculations for several field theories [Φ^4_{1+1} model,[7] Thirring model,[8] QED_{1+1} (Ref. 9)], a rigorous proof is lacking in those cases. Nevertheless, the idea is supported by practical experience.

However, there is a critical point, which deserves some discussion. What we have done is to approximate the functional $W[\mathcal{J}]$ by the functional $W^{(N)}[\mathcal{J}]$. The approximation is defined by the topology introduced by the scalar product based on a functional integration over the source fields, where the integration extends from $-\infty$ to $+\infty$. However, for the purpose of computing the Green's function, one needs $W[\mathcal{J}]$ and its functional derivatives at $\mathcal{J}=0$. Of course we can compute the functional $W^{(N)}[\mathcal{J}]$ at $\mathcal{J}=0$, but is this a good approximation? Certainly not in a mathematically rigorous sense. Considering as an example the approximation of ordinary functions by polynomials (Hermite) with a Gaussian weight,

$$\int_0^{\infty}dt\,\exp(-t^2)\left[2t[P_n(t)-f(t)]+\frac{d}{dt}[P_n(t)-f(t)]\right]$$
$$=-P_n(0)+f(0) , \quad (29)$$

which shows that in order to establish convergence at the point $t=0$, one needs convergence of the function plus its derivative in the L_2 sense. In principle, we could simulate this also in our functional space, by including a term with a functional derivative in the scalar product. However, this would make the formalism quite clumsy. Nevertheless we think that the functional approximation scheme proposed above is useful, because, after expansion in the coupling constant, it coincides with ordinary perturbation theory, where $J=0$ is taken.

VI. EXAMPLE: SIX-LOOP EXPANSION $N=1$

The case $N=0$ is trivial. It is completely determined by W^{free} and the normalization of $\mathcal{N}^{(0)}$: i.e.,

$$W^{(0)}[\mathcal{J}]=W^{\text{free}}[\mathcal{J}]\,. \tag{30}$$

Let us consider now the simplest nontrivial case, $N=1$. Using the abbreviation $|f_n\rangle \equiv |n\rangle$, the matrices A and B, given by Eq. (19), read

$$A=\begin{bmatrix}\langle 0|1\rangle & \langle 0|2\rangle\\ \langle 0|2\rangle & \langle 0|3\rangle\end{bmatrix},\qquad B=\begin{bmatrix}\langle 0|0\rangle & \langle 0|1\rangle\\ \langle 0|1\rangle & \langle 0|2\rangle\end{bmatrix}. \tag{31}$$

Thus we have to compute the matrix elements $\langle 0|0\rangle,\ldots,\langle 0|3\rangle$. It will turn out that one can extract by factorization a common factor $\langle 0|0\rangle$ from the matrix elements $\langle 0|1\rangle,\ldots,\langle 0|3\rangle$. This has the following consequences: (a) Then Eq. (23) implies that the eigenvalues α_n are independent of $\langle 0|0\rangle$; (b) the matrix AB^{-1} is independent of $\langle 0|0\rangle$. Then Eq. (24) implies that the eigenvectors η are independent of $\langle 0|0\rangle$. However, the normalization of η depends on $\langle 0|0\rangle$ via Eq. (25). (c) Using Eq. (25), one can rewrite the coefficient matrix of the generating functional, Eq. (27):

$$\eta^T \exp(\alpha)\eta^* B=\eta^T\exp(\alpha)(\eta^T)^{-1}\,. \tag{32}$$

Here the normalization of η cancels out; hence, the whole term and hence the generating functional $W^{(N)}[\mathcal{J}]$ is independent of $\langle 0|0\rangle$. The above remarks hold in general for every common factor extracted from A and B. This will play a role when eliminating the volume dependence of the loop diagrams.

Now let us turn to the computation of the matrix elements.

(a) $\langle 0|0\rangle$. Although this matrix element is irrelevant, we will compute it in order to review the functional integration technique:

$$\begin{aligned}\langle 0|0\rangle=\langle W^{\text{free}}|W^{\text{free}}\rangle&=[i]\int d[\mathcal{J}]\exp\left[-\int d^4x\, d^4y\, \mathcal{J}(x)\Delta(x,y)\mathcal{J}(y)\right]\\ &=\lim_{M\to\infty} i^M\int_{-\infty}^{\infty}d\mathcal{J}_1\cdots\int_{-\infty}^{\infty}d\mathcal{J}_M\exp\left[-\sum_{i,j=1}^{M}\mathcal{J}_i\Delta_{ij}\mathcal{J}_i\right].\end{aligned} \tag{33}$$

Because Δ is a positive operator, one can compute its square root and introduce the variable transformation

$$\mathcal{J}_i'=\sum_{j=1}^{M}\sqrt{\Delta_{ij}}\mathcal{J}_j\,. \tag{34}$$

Hence we have

$$\langle 0|0\rangle=\lim_{M\to\infty} i^M\int_{-\infty}^{\infty}d\mathcal{J}_1'\cdots\int_{-\infty}^{\infty}d\mathcal{J}_M'\det\left[\frac{\delta\mathcal{J}_i}{\delta\mathcal{J}_j'}\right]\times\exp(-\mathcal{J}_i'J_i')\,. \tag{35}$$

One computes

$$\det\left[\frac{\delta\mathcal{J}_i}{\delta\mathcal{J}_j'}\right]=\frac{1}{\det(\sqrt{\Delta})} \tag{36}$$

and

$$\int_{-\infty}^{\infty}dx\,\exp(-x^2)=\sqrt{\pi}\,. \tag{37}$$

Thus we obtain finally

$$\langle 0|0\rangle=\lim_{M\to\infty}\frac{(i\sqrt{\pi})^M}{\sqrt{\det(\Delta)}}\,. \tag{38}$$

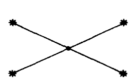
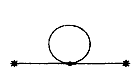
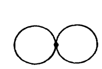
(b) $\langle 0|1\rangle$. To evaluate this matrix element one has to do a functional integral with functional derivatives, which can be done effectively by partial integration. But because the functional $|1\rangle=A|W^{\text{free}}[\mathcal{J}]\rangle$ has been given in the literature,[4] we want to start from that expression:

$$AW^{\text{free}}[\mathcal{J}]=\omega_1[\mathcal{J}]W^{\text{free}}[\mathcal{J}]\,, \tag{39}$$

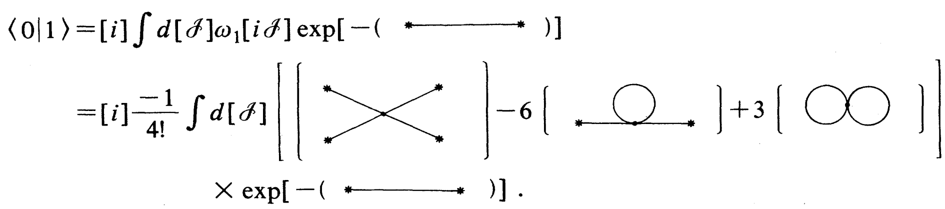
where

$$\begin{aligned}\omega_1[\mathcal{J}]=\frac{-\lambda}{4!}\Big[&\int d^4y\, d^4x_1\cdots d^4x_4\Delta(x_1,y)\cdots\Delta(x_4,y)\mathcal{J}(x_1)\cdots\mathcal{J}(x_4)\\ &+6\int d^4y\, d^4x_1 d^4x_2\Delta(x_1,y)\Delta(y,y)\Delta(y,x_2)\mathcal{J}(x_1)\mathcal{J}(x_2)+3\int d^4y\,\Delta^2(y,y)\Big]\end{aligned} \tag{40}$$

or, expressed in graphs, where the dot denotes a vertex, the straight line denotes the propagator Δ, and the asterisk denotes a source $\mathcal{J}$,

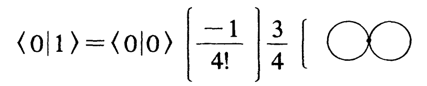
$\omega_1[\mathcal{J}]=\frac{-1}{4!}\Big[\Big[$  $\Big]+6\Big[$  $\Big]+3\Big[$  $\Big]\Big]\,.$ (41)

Thus we can write the matrix element (note the change of sign for terms with two sources)



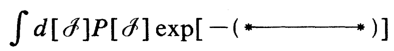
(42)

After a lengthy, but straightforward calculation, along the lines of Eqs. (33)–(38), one obtains



(43)

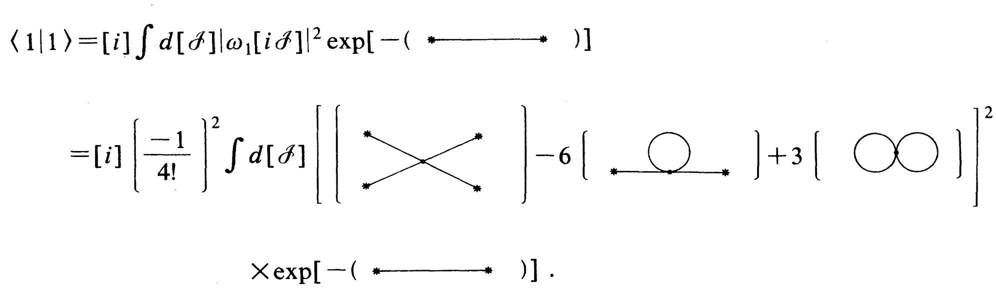
Here we would like to make a few comments. Firstly, the technique to do these functional integrals over source fields is the same as applied in stochastic quantization, e.g., when computing an n-point Green's function as an average value over Gaussian white-noise sources η (Ref. 10). In order to carry out a functional integral over sources ϑ of the type



(44)

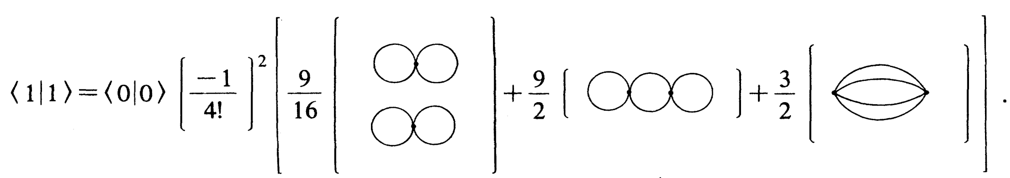
where $P[\vartheta]$ is a polynomial in ϑ, represented by a graph with external sources, one can use the following graphical rule. (a) Take two sources of the graph, join them and transform it into a line. Each such line gives a factor $\frac{1}{2}$. Do this for all sources. A graph with an odd number of sources will give a zero contribution. (b) Determine an overall factor, counting the number of possibilities to join two sources. Secondly, the total contribution coming from all three terms of the polynomial in Eq. (42), is identical to the contribution coming alone from the term with the highest number of sources ϑ (first term). This feature is also found in the other matrix elements $\langle 0|2\rangle$ and $\langle 0|3\rangle$. Thirdly, as mentioned above, the matrix element $\langle 0|1\rangle$ factors out the matrix element $\langle 0|0\rangle$, which is singular but irrelevant. Finally, the matrix element $\langle 0|1\rangle$ contains more singularities coming from the two-loop graph. There is an infinite-volume factor plus an ultraviolet divergence. The treatment of these singularities will be the theme of the next section.

(c) $\langle 0|2\rangle$. Because the functional operator A is Hermitian, one has $\langle 0|2\rangle = \langle 1|1\rangle$. Thus we can write, in analogy to Eq. (42),



(45)

Carrying out the calculation yields the result



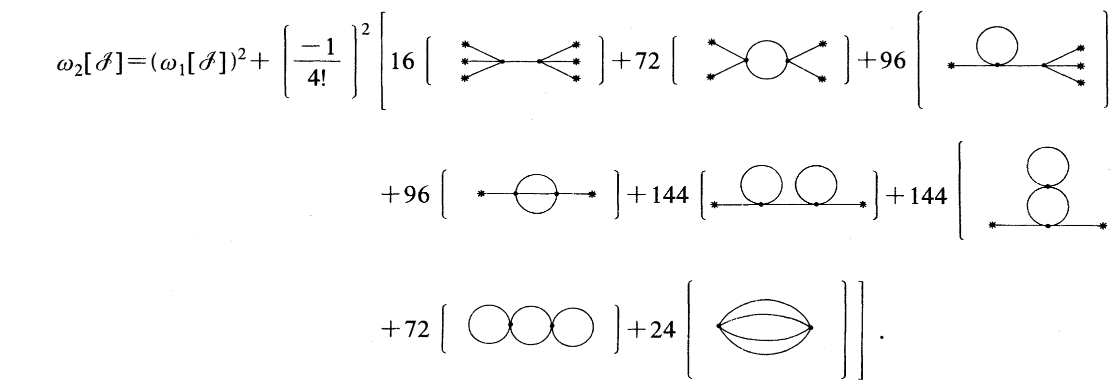
(46)

As stated above this matrix element factors out the matrix element $\langle 0|0\rangle$. Secondly, it is identical to the result that would have been obtained by taking into account only the term of the polynomial, Eq. (45), with the highest number of sources. The result contains an unlinked four-loop diagram, which has a square of the volume behavior.

(d) $\langle 0|3\rangle$. This is equal $\langle 1|2\rangle$. The functional $|2\rangle = A^2|W^{\text{free}}|\vartheta|\rangle$ has been computed in Ref. 4:

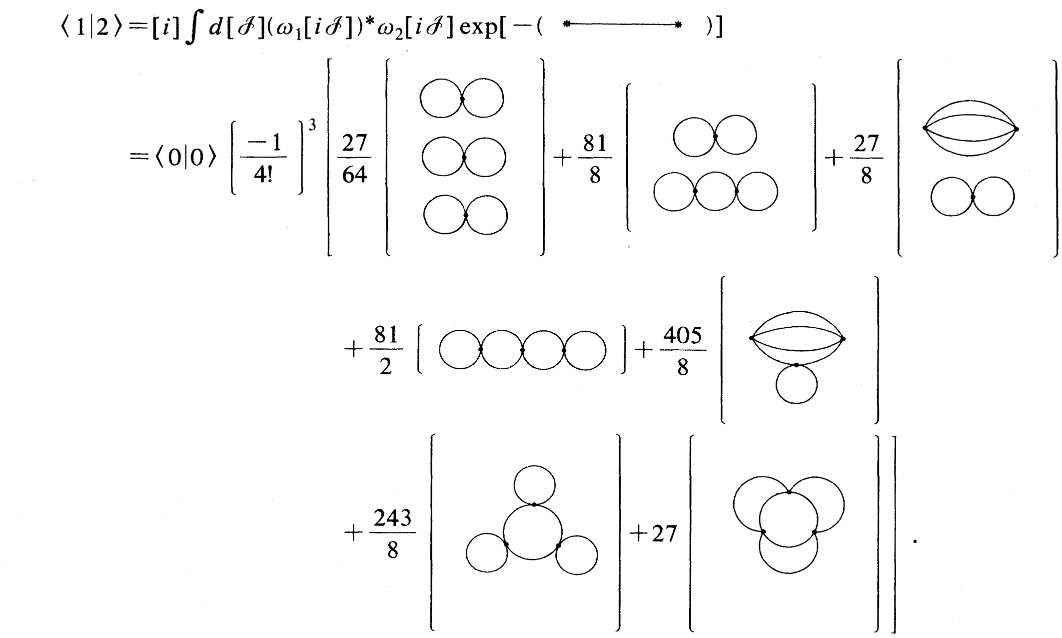
$$A^2 W^{\text{free}}[\vartheta] = \omega_2[\vartheta] W^{\text{free}}[\vartheta] , \tag{47}$$

where



(48)

Then we compute the matrix element



(49)

Again, this matrix element factors $\langle 0|0\rangle$. The result is identical to the one that would have been obtained by taking into account in the polynomial $\omega_1\omega_2$ only the term with the highest number of sources. In the result there is a contribution from an unlinked six-loop diagram, which has a cube of the volume behavior, and there are another two unlinked diagrams with a square of the volume behavior. Hence, the expansion parameter $N=1$ produces 6 as a maximal number of loops. The maximal number of loops for an arbitrary expansion parameter N would be $4N+2$.

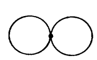
VII. RENORMALIZATION

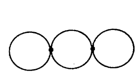
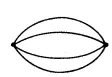
In order to extract from the matrix elements $\langle 0|0\rangle,\ldots,\langle 0|3\rangle$ the information relevant for the computation of physical observables, e.g., the n-point Green's function (in the Euclidean region), one has to obtain finite numbers for those matrix elements. As they stand, they contain three kinds of singularities: (a) the singular matrix element $\langle 0|0\rangle$ itself, (b) the dependence on the infinite volume, and (c) the ultraviolet divergence of the loop diagrams. In this section we want to discuss how to deal with these singularities.

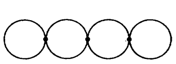
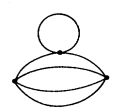
(a) As has been pointed out in the last section, a common factor appearing in the matrix elements $\langle 0|0\rangle,\ldots,\langle 0|3\rangle$ can be factored out and hence does not appear in the generating functional [compare Eq. (32)], nor in the Green's function. Thus we redefine the matrix elements $\langle 0|0\rangle,\ldots,\langle 0|3\rangle$ by division of $\langle 0|0\rangle$.

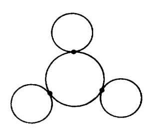
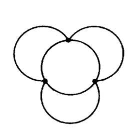
(b) We have observed that the matrix elements $\langle 0|2\rangle$ and $\langle 0|3\rangle$ (and the following ones, which would be involved for an expansion parameter $N\geq 2$) have contributions from unlinked diagrams, which invoke an infinite-volume factor to the orders 2 and 3, respectively. The physical observables such as scattering amplitudes are given by connected n-point Green's functions, which do not contain any volume dependence. In standard perturbation theory, expanding in the coupling constant, this is achieved by simply discarding the disconnected diagrams. Also in many-body theory, when computing the ground-state energy of a many-body system, one takes into account only the linked loop diagrams (Goldstone), which have a dependence on the volume to first order.[11] In the final expression for the ground-state energy density, the volume factor is canceled out. Also in our formalism we have to get rid of all volume dependence. To do this, we discard all unlinked loop diagrams in the matrix elements. Then we are left with terms which are all of first order in the volume. Then the volume dependence is completely canceled out in the expression for the generating functional, because we know that any common factor appearing in all matrix elements is eliminated from the generating functional. Thus we obtain the following redefined matrix elements, represented by loop diagrams, where we implicitly understand that the volume factor has been divided out:

$\langle 0|0\rangle = 0\,,$

$\langle 0|1\rangle = \left[\frac{-1}{4!}\right]\frac{3}{4}\Big[$  $\Big]\,,$

$\langle 0|2\rangle = \left[\frac{-1}{4!}\right]^2\Bigg[\frac{9}{2}\Big[$  $\Big]+\frac{3}{2}\Big[$  $\Big]\Bigg]\,,$

$\langle 0|3\rangle = \left[\frac{-1}{4!}\right]^3\Bigg[\frac{81}{2}\Big[$  $\Big]+\frac{405}{8}\Big[$  $\Big]$

$+\frac{243}{8}\Big[$  $\Big]+27\Big[$  $\Big]\Bigg]\,.$ (50)

Now we observe that the maximal number of loops is 4, contributing in the case $N=1$.

(c) The singularities left are the ultraviolet divergences. Note that in the case of the expansion parameter $N=1$, we have to renormalize a first-order loop diagram for the functional $|1\rangle$ [Eq. (41)] plus first-, second-, and third-order loop diagrams for the matrix elements [Eqs. (43), (46), and (49)]. We follow the discussion of renormalization given in Ref. 12, using dimensional regularization as suggested by 't Hooft and Veltman.[1] We rewrite the action

$$S_\omega[\phi] = -\int d^{2\omega}x\left[\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi+\tfrac{1}{2}m^2\phi^2+\frac{\lambda}{4!}(\mu^2)^{2-\omega}\phi^4\right]\,. \tag{51}$$

We will expand in $\epsilon=2-\omega$ around $\epsilon=0$. A new dimensionless vertex strength has been introduced via

$$\lambda_{\text{old}}=\lambda_{\text{new}}(\mu^2)^{2-\omega}\,, \tag{52}$$

where μ has the dimension of mass. Thus, corresponding to our original Lagrangian

$$\mathcal{L}=-\left[\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\mu^{2\epsilon}\phi^4\right]\,, \tag{53}$$

we add a counterterm Lagrangian

$$\mathcal{L}_{\text{ct}}=-\left[\tfrac{1}{2}A(\partial_\mu\phi)^2+\tfrac{1}{2}m^2B\phi^2+\frac{\lambda}{4!}\mu^{2\epsilon}C\phi^4+D\right] \tag{54}$$

and the renormalized Lagrangian is

$$\mathcal{L}_{\text{ren}}=\mathcal{L}+\mathcal{L}_{\text{ct}}\,. \tag{55}$$

In a first step we have to renormalize the infinities of the functionals. In the case $N=1$, only the functional f_1 has infinities. Let us look at the first-order one-loop diagram [Eq. (41)]. This leads to a mass renormalization. We put $A=C=D=0$ in Eq. (54): i.e.,

$$\mathcal{L}_{\text{ct}}=-\tfrac{1}{2}m^2B\phi^2\,. \tag{56}$$

Then the renormalized functional is given by

$$f_1^{\text{ren}}[\mathcal{J}]=A_{\text{ren}}W^{\text{free}}[\mathcal{J}]\,, \tag{57}$$

where

$$A_{\text{ren}}=A+A_{\text{ct}}=\int d^4x(-)\frac{\lambda}{4!}\mu^{2\epsilon}\left[\frac{\partial}{\partial\mathcal{J}(x)}\right]^4 -\tfrac{1}{2}m^2B\left[\frac{\partial}{\partial\mathcal{J}(x)}\right]^2\,. \tag{58}$$

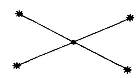
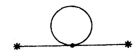
The counterterm gives

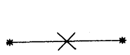
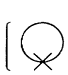
$$A_{\mathrm{ct}} W^{\mathrm{free}}[\mathcal{J}] = \omega_1^{\mathrm{ct}}[\mathcal{J}] W^{\mathrm{free}}[\mathcal{J}] \;, \tag{59}$$

where

$$\omega_1^{\mathrm{ct}}[\mathcal{J}] = -\tfrac{1}{2} m^2 B \int d^4 z \left[\Delta(z,z) + \left[\int d^4 x \mathcal{J}(x) \Delta(x,z) \right]^2 \right] . \tag{60}$$

Expressing the renormalized functional in terms of graphs

$$\omega_1^{\mathrm{ren}}[\mathcal{J}] = \omega_1[\mathcal{J}] + \omega_1^{\mathrm{ct}}[\mathcal{J}] = \frac{-1}{4!} \Bigg[\Bigg[$$  $$\Bigg] + 6 \Bigg[$$  $$\Bigg] + 3 \Bigg[$$  $$\Bigg] \Bigg]$$

$$+ -\tfrac{1}{2} m^2 B \Bigg[($$  $$) + \Bigg[$$  $$\Bigg] \Bigg] . \tag{61}$$

The number B is then determined by requiring that the contribution from the graphs with two sources becomes finite when $\epsilon \to 0$, i.e., using dimensional regularization:

$$-\frac{6}{4!}(-)\hat{\lambda} m^2 \left[\frac{1}{\epsilon} + \psi(2) - \ln(\hat{m}^2) + O(\epsilon) \right] - \tfrac{1}{2} m^2 B = \text{finite} \;; \tag{62}$$

hence,

$$B = \tfrac{1}{2} \hat{\lambda} \left[\frac{1}{\epsilon} + F_1(\epsilon, \hat{m}^2) \right] . \tag{63}$$

Here the following abbreviations have been used: $\hat{\lambda} = \lambda/(4\pi)^2$, $\hat{m}^2 = m^2/(4\pi\mu^2)$, and $\psi(x)$ being the logarithmic derivative of the gamma function. The function F_1 is arbitrary, but has to be analytic in $\epsilon = 0$. So far we have succeeded to renormalize the one-loop term of the functional f_1. However, there is also a two-loop vacuum diagram. This term will not contribute to a scattering amplitude, but only to the vacuum-vacuum amplitude and hence we neglect it. In principle it will be canceled by the normalization factor $\mathcal{N}$.

In a second step we have to renormalize the infinities of the first-, second-, and third-order loop diagrams occurring in the matrix elements. Let us look at the first-order two-loop diagram occurring in $\langle 0|1 \rangle$. Now we put $A = C = 0$,

$$\mathcal{L}_{\mathrm{ct}} = -(\tfrac{1}{2} m^2 B \phi^2 + D) \;, \tag{64}$$

where B is given by Eq. (63). Now we have to compute the matrix element $\langle 0|1_{\mathrm{ren}} \rangle$, including the counterterm B and hence determine the number D, such that result becomes finite when $\epsilon \to 0$. The result (after dividing by $\langle 0|0 \rangle$ and the volume) is

$$\langle 0|1_{\mathrm{ren}} \rangle = \frac{\hat{\lambda} \hat{m}^4 \mu^{4-2\epsilon}}{32} \left[\frac{3}{\epsilon^2} + \frac{1}{\epsilon}[2\psi(2) + 4F_1(0) - 2\ln(\hat{m}^2)] + \left[\frac{\pi^2}{3} + \psi^2(2) - \psi'(2) + 4\psi(2) F_1(0) + F_1'(0) - F_1(0)\ln(\hat{m}^2) \right] + O(\epsilon) \right] - D = \text{finite} \;, \tag{65}$$

where $F_1(0)$ and $F_1'(0)$ denote the first two Taylor coefficients of $F_1(\epsilon, \hat{m}^2)$ at $\epsilon = 0$. This determines the counterterms D:

$$D = \mathrm{Vol} \frac{\hat{\lambda} \hat{m}^4 \mu^{4-2\epsilon}}{32} \left[\frac{3}{\epsilon^2} + \frac{1}{\epsilon}[2\psi(2) + 4F_1(0) - 2\ln(\hat{m}^2)] + G_1(\epsilon, \hat{m}^2) \right] , \tag{66}$$

where G_1 is arbitrary but analytic in $\epsilon = 0$. This completes the renormalization at the first-order two-loop level. One has to carry out the renormalization also for the second-order three-loop and third-order four-loop diagrams, which we do not do here.

VIII. CONCLUSION

We have suggested a functional approximation of the generating functional of Euclidean n-point Green's functions. It is based on the choice of a finite-dimensional space of functionals of the source field $\mathcal{J}$. We define a distance in the functional space by introducing a scalar product. That allows us, starting from $A = \int d^4 x \mathcal{L}^{\mathrm{int}}[\partial/\partial\mathcal{J}(x)]$, to obtain an approximation $A^{(N)}$ in the finite-dimensional functional space and hence to compute $\exp(A^{(N)})$. In principle, there is a large arbitrariness in the choice of the functional space. Here we have chosen to consider the functional space generated by the functionals which occur in the expansion in the coupling constant. This leads to diagrams familiar from perturbation theory. In particular the operator $A^{(N)}$ corresponds to a finite real symmetric matrix, the matrix elements of which are given by loop diagrams. These can be computed, the matrix of $A^{(N)}$ can be diagonalized, and hence $\exp(A^{(N)})$ can be computed in its eigenrepresentation. We have given an explicit example for the case of a two-dimensional functional space ($N=1$), and computed all the loop diagrams involved. We have shown how to renormalize, doing it explicitly up to the two-loop level.

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