Trace anomaly in $\lambda\phi^4$ theory near a fixed point

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We make a number of observations on the trace anomaly in $\lambda\phi^4$ theory near a fixed point. We show that the trace anomaly for the unique finite energy-momentum tensor may not vanish but may in fact blow up near a fixed point. We make comments on the renormalization-group equation satisfied by Green's functions of $\theta_{\mu\nu}$.

I. INTRODUCTION

The energy-momentum tensor in scalar ϕ^4 theory has been studied extensively.[1-12] Callan, Coleman, and Jackiw[2] studied the canonical energy-momentum tensor and showed that it did not have finite matrix elements in one-loop order and that an improvement term of the form $(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2$ was needed to make the tensor finite to one-loop order. Freedman, Muzinich, and Weinberg[4,5] studied it further and showed that an improvement term of the form $f(\lambda)(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2$, where $f(\lambda)$ is a series in the renormalized coupling constant λ, makes $\theta_{\mu\nu}$ finite to two-loop order but that such a "finite improvement program" fails beyond two-loop order. Collins[6,7] considered the problem from many angles and showed, in particular, that there is a unique $\theta_{\mu\nu}$ of the form

$$\theta_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\mathcal{L}+H_0(\epsilon)(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2 \qquad (1.1)$$

[where $H_0(\epsilon)$ is a series in $\epsilon=4-n$ with only nonnegative powers of ϵ], which is finite to all orders in perturbation theory. Here $H_0(\epsilon)$ is a *unique* series in ϵ, which is determined successively in perturbation. Collins further showed the above $\theta_{\mu\nu}$ is the only finite $\theta_{\mu\nu}$ of the form

$$\theta_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\mathcal{L}+H_0(\epsilon,\lambda,m)(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2 \ , \qquad (1.2)$$

where $H_0(\epsilon,\lambda,m)$ is a finite function of renormalized parameters λ and m at $\epsilon=0$. It was shown in Ref. 11 that the energy-momentum tensor of Eq. (1.1) is also a unique energy-momentum tensor of the form

$$\theta_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\mathcal{L}+H_0\left[\epsilon,\lambda_0\mu^{-\epsilon},\frac{m_0^2}{\mu^2}\right](\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2, \qquad (1.3)$$

where $H_0(\epsilon,\lambda_0\mu^{-\epsilon},m_0^2/\mu^2)$ is a finite function of bare parameters λ_0 and m_0^2 at $\epsilon=0$. (Here μ is the arbitrary parameters in dimensional regularization.)

Brown[9] gave an alternate construction for a finite energy-momentum tensor, but this energy-momentum tensor has the property that the improvement term is nontrivially renormalized in higher orders and contains negative powers of ϵ. This should be contrasted with $\theta_{\mu\nu}$ of (1.1) which needs no counterterms.

Thus $\theta_{\mu\nu}$ of Eq. (1.1) is an energy-momentum tensor having a unique significance, and it has a nontrivial trace anomaly.

Schroer[12] has given the following criterion[7] for $\theta_{\mu\nu}$ to be acceptable as an energy-momentum tensor which enters the generators of conformal transformations: Its trace should be soft at a fixed point. Collins[6,7] has argued this to be the case for $\theta_{\mu\nu}$ of Eq. (1.1). In this work, we shall take up a closer scrutiny of this claim and show that generally this may not be true. We find that the trace θ^μ_μ, instead of vanishing, may, in fact, blow up at a fixed point. This is argued in Sec. III.

Normally, when one deals with the scaling equation for Green's functions of the theory, or those of operators at the fixed point [see Eq. (4.1)] one generally puts to zero the $\beta(\lambda)(\partial/\partial\lambda)$ term because $\beta(\lambda^*)=0$. In Sec. IV we shall take up the study of the scaling equation for the Green's functions of θ^μ_μ and show explicitly that this term proportional to $\beta(\lambda)$ cannot be dropped, as it yields a nonzero contribution to the equation. In Sec. V we derive the result of Sec. IV in another way. The results of Sec. III rely on the sign of $\gamma_m(\lambda^*)$. Hence, in Sec. VI, we give a large-N calculation of $\gamma_m(\lambda)$ in an O(N)-invariant scalar theory and show that, at least in this theory, $\gamma_m(\lambda^*)$ is positive for $\lambda^*>0$ as was required in Sec. III.

Certain applications of the new results obtained in Sec. III and the connections of these results to dilation and conformal identities are under study by us and will be reported elsewhere.

II. PRELIMINARY

We shall work in the context of $\lambda\phi^4$ theory whose Lagrange density is given by

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m_0^2\phi^2-\frac{\lambda_0}{4!}\phi^4, \quad S=\int d^n x\,\mathcal{L} \ . \qquad (2.1)$$

We shall use the minimal-subtraction scheme (MS scheme) throughout to determine the renormalization constants for $\mathcal{L}$ and for operators. The renormalization transformations are

$$\phi=Z^{1/2}\Phi, \quad m_0^2=Z_m m^2, \quad \lambda_0=\mu^\epsilon\lambda Z_\lambda \ . \qquad (2.2)$$

Here Z, Z_m, Z_λ are independent of m in the MS scheme.

The following set of operators[7,9] is closed under renormalization:

$$O_1=\phi(\partial^2+m_0^2)\phi,\quad O_2=\frac{\delta S}{\delta\phi}\phi\ ,\qquad O_3=m_0^2\phi^2,\quad O_4=\partial^2\phi^2\ . \tag{2.3}$$

(Our choice of O_1 differs somewhat from that in Ref. 7.)

O_2 and O_3 are finite operators and O_4 is multiplicatively renormalized:[7,9]

$$\{\partial^2\phi^2\}^{\rm UR}=Z_m^{-1}\{\partial^2\phi^2\}^{\rm R}\ . \tag{2.4}$$

We define the renormalization matrix by

$$\{O_i\}^{\rm UR}=\sum_j Z_{ij}\{O_j\}^{\rm R}\ . \tag{2.5}$$

As shown in Refs. 7 and 9, Z_{ij} has the structure

$$Z_{ij}=\begin{bmatrix} 1-\dfrac{\beta(\lambda)}{\lambda\epsilon} & -4\left[\dfrac{\beta(\lambda)}{4\lambda\epsilon}-\dfrac{\gamma(\lambda)}{\epsilon}\right] & \dfrac{4\gamma_m(\lambda)}{\epsilon} & Z_{14}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & Z_m^{-1}\end{bmatrix}, \tag{2.6}$$

where $\beta(\lambda)$, $\gamma(\lambda)$, $\gamma_m(\lambda)$ have been defined by the standard renormalization-group definitions:

$$\beta(\lambda,\epsilon)=-\lambda\epsilon+\beta(\lambda)=\mu\frac{\partial\lambda}{\partial\mu}\bigg|_{\lambda_0,m_0,\epsilon},\qquad \gamma(\lambda)=\mu\frac{\partial}{\partial\mu}\ln Z\bigg|_{\lambda_c,m_0,\epsilon},\qquad \gamma_m(\lambda)=-\frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_m\bigg|_{\lambda_0,m_0,\epsilon}\ . \tag{2.7}$$

As Collins has shown the following energy-momentum tensor has finite matrix elements to all orders in λ:

$$\theta_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\mathcal{L}+\frac{g(\epsilon)}{1-n}(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2\ , \tag{2.8}$$

where $g(\epsilon)$ is a series in non-negative powers of $\epsilon=4-n$. We redefine

$$g(\epsilon)=\frac{n-2}{4}-\tilde{g}(\epsilon)\frac{\epsilon}{4}\ . \tag{2.9}$$

Then it can be shown that $\tilde{g}(\epsilon)$ begins as $O(\epsilon^2)$.

It is easily shown that[7,9]

$$\langle\theta^\mu_\mu\rangle=-\frac{\epsilon}{4}\langle O_1\rangle^{\rm UR}-\frac{n}{4}\langle O_2\rangle^{\rm R}+\langle O_3\rangle^{\rm R}-\frac{\epsilon}{4}\tilde{g}(\epsilon)Z_m^{-1}\langle O_4\rangle^{\rm R} \tag{2.10}$$

or, using the expressions for Z_{ij} ($j=1,2,3$), this becomes

$$\langle\theta^\mu_\mu\rangle=\frac{\beta(\lambda)}{4\lambda}\langle O_1\rangle^{\rm R}+\left[\frac{\beta(\lambda)}{4\lambda}-\gamma\right]\langle O_2\rangle^{\rm R}-\gamma_m\langle O_3\rangle^{\rm R}-\tfrac{1}{4}X^{(1)}\langle O_4\rangle^{\rm R}-\langle O_2\rangle^{\rm R}+\langle O_3\rangle^{\rm R}+O(\epsilon)\ , \tag{2.11}$$

where

$$X=Z_{14}+\tilde{g}(\epsilon)Z_m^{-1} \tag{2.12}$$

and $X^{(1)}$ denotes the coefficient of simple pole terms in X; viz.,

$$X^{(1)}=Z_{14}^{(1)}+K(\lambda)\ . \tag{2.13}$$

$K(\lambda)$ being the coefficient of simple pole in $\tilde{g}(\epsilon)Z_m^{-1}$.

It can be shown that Z_{14} satisfies the renormalization-group (RG) equations

$$\mu\frac{\partial}{\partial\mu}Z_{14}=[-\lambda\epsilon+\beta(\lambda)]\frac{\partial Z_{14}}{\partial\lambda}=2\gamma_m Z_{14}+\gamma_{14}Z_{11} \tag{2.14}$$

and consequently X satisfies

$$\mu\frac{\partial}{\partial\mu}X=[-\lambda\epsilon+\beta(\lambda)]\frac{\partial X}{\partial\lambda}=2\gamma_m X+\gamma_{14}Z_{11}\ . \tag{2.15}$$

Equation (2.14) implies that

$$\gamma_{14}(\lambda)=-\lambda\frac{\partial}{\partial\lambda}Z_{14}^{(1)}\ . \tag{2.16}$$

$\tilde{g}(\epsilon)$ is chosen in perturbation theory requiring that X has no worse than simple poles. Then Eq. (2.15) implies that

$$\beta(\lambda)\frac{\partial}{\partial\lambda}K(\lambda)=2\gamma_m X^{(1)}=2\gamma_m(\lambda)[Z_{14}^{(1)}(\lambda)+K(\lambda)]\ . \tag{2.17}$$

$K(\lambda)$ is chosen successively in perturbation series from Eq. (2.17) and since terms in $K(\lambda)$ and $\tilde{g}(\epsilon)$ are related in a one-to-one manner, Eq. (2.17) fixes $\tilde{g}(\epsilon)$ uniquely in perturbation series given $Z_{14}^{(1)}$.

The anomalous part of $\langle\theta^\mu_\mu\rangle$ is then [See Eq. (2.11)]

$$\langle\theta^\mu_\mu\rangle_{\rm anom}=\frac{\beta(\lambda)}{4\lambda}\langle O_1\rangle^{\rm R}+\left[\frac{\beta(\lambda)}{4\lambda}-\gamma\right]\langle O_2\rangle^{\rm R}-\gamma_m\langle O_3\rangle^{\rm R}-\tfrac{1}{4}[Z_{14}^{(1)}+K(\lambda)]\langle O_4\rangle^{\rm R} \tag{2.18}$$

$$=\frac{\beta(\lambda)}{4\lambda}\langle O_1\rangle^{\rm R}+\left[\frac{\beta(\lambda)}{4\lambda}-\gamma\right]\langle O_2\rangle^{\rm R}-\gamma_m\langle O_3\rangle^{\rm R}-\frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{d}{d\lambda}K(\lambda)\langle O_4\rangle^{\rm R}\ . \tag{2.19}$$

Green's functions $\langle O_2\rangle^{\rm R}$ vanish on shell at nonzero momentum q. In the massless limit $\langle O_3\rangle^R = m^2\langle\phi^2\rangle^R$ also vanishes. Thus it would appear, on the face of it, that at a fixed point λ^* for which $\beta(\lambda^*)=0$, the trace anomaly would always vanish as the coefficients of $\langle O_1\rangle^R$ and $\langle O_4\rangle^R$ are proportional to $\beta(\lambda)$. However, as shown in the Sec. III, this is not always true.

III. EXPRESSION FOR TRACE ANOMALY IN THE NEIGHBORHOOD OF A FIXED POINT

In this section we shall deal with the coefficient of $\langle O_4\rangle^R$ in anomaly equation (2.19). It should be emphasized that as $K(\lambda)$ is not arbitrary but is restricted by the requirement that it should satisfy Eq. (2.17) which involves $\beta(\lambda)$, $K(\lambda)$ *can* show an unusual behavior near $\lambda=\lambda^*$. There is no reason to expect that like other quantities $K(\lambda)$ will be smooth near a fixed point λ^*. This is confirmed by the calculations below.

We shall do our calculations in the context of a fixed point $\lambda=\lambda^*\neq 0$, for which $\beta(\lambda^*+\delta)$ is negative, for δ small and positive. Thus if running coupling constant is in a region $\bar\lambda(\mu)>\lambda^*$, then as $\mu\to\infty$, $\bar\lambda(\mu)\to\lambda^*$ from above, under certain assumptions regarding the behavior of $\beta(\lambda)$ near $\lambda=\lambda^*$. Thus for an arbitrary mass scale μ $(0<\mu<\infty)$, $\bar\lambda(\mu)$ will be restricted to be above λ^*. (A similar analysis can be carried out in other cases depending on the kind of fixed point and the behavior of β near it.)

We shall assume that $\beta(\lambda)$ is analytic at λ^*. Thus,

$$\beta(\lambda)\approx a(\lambda-\lambda^*)^n \tag{3.1}$$

for λ sufficiently close to λ^*. Here n is a positive integer and a is a negative constant. Equation (2.17) has an exact solution:

$$K(\lambda)=K(\lambda^*+\delta)+\exp\left[\int_{\lambda^*+\delta}^{\lambda}\frac{2\gamma_m(\lambda')}{\beta(\lambda')}d\lambda'\right]\times\left[\int_{\lambda^*+\delta}^{\lambda}\frac{2\gamma_m(\lambda')Z_{14}^{(1)}(\lambda')}{\beta(\lambda')}\times\exp\left[-\int_{\lambda^*+\delta}^{\lambda'}\frac{2\gamma_m(\lambda'')}{\beta(\lambda'')}d\lambda''\right]\right], \tag{3.2}$$

where δ is an arbitrarily small but positive number. An approximate evaluation of the right-hand side of Eq. (3.1) depends on the value of n. We shall consider two distinct cases: (I) $n=1$ and (II) $n\geq 2$.

(A) Case I: $n=1$. We shall assume that $\gamma_m(\lambda)$ is a smooth function of λ at $\lambda=\lambda^*$. We may then approximate

$$\int_{\lambda^*+\delta}^{\lambda'}\frac{2\gamma_m(\lambda'')}{\beta(\lambda'')}d\lambda''\approx\int_{\lambda^*+\delta}^{\lambda'}\frac{2\gamma_m(\lambda^*)}{a(\lambda''-\lambda^*)}d\lambda'' =\frac{2\gamma_m(\lambda^*)}{a}\ln\frac{\lambda'-\lambda^*}{\delta}.$$

Thus,

$$\exp\left[-\int_{\lambda^*+\delta}^{\lambda'}\frac{2\gamma_m(\lambda'')}{\beta(\lambda'')}d\lambda''\right]=\left[\frac{\lambda'-\lambda^*}{\delta}\right]^{-2\gamma_m(\lambda^*)/a} =\left[\frac{\lambda'-\lambda^*}{\delta}\right]^{-\alpha}.$$

Assuming further that $Z_{14}^{(1)}(\lambda)$ is a smooth function at $\lambda=\lambda^*\neq 0$,

$$\int_{\lambda^*+\delta}^{\lambda}\frac{2\gamma_m(\lambda')Z_{14}^{(1)}(\lambda')}{\beta(\lambda')}\exp\left[-\int_{\lambda^*+\delta}^{\lambda'}\frac{2\gamma_m(\lambda'')}{\beta(\lambda'')}d\lambda''\right] \approx\frac{2\gamma_m(\lambda^*)Z_{14}^{(1)}(\lambda^*)}{a\delta^{-\alpha}}\times\int_{\lambda^*+\delta}^{\lambda}(\lambda'-\lambda^*)^{-2\gamma_m(\lambda^*)/a-1}d\lambda' =-Z_{14}^{(1)}(\lambda^*)\frac{(\lambda-\lambda^*)^{-\alpha}}{\delta^{-\alpha}}+Z_{14}^{(1)}(\lambda^*). \tag{3.3}$$

This yields

$$K(\lambda)\simeq K(\lambda^*+\delta)-Z_{14}^{(1)}(\lambda^*)+Z_{14}^{(1)}(\lambda^*)\left[\frac{\lambda-\lambda^*}{\delta}\right]^{\alpha}. \tag{3.4}$$

This requires that

$$K(\lambda)+Z_{14}^{(1)}(\lambda^*)=(\lambda-\lambda^*)^{\alpha}[K(\lambda^*+\delta)(\lambda-\lambda^*)^{-\alpha}+Z_{14}^{(1)}(\lambda^*)\delta^{-\alpha}]. \tag{3.5}$$

The left-hand side is independent of δ and so is the factor $(\lambda-\lambda^*)^{\alpha}$ on the right-hand side. This requires that the term in square brackets is also independent of δ (to the leading order). One can, therefore, express Eq. (3.5) as

$$K(\lambda)+Z_{14}^{(1)}(\lambda^*)=-4C(\lambda-\lambda^*)^{\alpha}. \tag{3.6}$$

Now, from Eq. (2.16) it follows that

$$Z_{14}^{(1)}(\lambda)=Z_{14}^{(1)}(\lambda^*)-\frac{\gamma_{14}(\lambda^*)}{\lambda^*}(\lambda-\lambda^*)+O((\lambda-\lambda^*)^2). \tag{3.7}$$

Equations (2.17), (3.6), and (3.7) then imply that, for $-n<\alpha<-n+1$, $n=0,1,\ldots$,

$$-\frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{d}{d\lambda}K(\lambda) \simeq C(\lambda-\lambda^*)^{\alpha} +\text{terms of order }(\lambda-\lambda^*)^{\alpha+1}\cdots(\lambda-\lambda^*)^{\alpha+n} +\frac{1}{4}\frac{\gamma_{14}(\lambda^*)}{\lambda^*}(\lambda-\lambda^*)+\cdots$$

and, for $2>\alpha>0$,

$$-\frac{1}{8}\beta(\lambda)\frac{dK(\lambda)}{d\lambda}=C(\lambda-\lambda^*)^{\alpha}+\frac{1}{4}\frac{\gamma_{14}(\lambda^*)}{\lambda^*}(\lambda-\lambda^*) +\text{higher-order terms},$$

while, for $\alpha \geq 2$,

$$\frac{1}{8}\frac{\beta(\lambda)dk(\lambda)}{d\lambda} = \frac{1}{4}\frac{\gamma_{14}(\lambda^*)}{\lambda^*}(\lambda-\lambda^*) + \text{terms of order } (\lambda-\lambda^*)^2 \cdots (\lambda-\lambda^*)^{[\alpha]} + C(\lambda-\lambda^*)^{\alpha} + \cdots \,. \tag{3.8}$$

[The terms in Eq. (3.8), not explicitly calculated, are not needed in future discussions.]

The above expression shows that if $\alpha=2\gamma_m(\lambda^*)/a$ is negative, the coefficient of $\langle O_4\rangle^{\mathrm{R}}$ in fact blows up as $\lambda\to\lambda^*$. Thus, the trace anomaly instead of vanishing at a fixed point, in fact, blows up near $\lambda=\lambda^*$ (assuming that $\langle O_4\rangle^{\mathrm{R}}$ is smooth near $\lambda=\lambda^*$). In this connection see Sec. IV). Moreover, the dominant behavior of the trace anomaly is, in the case $\alpha<1$, solely determined by $\alpha=2\gamma_m(\lambda^*)/a$ and does not depend on, say, extra anomalous dimensions.

(B) Case II: $n\geq 2$. This case can be dealt with in a similar manner. We sketch the derivation, stating the approximations made:

$$\int_{\lambda^*+\delta}^{\lambda'}\frac{2\gamma_m(\lambda'')}{\beta(\lambda'')}d\lambda'' \approx \int_{\lambda^*+\delta}^{\lambda'}\frac{2\gamma_m(\lambda^*)}{a(\lambda''-\lambda^*)^n}d\lambda'' = \frac{\alpha}{-n+1}\left[\frac{1}{(\lambda'-\lambda^*)^{n-1}}-\frac{1}{\delta^{(n-1)}}\right],$$

$$\int_{\lambda^*+\delta}^{\lambda}\frac{2\gamma_m(\lambda')Z_{14}^{(1)}(\lambda')}{\beta(\lambda')}\exp\left[-\int_{\lambda^*+\delta}^{\lambda'}\frac{2\gamma_m(\lambda'')}{\beta(\lambda'')}d\lambda''\right] \tag{3.9}$$

$$= \int_{\lambda^*+\delta}^{\lambda}d\lambda'\frac{2\gamma_m(\lambda')Z_{14}^{(1)}(\lambda')}{\beta(\lambda')}\exp\left[\frac{\alpha}{n-1}\frac{1}{(\lambda'-\lambda^*)^{n-1}}\right]\exp\left[\frac{-\alpha}{(n-1)\delta^{n-1}}\right]$$

$$\approx \int_{\lambda^*+\delta}^{\lambda}\frac{2\gamma_m(\lambda^*)}{a}Z_{14}^{(1)}(\lambda^*)\exp\left[-\frac{\alpha}{(n-1)\delta^{n-1}}\right]\frac{\exp\left[\dfrac{\alpha}{n-1}\dfrac{1}{(\lambda'-\lambda^*)^{n-1}}\right]}{(\lambda'-\lambda^*)^n}d\lambda' \,.$$

A change of variables $-1/(n-1)(\lambda'-\lambda^*)^{n-1}=\xi$ gives

$$\alpha Z_{14}^{(1)}(\lambda^*)\exp\left[-\frac{\alpha}{(n-1)\delta^{n-1}}\right]\int_{\lambda^*+\delta}^{\lambda}e^{-\alpha\xi}d\xi$$

$$= -Z_{14}^{(1)}(\lambda^*)\exp\left[-\frac{\alpha}{(n-1)\delta^{n-1}}\right]\left[\exp\left[\alpha\frac{1}{(n-1)(\lambda-\lambda^*)^{n-1}}\right]-\exp\left[\frac{\alpha}{(n-1)\delta^{n-1}}\right]\right]$$

and thus

$$K(\lambda)+Z_{14}^{(1)}(\lambda^*) = Z_{14}^{(1)}(\lambda^*)\exp\left[-\frac{\alpha}{(n-1)}\frac{1}{(\lambda-\lambda^*)^{n-1}}\right]\exp\left[\frac{\alpha}{(n-1)\delta^{n-1}}\right]+K(\lambda^*+\delta) \tag{3.10}$$

$$= \exp\left[-\frac{\alpha}{(n-1)}\frac{1}{(\lambda-\lambda^*)^{n-1}}\right]\left[Z_{14}^{(1)}(\lambda^*)\exp\left[\frac{\alpha}{(n-1)\delta^{n-1}}\right]+K(\lambda^*+\delta)\exp\left[\frac{\alpha}{n-1}\frac{1}{(\lambda-\lambda^*)^{n-1}}\right]\right]. \tag{3.11}$$

As in the earlier case, the term in square brackets must be independent of δ (to the leading order) and thus

$$K(\lambda)+Z_{14}^{(1)}(\lambda^*) = -4C'\exp\left[-\frac{\alpha}{n-1}\frac{1}{(\lambda-\lambda^*)^{n-1}}\right]. \tag{3.12}$$

This, as before, yields

$$-\frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{d}{d\lambda}K(\lambda) \simeq C'\exp\left[-\frac{\alpha}{(n-1)(\lambda-\lambda^*)^{n-1}}\right], \quad \alpha<0 \tag{3.13a}$$

$$\simeq \frac{1}{4}\frac{\gamma_{14}(\lambda^*)}{\lambda^*}(\lambda-\lambda^*)+\cdots, \quad \alpha>0 \,. \tag{3.13b}$$

Again if $\alpha=2\gamma_m(\lambda^*)/a$ is negative the coefficient of $\langle O_4\rangle^{\mathrm{R}}$ in the expression for θ_μ^μ diverges exponentially as $\lambda\to\lambda^*$ (from above). Thus, in this case also the trace anomaly blows up as $\lambda\to\lambda^*$. The behavior of the trace anomaly is again solely determined by $\gamma_m(\lambda^*)$ and a. On the other hand, if $\alpha>0$, Eq. (3.13b) yields the correct behavior of the anomaly coefficient near $\lambda=\lambda^*$.

IV. SCALING EQUATION

Consider, for simplicity, a multiplicatively renormalizable operator O, having anomalous dimension γ_0. The scaling equation derived from RG equation and dimensional analysis reads[13]

$$\left[\sum_{\nu}\sum_{i=1}^{n} p_{i\nu}\frac{\partial}{\partial p_{i\nu}}-\beta(\lambda)\frac{\partial}{\partial\lambda}+m(1-\gamma_m)\frac{\partial}{\partial m}+n(1+\gamma)-4+\gamma_0\right]\Gamma_0^{(n)}(p_1,p_2,\ldots,p_n,\lambda,m,\mu)=0 . \tag{4.1}$$

It is usually assumed that *at* the fixed point $\beta(\lambda^*)=0$ and hence the second term can be dropped when λ is in the neighborhood of λ^*. This leads, in the massless limit, to

$$\left[\sum_{i=1}^{n}\sum_{\nu} p_{i\nu}\frac{\partial}{\partial p_{i\nu}}+n(1+\gamma)-4+\gamma_0\right]\times\Gamma_0^{(n)}(p_1,\ldots,p_n,\lambda^*,0,\mu)=0 . \tag{4.2}$$

The above equation would imply that, in the large momentum limit, $\Gamma_0^{(n)}$ scales by the scaling dimension $-n(1+\gamma)+4-\gamma_0$ at the fixed point.

In this section, we wish to present an example of a case where the $\beta(\lambda)(\partial/\partial\lambda)$ term cannot be dropped and instead leads to a nontrivial contribution and a *different* scaling dimension as compared to that in Eq. (4.2). Consider $O=\theta_\mu^\mu$. O is a finite operator and hence $\gamma_0=0$. For simplicity, consider the massless case. Then $\langle O_3\rangle^{\rm R}=0$. Further, for concreteness, consider the possibility $n=1$ in Eq. (3.1) and let $\gamma_m(\lambda^*)>0$. Then the leading behavior of Γ_0 near $\lambda=\lambda^*$ is given solely by the term proportional to $\langle O_4\rangle^{\rm R}$ (provided it does not vanish):

$$\Gamma_0^{(n)}\approx-\frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{d}{d\lambda}K(\lambda)\Gamma_{O_4}^{(n)}$$
$$\approx C(\lambda-\lambda^*)^{\alpha}\Gamma_{O_4}^{(n)}(p_1,p_2,\ldots,p_n,\lambda^*,\mu),\quad \alpha<0 \tag{4.3}$$

assuming that Γ_{O_i} are smooth functions of λ near $\lambda=\lambda^*$ (a comment about this is made later). Then one has, from Eq. (4.1),

$$\left[\sum_{i=1}^{n}\sum_{\nu} p_{i\nu}\frac{\partial}{\partial p_{i\nu}}-\beta(\lambda)\frac{\partial}{\partial\lambda}+n(1+\gamma)-4\right]\times\Gamma_0^{(n)}(p_1,\ldots,p_n,\lambda,\mu)=0 . \tag{4.4}$$

Now the term

$$-\beta(\lambda)\frac{\partial}{\partial\lambda}\Gamma_0^{(n)}\approx-\beta(\lambda)\frac{\partial}{\partial\lambda}C(\lambda-\lambda^*)^{\alpha}\Gamma_{O_4}^{(n)}$$
$$\approx-a(\lambda-\lambda^*)C\alpha(\lambda-\lambda^*)^{\alpha-1}\Gamma_{O_4}^{(n)}$$
$$=-2\gamma_m(\lambda^*)C(\lambda-\lambda^*)^{\alpha}\Gamma_{O_4}^{(n)}$$
$$\simeq-2\gamma_m(\lambda^*)\Gamma_O^{(n)}(p_1,\ldots,p_n,\lambda,\mu) \tag{4.5}$$

contributes nontrivially to the equation, leading to the different scaling dimension $2\gamma_m(\lambda^*)-n(1+\gamma)+4$.

In the above discussion, we have seen that $\Gamma_0^{(n)}$ has a nontrivial behavior near the fixed point if $\alpha<0$. This observation, based on Eq. (4.3), relies on the fact $\Gamma_{O_4}^{(n)}$ is a smooth function near $\lambda=\lambda^*$. If $\Gamma_{O_4}^{(n)}$ had a nontrivial behavior near $\lambda=\lambda^*$ such as, for example, to cancel the singular behavior of the anomaly coefficient $-\frac{1}{8}[\beta(\lambda)/\gamma_m(\lambda)](d/d\lambda)K(\lambda)$ our conclusion would not be valid. It is normally assumed that, for any operator O, $\Gamma_O^{(n)}$ has a smooth behavior near a fixed point. But this cannot be expected of θ_μ^μ because it contains a $[g(\epsilon)/(1-n)]\partial^2\phi^2$ term which has been constructed in a nontrivial manner in perturbation series. $\langle\theta_\mu^\mu\rangle$ explicitly contains $K(\lambda)$ which has been chosen to satisfy a differential condition of Eq. (2.17). This is the justification for allowing $\Gamma_O^{(n)}$ to possess a nontrivial behavior near $\lambda=\lambda^*$, while $\Gamma_{O_i}^{(n)}$ ($i=1,2,3,4$) have been assumed to be smooth and nonvanishing near $\lambda=\lambda^*$.

A similar conclusion holds in the cases $n\geq 2$ and $\alpha<0$ as is verified easily.

V. ALTERNATE DERIVATION OF THE SCALING DIMENSION OF $\Gamma_O^{(n)}$

In the preceding section we showed that the scaling dimension for $\Gamma_O^{(n)}$ differs from that naively expected. In this section, we shall present an alternative but extremely straightforward derivation of these results.

From Eq. (2.11) we have, for $m_0=O$,

$$\Gamma_O^{(n)}(e^tp_i,\lambda,\mu)=\frac{\beta(\lambda)}{4\lambda}\Gamma_{O_1}^{(n)}(e^tp_i,\lambda,\mu)$$
$$-\left[1+\gamma-\frac{\beta(\lambda)}{4\lambda}\right]\Gamma_{O_2}^{(n)}(e^tp_i,\lambda,\mu)$$
$$-\frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{d}{d\lambda}K(\lambda)\Gamma_{O_4}^{(n)}e^tp_i,\lambda,\mu) . \tag{5.1}$$

We want to see the behavior of $\Gamma_O^{(n)}$ (p_i,λ,μ) under small rescaling of p_i when $\lambda\approx\lambda^*$.

If $\gamma_m(\lambda^*)>0$, coefficient of $\Gamma_{O_1}^{(n)}$ in Eq. (5.1) is relatively small and that of $\Gamma_{O_2}^{(n)}$ is approximately $-[1+\gamma(\lambda^*)]$, a finite number while that of $\Gamma_{O_4}^{(n)}$ is large. Hence, provided that $\Gamma_{O_4}^{(n)}$ is not zero or negligible,

$$\Gamma_O^{(n)}(e^tp_i,\lambda,\mu)\approx-\frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{d}{d\lambda}K(\lambda)\Gamma_{O_4}^{(n)}(e^tp_i,\lambda,\mu) . \tag{5.2}$$

Now $O_4=\partial^2\phi^2$ is a multiplicatively renormalizable operator with anomalous dimension $\gamma_{O_4}=2\gamma_m$. Hence,[13]

$$\Gamma_{O_4}^{(n)}(e^t p_i,\lambda,\mu)\simeq\exp\left[\int_0^t dt'\{[2\gamma_m(\lambda(t'))]-n\gamma(\lambda(t'))\}+(4-n)t\right]\Gamma_{O_4}^{(n)}(p_i,\lambda,\mu) .$$

For $\lambda\simeq\lambda^*$ and t small this becomes

$$\simeq\exp(\{4+2\gamma_m(\lambda^*)-n[1+\gamma(\lambda^*)]\}t)\Gamma_{O_4}^{(n)}(p_i,\lambda,\mu) . \tag{5.3}$$

This implies

$$\Gamma_O^{(n)}(e^t p_i,\lambda,\mu)\approx\exp(\{4+2\gamma_m(\lambda^*)-n[1+\gamma(\lambda^*)]\}t)\times\Gamma_O^{(n)}(p_i,\lambda,\mu) , \tag{5.4}$$

giving rise to the same anomalous dimension as in Sec. IV. It is, however, important to note that in Sec. IV, the term $2\gamma_m(\lambda^*)$ arose out of the coefficient of Γ_{O_4} through the term $\beta(\lambda)(\partial/\partial\lambda)\Gamma_0$; while in Eq. (5.4), it arises as the anomalous dimension of O_4.

VI. $\gamma_m(\lambda^*)$ IN THE LARGE-N LIMIT

Certain interesting observations made in Secs. III, IV, and V were dependent on the sign of $\gamma_m(\lambda^*)$. As the nonzero fixed points of $\lambda\phi^4$ theory are not known, and knowledge of $\gamma_m(\lambda^*)$ would require an exact calculation, no definite conclusions about sign of $\gamma_m(\lambda^*)$ can be stated. However, it *is* possible to obtain $\gamma_m(\lambda^*)$ in a related O(N)-invariant ϕ^4 theory, at least in the large-N limit (keeping λN fixed). In this case, it turns out, as outlined below, that γ_m receives contributions only in $O(\lambda)$ and the result is indeed positive for $\lambda^*>0$. [λ^* must be positive for $H(\lambda^*)$ to have a lower bound.]

It is possible to show, by an analysis of graphs that the only graphs that make leading contributions to $\gamma_m(\lambda)$ in each order are of the kind shown in Fig. 1 *together* with the counterterm graphs that correspond to subtractions of this graph. This set of graphs can be easily summed[14] to give the result (here a is the loop expansion parameter):

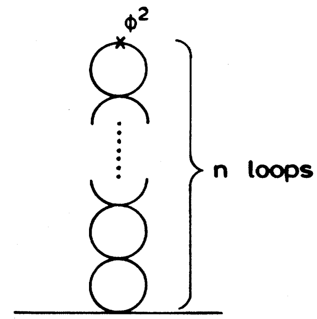

FIG. 1. Diagram contributing to Z_m in the n-loop approximation in the large-N limit.

$$Z_m=\sum_{n=O}^{\infty}a^n z_m^{(n)}\lambda^n$$

with $z_m^{(n)}(\epsilon)=A_n/\epsilon^n$ giving only an ϵ^{-n} term contribution to $z_m^{(n)}(\epsilon)$. Thus, the simple pole contribution comes only in one-loop order and hence $\gamma_m(\lambda)$ receives a contribution only from the one-loop approximation. The result is

$$\gamma_m(\lambda)=\frac{1}{16\pi^2}\frac{N\lambda}{2}+\text{nonleading terms} . \tag{6.1}$$

For $\lambda=\lambda^*>0$, $\gamma_m(\lambda^*)>0$ in the limit $N\lambda$ fixed and $N\to\infty$.

[1]J. Lowenstein, Phys. Rev. D **4**, 2281 (1971); E. C. Poggio, Ann. Phys. (N.Y.) **81**, 481 (1973); K. Yamada, Phys. Rev. D **10**, 599 (1974).
[2]C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys.(N.Y.) **59**, 42 (1970).
[3]S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) **67**, 552 (1971).
[4]D. Z. Freedman, I. J. Muzinich, and E. J. Weinberg, Ann. Phys. (N.Y.) **87**, 95 (1974).
[5]D. Z. Freedman and E. J. Weinberg, Ann. Phys. (N.Y.) **87**, 354 (1974).
[6]J. C. Collins, Phys. Rev. Lett. **36**, 1518 (1976).
[7]J. C. Collins, Phys. Rev. D **14**, 1965 (1976).
[8]L. S. Brown, Phys. Rev. D **14**, 1469 (1977).
[9]L. S. Brown, Ann. Phys. (N.Y.) **136**, 135 (1980).
[10]L. S. Brown and J. C. Collins, Ann. Phys. (N.Y.) **130**, 215 (1980).
[11]S. D. Joglekar and Anuradha Misra, Ann. Phys. (N.Y.) **185**, 231 (1988).
[12]B. Schroer, Lett. Nuovo Cimento **2**, 867 (1971).
[13]D. J. Gross, in *Methods in Field Theory*, proceedings of the Les Houches Summer School, Les Houches, France, 1975, edited by R. Balian and J. Zinn-Justin (Les Houches Summer School Proceedings, Vol. 28) (North-Holland, Amsterdam, 1976).
[14]See, e.g., D. J. Amit, *Field Theory: The Renormalization Group and Critical Phenomena*, 2nd. ed. (World Scientific, Singapore, 1984), p. 133.