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Operator approach to bosonic string: Multiloop calculation and b-ghost insertion

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We construct the g-loop N-string vertex using the Feynman-like rules in the operator formalism with bosonized bc ghosts. The measure factor is analyzed in detail, which consists of the ghost g vacuum and the contour integration associated with the b-ghost insertion. The general form of the g vacuum is derived using the handle operators. The handle operator is formulated by the geometrical quantities: period matrix, first Abelian integral, prime form, $\frac{1}{2}$ differential, and Riemann constant. Explicit results are given up to two-loop level. The contour integrations associated with the ghost insertions can be performed explicitly in the one-loop case, in which we reproduce the well-known formula of the one-loop N-tachyon amplitude.

I. INTRODUCTION

The theory of strings has been intensively investigated, since it is a candidate for a consistent formulation of quantum gravity. However, there are still several points which are not well understood when constructing the quantum theory of strings. In order to improve our comprehension of string theory, it is important to investigate the structure of the multiloop amplitude from various sides. One recent aspect which has revealed some mathematical structure of the string amplitude is its deep connection with the algebraic geometry. On the other hand, from the physical point of view, string theory can be understood as the theory of an infinite number of fields, i.e., the dual resonance model. In the dual resonance model we can construct the multiloop amplitude by a simple operator algebra. Therefore, these two approaches may show us complementary aspects of string theory.

In this paper we follow the idea of Feynman-like rules as originally developed in the dual resonance model[1–4] to analyze the multiloop amplitudes. The basic objects, i.e., the three-string vertex[2,5,6] and the propagator, are completed by adding the ghost contribution to ensure the Becchi-Rouet-Stora-Tyutin (BRST) invariance.[7–10] The case of the N-string vertices including the superstring[10–19] and the diagrammatic approaches based on the BRST invariance[20–26] were already considered. The construction of the BRST-invariant multiloop amplitude in such an approach has been investigated for both bosonic and supersymmetric strings by many authors.[17,20–23,27–32] In a previous paper,[32] using Lovelace's and also Olive's parametrization, we presented a formulation of these Feynman-like rules in the framework of the operator approach. There, the factorization properties and duality of the tree vertices were analyzed by constructing the tree N-string vertex. It was shown how the measure factor is generated from the contribution of the ghosts in a BRST-invariant way.

In our formulation, the bc ghosts are bosonized. Such a formulation is necessary when generalizing our method to the superstring case, especially for including the Ramond sector. In the bosonized formulation, the b-ghost insertions which is necessary to cancel the background ghost charge, is made into the propagator together with a contour integration. In this way, the analogy with the algebraic geometrical approach becomes transparent, where we have the b-ghost insertions together with Beltrami differentials.[32] The contour integration appearing in the definition of our propagator should be performed after the sewing process for constructing the amplitude is completed. In this respect our formalism is different from the one with nonbosonized ghosts.[17,20]

Here, our aim is to apply the above described formulation to construct the multiloop N-string vertex. We shall formulate the results of our approach in such a form that they show a close correspondence with the expressions obtained by the algebraic geometry approach.[33–38] For this construction we use Lovelace's parametrization, since the BRST invariance of the loop amplitude holds straightforwardly.

In Sec. II we give a brief description of the Feynman-like rules. In Sec. III we construct the g-loop vacuum and express it in terms of the period matrix, the first Abelian integrals, the prime form, the $g/2$ differential and the Riemann constants. In Sec. IV we analyze the measure part arising from the ghost contribution in detail, when the external legs are on shell. The integration associated with the ghost insertion can be performed explicitly for the one-loop case ($g=1$), in which our result reproduces the well-known formula.

II. FEYNMAN-LIKE RULES AND MULTILOOP STRING VERTICES

In this section we give the general construction of the g-loop N-string vertex from the Feynman-like rules. Using the string-emission operator we derive the g-loop vacuum in terms of handle operators. For this end, we use the result of the tree $(2g+N)$-string vertex $(\mathbf{V}^{\text{tree},2g+N}|$ given in our previous paper.[32] The loops are constructed by connecting pairwise $2g$ legs of the tree string vertex by propagators. From the point of view of the Feynman-like rules, this can be represented as

$$(\mathbf{V}^{g\text{ loop},N}| = \prod_{t=1}^{2g+N-2}\prod_{i=1}^{3g+N-3}(\mathbf{V}_t^3|\mathbf{P}_i)$$

$$= \prod_{i=1}^{g}(\mathbf{V}^{\text{tree},2g+N}|\mathbf{P}_i)\ , \tag{2.1}$$

where $|\mathbf{P}_i)$ denotes the propagators and $(\mathbf{V}_t^3|$ the three-string vertices. There are various ways to construct the tree $(2g+N)$-string vertex, depending on the decomposition of the vertex into three-string vertices and propagators. Here, we choose the tree string vertex of the peripheral type for the vertex $(\mathbf{V}^{\text{tree},2g+N}|$, see Fig. 1. We shall discuss in Sec. V about the independence of the way of the construction of the tree vertex $(\mathbf{V}^{\text{tree},2g+N}|$.

In Lovelace's parametrization, the three-string vertex $(\mathbf{V}_t^3|$ is defined with the projective transformation $Y_r(z)$ associated with its legs (r) such as

$$Y_r(z)=\begin{bmatrix} 0 & 1 & \infty \\ z_r & z_{r+1} & z_{r-1}\end{bmatrix}\ , \tag{2.2}$$

with the convention $r+3=r$. z_r is the Koba-Nielsen variable of the corresponding external leg.

The propagator $|\mathbf{P})$ is defined with a b-ghost insertion $b(\zeta)$ as

$$|\mathbf{P})=\int\frac{dx}{x(1-x)}\oint\frac{d\zeta}{2\pi i}\zeta(1-\zeta)b(\zeta)|\mathcal{P}\rangle\ , \tag{2.3}$$

where $|\mathcal{P}\rangle$ denotes the operator part of the propagator as introduced in Ref. 32, x in Eq. (2.3) is the Chan variable and the operator $|\mathcal{P}\rangle$ is given by the canonical form of the Möbius transformation:

$$\gamma(z)=\begin{bmatrix} 0 & 1 & \infty \\ x & 0 & 1\end{bmatrix}\ . \tag{2.4}$$

The contour integration with respect to ζ is to be performed around the propagator.

Then, the tree $(2g+N)$-string vertex $(\mathbf{V}^{\text{tree},2g+N}|$ is written as[32]

$$(\mathbf{V}^{\text{tree},2g+N}| = \prod_{t=1}^{2g+N-2}\prod_{i=1}^{2g+N-3}(\mathbf{V}_t^3|\mathbf{P}_i)$$

$$=\int dV^{2g+N}$$

$$\times\prod_{i=1}^{2g+N-3}\oint\frac{dy_i}{2\pi i}F_i^{\text{tree}}(y_i)$$

$$\times\langle\mathcal{V}^{\text{tree},2g+N};\{y\}|\omega\ , \tag{2.5}$$

where $\langle\mathcal{V}^{\text{tree},2g+N};\{y\}|$ is the b-ghost-inserted symmetric N-string vertex:[12]

$$\langle\mathcal{V}^{\text{tree},2g+N};\{y\}|=\langle\mathcal{V}^{\text{tree},2g+N}|$$

$$\times\prod_{i=1}^{2g+N-3}[\partial Y_s^{-1}(y_i)]^2$$

$$\times b^{(s)}(Y_s^{-1}(y_i))\ . \tag{2.6}$$

The projective transformation $Y_i(z)$ of the $(2g+N)$-string vertex is defined by Eq. (2.2) with the convention $2g+N+r=r$. The function $F_i^{\text{tree}}(y_i)$ is given by

$$F_i^{\text{tree}}(y_i)=\frac{(y_i-z_1)(y_i-z_{i+2})}{z_1-z_{i+2}}\ . \tag{2.7}$$

The factor dV^{2g+N} in Eq. (2.5) is

$$dV^{2g+N}=\frac{\prod_{s=1}^{2g+N}dz_s}{dV_{abc}}\frac{1}{\prod_{r=1}^{2g+N}(z_r-z_{r+1})}\ , \tag{2.8}$$

$$dV_{abc}=\frac{dz_a dz_b dz_c}{(z_a-z_b)(z_b-z_c)(z_c-z_a)}\ , \tag{2.9}$$

where $2g+N+r=r$. Since we are using Lovelace's parametrization, the tree vertex is symmetric up to a gauge transformation ω (Refs. 20 and 32). The gauge factor ω depends on the way of the construction of the tree amplitude and is given by the projective transformation with fixed points 0 and 1. It may seem that the existence of the gauge transformation ω complicates the tree vertex. However, this is not the case since the gauge transformation associated with each leg becomes the identity when external legs are coupled with physical states, or when a leg is sewn with the twisted propagator to make a loop, i.e.,

$$\omega^{(s)}|\text{phys})_s=|\text{phys})_s \text{ and } \omega^{(s)}|\mathbf{P})_{sr}=|\mathbf{P})_{sr}\ . \tag{2.10}$$

To evaluate the multiloop amplitude, we express the operator part of the string vertex using the string-emission operators:

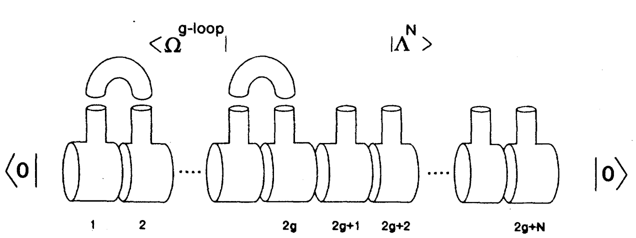

FIG. 1. The g-loop N-string vertex is represented in terms of string-emission operator. By joining a pair of adjacent legs of the tree $(2g+N)$-string vertex by a propagator we get a handle operator. Thus, the g-loop vacuum $\langle\Omega^{g\text{ loop}}|$ is constructed by a product of g handle operators. The operators giving the contribution of the external legs $|\Lambda^N\rangle$ is obtained by multiplying N string-emission operators.

$$\langle \mathcal{V}^{\text{tree},2g+N};\{y\}| = {}_A\left\langle 0 \left| \Upsilon^{(1)}_{AA_1}\Upsilon^{(2)}_{A_1A_2}\cdots\Upsilon^{(2g)}_{A_{2g-1}B}\prod_{i=1}^{2g+N-3} b^{(B)}(y_i)\Upsilon^{(2g+1)}_{BB_1}\Upsilon^{(2g+2)}_{B_1B_2}\cdots\Upsilon^{(2g+N)}_{B_{N-1}B_N}\right| 0\right\rangle_{B_N} . \tag{2.11}$$

In terms of the bosonized ghost σ, the ghost part of the string-emission operator $\Upsilon^{(r)}_{\text{ghost},AB}$ is defined as

$$\begin{aligned}\Upsilon^{(r)}_{\text{ghost},AB} = {}_r\langle q_\sigma=0|\ {}_B\Big\langle q_\sigma=0\Big|\delta(\alpha^{(B)}_{\sigma,0}+\alpha^{(r)}_{\sigma,0}-\alpha^{(A)}_{\sigma,0})\\ \times :\exp\left[\oint_{C_0}\frac{dz}{2\pi i}\oint_{C_0}\frac{dw}{2\pi i}\left\{\partial^{(B)}(w)\ln\left[\frac{w-Y_r(z)}{\sqrt{\partial Y_r(z)}}\right]\partial\sigma^{(r)}(z)\right.\right.\\ +\partial_w\left[\sigma^{(A)}\left[\frac{1}{w}\right]+Q\ln\left[\frac{1}{w}\right]\right]\ln\left[\frac{\Gamma(w)-Y_r(z)}{\sqrt{\partial\Gamma(w)\partial Y_r(z)}}\right]\partial\sigma^{(r)}(z)\\ \left.\left.+\partial_w\left[\sigma^{(A)}\left[\frac{1}{w}\right]+Q\ln\left[\frac{1}{w}\right]\right]\ln(1-wz)\partial\sigma^{(B)}(z)\right\}\right]:\Big|q_\sigma=0\Big\rangle_A .\end{aligned} \tag{2.12}$$

[Our bosonization formula is $b=e^{-\sigma}$ and $c=e^{\sigma}$ (Ref. 39). Since we discuss here the ghost contribution, we mainly give explicit formulas only for the ghost part. The details of the matter part can be found in Ref. 31.] The superscript (r) labels the external string while the indices A, B denote the internal strings. Q denotes the background ghost charge, and here $Q=-3$. Each string-emission operator is defined such that when we multiply an arbitrary on-shell state onto the external string (r) it reduces to the vertex operator for the emission of the corresponding state.

The g-loop vertex is obtained by multiplying g propagators with b-ghost insertions given in Eq. (2.3). This b-ghost insertion can again be shifted to the internal string.

It is convenient to represent the loops using the handle operator defined as

$$\Omega_{AC} = \Upsilon^{(r)}_{AB}\Upsilon^{(s)}_{BC}(-1)^F|\mathcal{P}\rangle_{rs} , \tag{2.13}$$

where $(-1)^F = \exp(\pi i\alpha^{(s)}_{\sigma,0})$ which gives the proper spin structure for the ghost loop. Then, we can write the g-loop N-string vertex (2.1) as (Fig. 1)

$$\langle \mathbf{V}^{g\ \text{loop},N}| = \int dV^{g\ \text{loop},N}\prod_{s=1}^{g}\oint\frac{d\tilde{y}_s}{2\pi i}F^{\text{loop}}_s(\tilde{y}_s)\prod_{i=1}^{2g+N-3}\oint\frac{dy_i}{2\pi i}F^{\text{tree}}_i(y_i)\left\langle\Omega^{g\ \text{loop}}\left|\prod_{s=1}^{g}b(\tilde{y}_s)\prod_{i=1}^{2g+N-3}b(y_i)\right|\Lambda^N\right\rangle , \tag{2.14}$$

where $|\Lambda^N\rangle$ denotes the contribution of the N external strings defined by

$$|\Lambda^N\rangle_B = \Upsilon^{(2g+1)}_{BB_1}\Upsilon^{(2g+2)}_{B_1B_2}\cdots\Upsilon^{(2g+N)}_{B_{N-1}B_N}|0\rangle_{B_N} \tag{2.15}$$

and the $\langle\Omega^{g\ \text{loop}}|$ is the g-loop vacuum defined by

$${}_{A_g}\langle\Omega^{g\ \text{loop}}| = {}_A\langle 0|\Omega_{AA_1}\Omega_{A_1A_2}\cdots\Omega_{A_{g-1}A_g} . \tag{2.16}$$

In Eq. (2.14), the factor $dV^{g\ \text{loop},N}$ is

$$dV^{g\ \text{loop},N} = \prod_{i=1}^{g}\frac{d\tilde{x}_i}{\tilde{x}_i(1-\tilde{x}_i)}\frac{\prod_{s=1}^{2g+N}dz_s}{dV_{abc}}\frac{1}{\prod_{r=1}^{2g+N}(z_r-z_{r+1})} , \tag{2.17}$$

and $F^{\text{loop}}_s(\tilde{y}_s)$ is a c-number function originated from the factor $\zeta(1-\zeta)$ in the propagator (2.3) which is used to make the sth loop. Although the b-ghost insertion is symmetric with respect to the propagator leg into which it is made, the function $F^{\text{loop}}_s(\tilde{y}_s)$ is not. Depending on which leg the b ghost is inserted into, it is either given by

$$F^{\text{loop}}_s(\tilde{y}_s) = \frac{(\tilde{y}_s-z_{2s-1})(\tilde{y}_s-z_{2s})}{z_{2s-1}-z_{2s}} \tag{2.18}$$

or by

$$F^{\text{loop}}_s(\tilde{y}_s) = \frac{(\tilde{y}_s-z_{2s})(\tilde{y}_s-z_{2s+1})}{z_{2s}-z_{2s+1}} . \tag{2.19}$$

However, after the contour integration it turns out that both choices give the same result.

III. HANDLE OPERATOR AND g-LOOP VACUUM OF THE GHOST

In this section we give a brief description of the handle operator and the g-loop vacuum for the ghost part. Such an operator and a vacuum were constructed and analyzed in detail in Ref. 31 for the matter part (i.e., the coordinate fields and the fermion fields) of the superstring. The corresponding formulas for the ghosts can be obtained by simply applying the same technique.

A. One-loop calculations and the handle operator

The operator part of the propagator is given by

$$|\mathcal{P}_{\text{ghost}}\rangle_{rs}=:\exp\left\{\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\partial_x\left[\sigma^{(r)}\left[\frac{1}{x}\right]+Q\ln\left[\frac{1}{x}\right]\right]\ln\left[\frac{\Gamma\gamma(x)-y}{\sqrt{\partial\Gamma\gamma(x)}}\right]\partial_y\left[\sigma^{(s)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\right\}:|\text{vac}(2)\rangle_{rs}\ , \tag{3.1}$$

where the vacuum is

$$|\text{vac}(2)\rangle_{rs}=\delta(\alpha_{\sigma,0}^{(r)}+\alpha_{\sigma,0}^{(s)}+Q)|q=0\rangle_r|q=0\rangle_s\ . \tag{3.2}$$

Inserting the unit operator, the handle operator for the ghosts (2.13) is

$$\Omega_{AC}=\Upsilon_{AE}^{(r)}1_{EF}\Upsilon_{FC}^{(s)}(-1)^F|\mathcal{P}\rangle_{rs}\ , \tag{3.3}$$

with the unit operator

$$1_{EF}=\sum_{\hat\alpha_0}\prod_{n=1}^{\infty}d\hat a_n^* d\hat a_n\exp\left\{-\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\partial_y\left[\hat\sigma_{\le}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\ln(1-yx)\partial\hat\sigma_{\ge}(x)\right\}$$
$$\times|\hat\alpha_n,\hat\alpha_0\rangle_{EF}\langle\hat\alpha_n,-\hat\alpha_0-Q|\ . \tag{3.4}$$

Here, $\hat\alpha_n=\sqrt{|n|}\hat a_n$ $(n\neq 0)$ and the quantities with the caret are c numbers. The coherent states are defined as

$$\alpha_n|\hat\alpha_n,\hat\alpha_0\rangle=\hat\alpha_n|\hat\alpha_n,\hat\alpha_0\rangle\ \text{and}\ \langle\hat\alpha_n,-\hat\alpha_0-Q|\alpha_{-n}=\langle\hat\alpha_n,-\hat\alpha_0-Q|\hat\alpha_n\ ,$$
$$\alpha_0|\hat\alpha_n,\hat\alpha_0\rangle=\hat\alpha_0|\hat\alpha_n,\hat\alpha_0\rangle\ \text{and}\ \langle\hat\alpha_n,-\hat\alpha_0-Q|\alpha_0=\langle\hat\alpha_n,-\hat\alpha_0-Q|\hat\alpha_0\ ,$$

for $n\ge 1$. When the fields $\sigma^{(E)}(z)$ and $\sigma^{(F)}(z)$ act on these states, they are converted into the c-number fields $\hat\sigma(z)$.

The integration of the nonzero modes is a Gaussian which can be evaluated straightforwardly and leads to the following expression for the handle operator:

$$\Omega_{\text{ghost},AC}={}_C\Big\langle q_\sigma=0\Big|\delta(\alpha_{\sigma,0}^{(A)}-\alpha_{\sigma,0}^{(C)}+Q)(\det\mathbf{C}^{1\ \text{loop}})^{-1}$$
$$\times\sum_k:\exp\Bigg[i\pi k(k+Q)B$$
$$+2\pi ik\left[\frac{Q}{2}+\oint_{C_0}\frac{dz}{2\pi i}\left\{\partial_z\left[\sigma^{(A)}\left[\frac{1}{z}\right]+Q\ln\left[\frac{1}{z}\right]\right]\phi(\Gamma(z),z_0)+\partial\sigma^{(C)}(z)\phi(z,z_0)\right\}\right]$$
$$+\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\left\{\frac{1}{2}\partial\sigma^{(C)}(x)\partial\sigma^{(C)}(y)\ln\left[\frac{E(x,y)}{x-y}\right]\right.$$
$$+\frac{1}{2}\partial_x\left[\sigma^{(A)}\left[\frac{1}{x}\right]+Q\ln\left[\frac{1}{x}\right]\right]$$
$$\times\partial_y\left[\sigma^{(A)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\ln\left[\frac{xy}{x-y}E(\Gamma(x),\Gamma(y))\right]$$
$$\left.+\partial\sigma^{(C)}(x)\partial_y\left[\sigma^{(A)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\ln yE(x,\Gamma(y))\right\}$$
$$-Q\oint_{C_0}\frac{dz}{2\pi i}\left\{\partial_z\left[\sigma^{(A)}\left[\frac{1}{z}\right]+Q\ln\left[\frac{1}{z}\right]\right]\right.$$
$$\times\ln\left[\frac{\Gamma T\Gamma(z)\sqrt{\partial T(\xi)}}{\sqrt{\partial\Gamma T\Gamma(z)}}\prod_{n=0}^{\infty}(T^{n+1}\Gamma(z),T^{-1}\Gamma(0),T^n(0),\Gamma(0))\right]$$
$$\left.+\partial\sigma^{(c)}(z)\ln\prod_{n=0}^{\infty}(T^n(z),T^{-1}\Gamma(0),T^n(0),\Gamma(0))\right\}\Bigg]:\Big|q_\sigma=0\Big\rangle_A\ , \tag{3.5}$$

where $T(z)$ is defined by

$$T=Y_s\gamma\Gamma Y_r^{-1}\ , \tag{3.6}$$

which gives the Möbius transformation generating the Schottky group of the one-loop case. We denote the fixed points of the projective transformation $T(z)$ by ξ and η where $\xi=T^{-\infty}(v)$ and $\eta=T^{\infty}(u)$.

Recall that the matrixes Y_r and Y_s are the projective transformations of the two string-emission operators which were glued to form the handle operator. The factor $\det \mathbf{C}^{1\ \mathrm{loop}}$ is $\prod_{n=1}^{\infty}(1-K^n)$ with K being the multiplier of the transformation $T(z)$ (Ref. 31).

In Eq. (3.5) the summation over the loop momentum $\hat{\alpha}_0$ of the handle operator is replaced by the summation over k with $k=\hat{\alpha}_0-\alpha^{C}_{\sigma,0}$. The quantities B, $\phi(z,z_0)$, and $E(x,y)$ are the period matrix, the first Abelian integral, and the prime form for the one-loop case, respectively:

$$B=\frac{1}{2\pi i}\ln\prod_{k=-\infty}^{\infty}(T^{k+1}(a),T(b),T^{k}(a),b)\ , \tag{3.7}$$

$$\phi(z,z_0)=\frac{1}{2\pi i}\ln\prod_{k=-\infty}^{\infty}(T^{k}(z),T(a),T^{k}(z_0),a)\ , \tag{3.8}$$

$$E(x,y)=(x-y)\prod_{n=1}^{\infty}(T^{k}(x),y,T^{k}(y),x)\ . \tag{3.9}$$

Then, summing over the loop momentum, we obtain

$$\begin{aligned}\Omega_{\mathrm{ghost},\,AC}={}_C\Big\langle q_\sigma=0\Big|\,&\delta(\alpha^{(A)}_{\sigma,0}-\alpha^{(C)}_{\sigma,0}+Q)(\det\mathbf{C}^{1\ \mathrm{loop}})^{-1}\\
&\times{:}\vartheta\begin{bmatrix}0\\0\end{bmatrix}\left[B\left|\oint_{C_0}\frac{dz}{2\pi i}\left\{\partial\sigma^{(C)}(z)\phi(z,z_0)+\partial_z\left[\sigma^{(A)}\left[\frac{1}{z}\right]+Q\ln\left[\frac{1}{z}\right]\right]\phi(\Gamma(z),z_0)\right\}+Q\Delta\right.\right]\\
&\times\exp\left[\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\left\{\frac{1}{2}\partial\sigma^{(C)}(x)\partial\sigma^{(C)}(y)\ln\left[\frac{E(x,y)}{x-y}\right]\right.\right.\\
&\qquad+\frac{1}{2}\partial_x\left[\sigma^{(A)}\left[\frac{1}{x}\right]+Q\ln\left[\frac{1}{x}\right]\right]\partial_y\left[\sigma^{(A)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\\
&\qquad\times\ln\left[\frac{xy}{x-y}E(\Gamma(x),\Gamma(y))\right]\\
&\qquad\left.+\partial\sigma^{(C)}(x)\partial_y\left[\sigma^{(A)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\ln yE(x,\Gamma(y))\right\}\\
&\left.+Q\oint_{C_0}\frac{dz}{2\pi i}\left\{\partial_z\left[\sigma^{(A)}\left[\frac{1}{z}\right]+Q\ln\left[\frac{1}{z}\right]\right]\ln\frac{1}{z}S_1\left[\frac{1}{z}\right]+\partial\sigma^{(C)}(z)\ln S_1(z)\right\}\right]{:}\Big|q_\sigma=0\Big\rangle_A\ ,\end{aligned} \tag{3.10}$$

where we have introduced the quantities

$$\Delta=\tfrac{1}{2}(B+1)\ , \tag{3.11}$$

$$S_1(z)=\frac{1}{z-\xi}\ . \tag{3.12}$$

The quantity Δ is known as the Riemann constant and $S_1(z)$ can be identified with the half differential with no zeros or poles in the fundamental region for the genus-one surface.[40,41] For details see Appendix A. Hence, each term in the handle operator has a geometrical meaning.

Multiplying the vacuum to the handle operator from the left, we can immediately derive the g-loop vacuum for the one-loop order, ${}_C\langle\Omega^{1\ \mathrm{loop}}_{\mathrm{ghost}}|={}_A\langle 0|\Omega_{\mathrm{ghost},\,AC}$:

$$\begin{aligned}\langle\Omega^{1\ \mathrm{loop}}_{\mathrm{ghost}}|=(\det\mathbf{C}^{1\ \mathrm{loop}})^{-1}\langle -Q|{:}\vartheta\begin{bmatrix}0\\0\end{bmatrix}&\left[B\left|\oint_{C_0}\frac{dz}{2\pi i}\partial\sigma(z)\phi(z,z_0)+Q\Delta\right.\right]\\
&\times\exp\left[\frac{1}{2}\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\partial\sigma(x)\partial\sigma(y)\ln\left[\frac{E(x,y)}{x-y}\right]+Q\oint_{C_0}\frac{dz}{2\pi i}\partial\sigma(z)\ln S_1(z)\right]{:}\ .\end{aligned} \tag{3.13}$$

B. Multiloop calculations and the g-loop vacuum

The g-loop vacuum is obtained by contracting g of the handle operators as Eq. (2.16) indicates. To sew two adjacent handle operators we use the coherent state method by inserting the unit operator (3.4) as explained in Appendix B. The result can be expressed as

$$\langle \Omega_{\rm ghost}^{g\ \rm loop}| = (\det \mathbf{C}^{g\ \rm loop})^{-1} \langle -Qg| : \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left[B_{\mu\nu} \left| \oint_{C_0} \frac{dz}{2\pi i} \partial\sigma(z) \phi_\nu^{g\ \rm loop}(z,z_0) + Q\Delta_\nu \right. \right] \times \exp\left[\frac{1}{2} \oint_{C_0} \frac{dx}{2\pi i} \oint_{C_0} \frac{dy}{2\pi i} \partial\sigma(x)\partial\sigma(y) \ln\left[\frac{E^{g\ \rm loop}(x,y)}{x-y} \right] + Q \oint_{C_0} \frac{dz}{2\pi i} \partial\sigma(z) \ln S_g(z) \right] : , \tag{3.14}$$

where $B_{\mu\nu}^{g\ \rm loop}$ is the period matrix, $\phi_\nu^{g\ \rm loop}(z,z_0)$ the first Abelian integral, and $E^{g\ \rm loop}(x,y)$ the prime form in the Schottky parametrization. These terms in the exponent are obtained by the same methods as applied for the case of the g-loop vacuum of the matter part of the superstring.[31] The new terms characteristic for the contribution of the ghosts are collected into the two terms proportional to the background ghost charge Q as in Eq. (3.14). The explicit form of $S_g(z)$ and Δ_ν are given in Appendix B for the two-loop case. Unlike in the one-loop case, it is not easy to prove explicitly that they are the vector of Riemann constants and the $g/2$ differential for the multiloop case.

IV. THE MEASURE AND THE MULTILOOP STRING AMPLITUDE

The operator part derived above can be identified with the g vacuum. According to Feynman-like rules, the g-loop N-string vertex has $(3g-3+N)$ b-ghost insertions. The number of the inserted b ghosts, which corresponds to the number of propagators, is the correct number of b-ghost insertions to get the nonzero result.[39] In our formalism the b-ghost insertion is performed together with a contour integral accompanied by the function F. This has an analogue in the path-integral approach, where we insert a b ghost together with a Beltrami differential and integrate over the insertion point.

In the following, we consider the full expression of the g-loop N-string vertex in Eq. (2.14). For this we shall combine the operator part calculated above and the remaining c-number part. The result is presented in a form that the above described analogy may become transparent.

For this end we first put the external ghost string legs on the mass shell. This is done by saturating the external strings by physical states $|\text{phys}) \equiv \prod_r |\text{matter}\rangle_r \otimes |1;\text{ghost}\rangle_r$ in the g-loop N-string vertex (2.14), where $|1;\text{ghost}\rangle$ denotes the ghost-number one state.

We first note that

$$\Upsilon_{\text{matter},BC}^{(r)} \Upsilon_{\text{ghost},BC}^{(r)} (|1;\text{ghost}\rangle_r \otimes |\text{matter}\rangle_r) \times |0;\text{ghost}\rangle_C \otimes |0\rangle_C = c^{(B)}(z_r) V_{\text{matter},B}^{(r)}(z_r) |0;\text{ghost}\rangle_B \otimes |0\rangle_B , \tag{4.1}$$

where $|0;\text{ghost}\rangle$ and $|0\rangle$ are the SL$(2,C)$-invariant vacua for the ghost and the matter, respectively. $V_{\text{matter},B}^{(r)}$ denotes the on-shell vertex operator constructed with the field $X^{(B)}(z)$ creating the physical state $|\text{matter}\rangle_r$. Thus, the on-shell amplitude is obtained by choosing the tree part $|\Lambda^N\rangle$ of Eq. (2.15) as

$$|\Lambda^N\rangle_B = \prod_{r=2g+1}^{2g+N} [c^{(B)}(z_r) V_{\text{matter},B}^{(r)}(z_r)] \times |0;\text{ghost}\rangle_B \otimes |0\rangle_B . \tag{4.2}$$

Then, the resulting amplitude is

$$(\mathbf{V}^{g\ \text{loop},N}|\text{phys}) = \int dM^{g\ \text{loop},N} \left\langle \Omega_{\text{matter}}^{g\ \text{loop}} \left| \prod_{r=2g+1}^{2g+N} V_{\text{matter}}^{(r)} \right| 0 \right\rangle , \tag{4.3}$$

where we have included the ghost operator part into the measure $dM^{g\ \text{loop},N}$ as

$$dM^{g\ \text{loop},N} \equiv dV^{g\ \text{loop},N} \prod_{i=1}^{2g+N-3} \oint \frac{dy_i}{2\pi i} F_i^{\text{tree}}(y_i) \prod_{s=1}^{g} \oint \frac{d\tilde{y}_s}{2\pi i} F_s^{\text{loop}}(\tilde{y}_s) \times \left\langle \Omega_{\text{ghost}}^{g\ \text{loop}} \left| \prod_{s=1}^{g} b(\tilde{y}_s) \prod_{i=1}^{2g+N-3} b(y_i) \prod_{r=2g+1}^{2g+N} c(z_r) \right| 0;\text{ghost} \right\rangle , \tag{4.4}$$

with $dV^{g\text{ loop},N}$ given in Eq. (2.17). We analyze this measure in the following. The matrix element of the matter part $\langle \Omega^{g\text{ loop}}_{\text{matter}}| \prod_{r=2g+1}^{2g+N} V^{(r)}_{\text{matter}}|0\rangle$ is analyzed in Appendix C.

It is easy to show using Eq. (3.14) that the ghost operator part in Eq. (4.4) is

$$\left\langle \Omega^{g\text{ loop}}_{\text{ghost}} \left| \prod_{s=1}^{g} b(\tilde{y}_s) \prod_{i=1}^{2g+N-3} b(y_i) \prod_{r=2g+1}^{2g+N} c(z_r) \right| 0 \right\rangle$$

$$= (\det \mathbf{C}^{g\text{ loop}})^{-1} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left[B^{g\text{ loop}}_{\mu\nu} \left| \sum_r \phi^{g\text{ loop}}_\nu(z_r,z_0) - \sum_s \phi^{g\text{ loop}}_\nu(\tilde{y}_s,z_0) - \sum_i \phi^{g\text{ loop}}_\nu(y_i,z_0) + Q\Delta_\nu \right. \right]$$

$$\times \frac{\prod_{s<t} E(\tilde{y}_s,\tilde{y}_t) \prod_{s,i} E(\tilde{y}_s,y_i) \prod_{i<j} E(y_i,y_j) \prod_{r<q} E(z_r,z_q)}{\prod_{s,r} E(\tilde{y}_s,z_r) \prod_{i,r} E(y_i,z_r)} \left[\frac{\prod_s S_g(\tilde{y}_s) \prod_i S_g(y_i)}{\prod_r S_g(z_r)} \right]^{-Q} . \tag{4.5}$$

This should be compared to the expression of the ghost correlation function.[42,43] We expect that the quantities Δ_ν and $S_g(z)$ can be identified with the vector of Riemann constants and the $g/2$ differentials, respectively, given in the Schottky parametrization. Then, our g-loop vacuum $\langle \Omega^{g\text{ loop}}|$ can be identified with the g vacuum for the ghost.

The contour integration related with the external legs in Eq. (4.4) can be performed, since the analytic structure with respect to y_i is simple for the case where i is running from $2g-1$ to $2g+N-3$. In that case, the integrand has only poles at z_r with $r>2g+2$. Thus, the contour integration dy_i in Eq. (4.4) for these variables gives

$$\left[\prod_{i=2g-1}^{2g+N-3} \oint \frac{dy_i}{2\pi i} F^{\text{tree}}_i(y_i) \right] \left\langle \Omega^{g\text{ loop}}_{\text{ghost}} \left| \prod_{s=1}^{g} b(\tilde{y}_s) \prod_{j=1}^{2g+N-3} b(y_j) \prod_{r=2g+1}^{2g+N} c(z_r) \right| 0 \right\rangle$$

$$= \frac{\prod_{j=2g+1}^{2g+N} (z_j - z_{j+1})}{z_{2g+1} - z_1} \left\langle \Omega^{g\text{ loop}}_{\text{ghost}} \left| \prod_{s=1}^{g} b(\tilde{y}_s) \prod_{j=1}^{2g-2} b(y_j) c(z_{2g+1}) \right| 0 \right\rangle . \tag{4.6}$$

Here, the convention $z_{2g+N+r} = z_r$ is understood. Therefore, the measure (4.4) leads to

$$dM^{g\text{ loop},N} = d\check{V}^{g\text{ loop}} \prod_{s=2g+1}^{2g+N} dz_s \prod_{i=1}^{2g-2} \oint \frac{dy_i}{2\pi i} F^{\text{tree}}_i(y_i) \prod_{s=1}^{g} \oint \frac{d\tilde{y}_s}{2\pi i} F^{\text{loop}}_s(\tilde{y}_s) \left\langle \Omega^{g\text{ loop}}_{\text{ghost}} \left| \prod_{s=1}^{g} b(\tilde{y}_s) \prod_{j=1}^{2g-2} b(y_j) c(z_{2g+1}) \right| 0 \right\rangle , \tag{4.7}$$

with $d\check{V}^{g\text{ loop}}$ being

$$d\check{V}^{g\text{ loop}} = \prod_{t=1}^{g} \frac{d\tilde{x}_t}{\tilde{x}_t(1-\tilde{x}_t)} \frac{\prod_{s=1}^{2g} dz_s}{dV_{abc}} \frac{1}{(z_{2g+1}-z_1) \prod_{r=1}^{2g} (z_r - z_{r+1})} . \tag{4.8}$$

We now change the integration variables related with the loop configuration in Eq. (4.8) into the fixed points and the multipliers of the projective transformation associated with each handle. As defined in Eq. (3.6) for the handle operator, for the νth handle the projective transformation T_ν is

$$T_\nu \equiv Y_{2\nu} \gamma \Gamma Y^{-1}_{2\nu-1} = \begin{bmatrix} z_{2\nu} & z_{2\nu-1} & z_{2\nu-2} & Y_{2\nu-1}(\tilde{x}_\nu) \\ z_{2\nu} & z_{2\nu+1} & Y_{2\nu}(\tilde{x}_\nu) & z_{2\nu-1} \end{bmatrix} . \tag{4.9}$$

Hence, one set of fixed points is given by $\eta_\nu \equiv z_{2\nu}$. Let us denote the other set of fixed points by ξ_ν. Then, the multiplier K_ν is defined by

$$K_\nu \equiv (T_\nu(z), \eta_\nu, z, \xi_\nu) = (z_{2\nu+1}, \eta_\nu, z_{2\nu-1}, \xi_\nu) \tag{4.10}$$

$$= (z_{2\nu-1}, \eta_\nu, Y_{2\nu-1}(\tilde{x}_\nu), \xi_\nu) . \tag{4.11}$$

From (4.11), $\tilde{x}_\nu$ may be solved in terms of K_ν and $Y^{-1}_{2\nu-1}(\xi_\nu)$. Then, we use (4.10) to express $Y^{-1}_{2\nu-1}(\xi_\nu)$ in terms of K_ν and $Y^{-1}_{2\nu-1}(z_{2\nu-1})$. The result reads

$$K_\nu = \frac{\tilde{x}_\nu}{1-\tilde{x}_\nu} (z_{2\nu-1}, z_{2\nu+1}, z_{2\nu}, z_{2\nu-2}) . \tag{4.12}$$

Using (4.10) and (4.12) the integration variables may be changed from $(\tilde{x}_\nu, z_{2\nu-1})$ to (K_ν, ξ_ν). As a result we find that

$$\prod_{\nu=1}^{g} \frac{d\tilde{x}_\nu dz_{2\nu-1} dz_{2\nu}}{\tilde{x}_\nu(1-\tilde{x}_\nu)} = \prod_{\nu=1}^{g} \frac{dK_\nu d\xi_\nu d\eta_\nu}{K_\nu} \frac{(z_{2\nu-1}-\eta_\nu)(z_{2\nu-1}-z_{2\nu+1})}{(\xi_\nu-\eta_\nu)(z_{2\nu+1}-\xi_\nu)} . \tag{4.13}$$

Thus,

$$d\check{V}^{g\text{ loop}} = \prod_{\nu=1}^{g} \frac{dK_\nu d\xi_\nu d\eta_\nu}{dV_{abc} K_\nu} \frac{(z_{2\nu-1}-\eta_\nu)(z_{2\nu-1}-z_{2\nu+1})}{(\xi_\nu-\eta_\nu)(z_{2\nu+1}-\xi_\nu)} \frac{1}{(z_{2g+1}-z_1)\prod_{t=1}^{2g}(z_t-z_{t+1})} . \tag{4.14}$$

Hence, our final form of the g-loop N-string vertex is given by

$$(\mathbf{V}^{g\text{ loop},N}|\text{phys}) = \int d\check{V}^{g\text{ loop}} \prod_{s=2g+1}^{2g+N} \int dz_s \prod_{i=1}^{2g-2} \oint \frac{dy_i}{2\pi i} F_i^{\text{tree}}(y_i) \prod_{s=1}^{g} \oint \frac{d\tilde{y}_s}{2\pi i} F_s^{\text{loop}}(\tilde{y}_s)$$
$$\times \left\langle \Omega_{\text{ghost}}^{g\text{ loop}} \middle| \prod_{s=1}^{g} b(\tilde{y}_s) \prod_{i=1}^{2g-2} b(y_i) c(z_{2g+1}) \middle| 0 \right\rangle$$
$$\times \left\langle \Omega_{\text{matter}}^{g\text{ loop}} \middle| \prod_{r=2g+1}^{2g+N} V_{\text{matter}}^{(r)} \middle| 0 \right\rangle . \tag{4.15}$$

One-loop case

In the one-loop case we can further perform the contour integration rather straightforwardly, which gives the well-known modular-invariant result from our general formula. For this end we have to evaluate the ghost operator part so that one can see the pole structure of the amplitude. It is convenient to use the following formula which is equivalent to Eq. (3.13):

$$\langle \Omega_{\text{ghost}}^{1\text{ loop}}| = (\det \mathbf{C}^{1\text{ loop}})^{-1} \exp\left[-\frac{i\pi}{2}Q - \frac{i\pi}{4}Q^2 B\right] \langle -Q| \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \left[B \middle| \oint_{C_0} \frac{dz}{2\pi i} \partial\sigma(z)\phi(z,z_0) \right]$$
$$\times \exp\left[\frac{1}{2} \oint_{C_0} \frac{dx}{2\pi i} \oint_{C_0} \frac{dy}{2\pi i} \partial\sigma(x)\partial\sigma(y) \ln\left[\frac{E(x,y)}{x-y}\right] - \frac{Q}{2} \oint_{C_0} \frac{dz}{2\pi i} \partial\sigma(z) \ln(z-\xi)(z-\eta) \right] , \tag{4.16}$$

where we have used the relations of the ϑ functions.[41] Then, from Eq. (4.5) we get

$$\langle \Omega_{\text{ghost}}^{1\text{ loop}}| b(\tilde{y}) c(z_3)|0\rangle = (\det \mathbf{C}^{1\text{ loop}})^{-1} \exp\left[-\frac{i\pi}{2}Q - \frac{i\pi}{4}Q^2 B\right] \frac{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (B|\phi(z_3,\tilde{y}))}{E(\tilde{y},z_3)} \left[\frac{(z_3-\xi)(z_3-\eta)}{(\tilde{y}-\xi)(\tilde{y}-\eta)} \right]^{-Q/2} , \tag{4.17}$$

where the prime form is given as

$$E(z_3,\tilde{y}) = \frac{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (B|\phi(z_3,\tilde{y}))}{[\partial\phi(\tilde{y})\partial\phi(z_3)]^{1/2} \vartheta' \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (B|0)} , \tag{4.18}$$

with

$$\vartheta' \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (B|0) = \partial_z \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (B|z) \Big|_{z=0} = 2\pi K^{1/8} (\det \mathbf{C}^{1\text{ loop}})^3 . \tag{4.19}$$

Therefore, we get

$$\langle \Omega_{\text{ghost}}^{1\text{ loop}}| b(\tilde{y}) c(z_3)|0\rangle = (\det \mathbf{C}^{1\text{ loop}})^2 \frac{1}{K} \frac{(z_3-\eta)(z_3-\xi)(\xi-\eta)}{(\tilde{y}-\eta)^2(\tilde{y}-\xi)^2} . \tag{4.20}$$

Using this, the contour integration in Eq. (4.15) can be performed:

$$\oint \frac{d\tilde{y}}{2\pi i} F^{\text{loop}}(\tilde{y}) \langle \Omega_{\text{ghost}}^{1\text{ loop}}| b(\tilde{y}) c(z_3)|0\rangle = (\det \mathbf{C}^{1\text{ loop}})^2 \frac{1}{K} \frac{(z_3-\eta)(z_3-\xi)}{(\eta-\xi)} , \tag{4.21}$$

where the same result is obtained whether we use (2.18) or (2.19) for $F^{\text{loop}}(\tilde{y})$.

On the other hand, Eq. (4.14) reads

$$d\check{V}^{1\ \mathrm{loop},N}=\frac{d\tilde{x}}{\tilde{x}(1-\tilde{x})}\,\frac{\prod_{r=1}^{N+2} dz_r}{dV_{abc}}\,\frac{1}{(z_3-z_1)(z_1-z_2)(z_2-z_3)}=\frac{dK}{K}\prod_{r\geq 3}^{N+1} dz_r\ , \tag{4.22}$$

where we have used

$$dV_{abc}=\frac{d\eta\, d\xi\, dz_{2g+N}}{(\xi-\eta)(\eta-z_{2g+N})(z_{2g+N}-\xi)}\ . \tag{4.23}$$

Consequently, we find that

$$dM^{1\ \mathrm{loop},N}=\frac{dK}{K^2}\prod_{r\geq 4}^{N+2} dz_r(\det\mathbf{C}^{1\ \mathrm{loop}})^2\times\frac{(z_{2g+N}-\eta)(z_{2g+N}-\xi)}{(\eta-\xi)}\ . \tag{4.24}$$

By making use of the projective invariance we may choose that

$$\eta=0,\quad \xi=\infty,\quad z_{2g+N}=1\ . \tag{4.25}$$

Then, Eq. (4.24) reproduces the well-known formula for the measure of the one-loop N-tachyon vertex. The integrand is given by Eq. (C6) in Appendix C.

V. DISCUSSION

In this paper we have constructed the g-loop N-string vertex in the operator formalism applying the Feynman-like rules. For this construction we used the tree vertex of the peripheral type and sewed g pairs of external legs by propagators. The measure $dM^{g\ \mathrm{loop},N}$ was analyzed in detail which includes the ghost contribution. The ghost operator part in the bosonized form leads us to the ghost g vacuum.

The building element of our multiloop vertex is the handle operator represented entirely in terms of geometrical quantities, i.e., first Abelian integral, period matrix, prime form, Riemann constant, and $\frac{1}{2}$ differential as in Eq. (3.10). Then, the g vacuum was generated simply by joining g handle operators. With this ghost g vacuum, any ghost correlation function can be easily written down, see Eq. (4.5), as in the case of the matter part.[31] The new terms characteristic to the ghost case are collected into the quantities Δ_ν and S_g, which should be identified with the vector of Riemann constants and the $g/2$ differential, respectively.

This identification has been proven in Appendix A for the one-loop case. We also have given there formulas of S_2 and Δ_ν for the two-loop case. This construction of the ghost g vacuum seems to give us the recursive construction of the $g/2$ differential out of the $(g-1)/2$-differential in the Schottky parametrization. However, for the expressions of the two-loop case it is not easy to prove the identification explicitly.

Our three-string vertex is formulated in Lovelace's parametrization. Thus, the BRST invariance of our g-loop vertex is seen from the fact that both the three-string vertex and the propagator are BRST invariant. Since the propagator is BRST invariant up to a total derivative, the resulting N-string g-loop vertex is also BRST invariant up to a total derivative with respect to the Chan variables. On the other hand, the BRST invariance is broken when using Olive's parametrization since then the integration over the Chan variable is performed after identifying the Chan variables of the propagator with the ones of the three-string vertex. Indeed, if we use the Olive vertex, we obtain the extra factor $\prod(1-K_\nu)^{-1}$ in $d\check{V}^{g\ \mathrm{loop}}$ in Eq. (4.14), which confirms previous results on this problem.[4,20,44]

Since the tree N-string vertex satisfies duality, we expect that the multiloop vertex obtained from a tree $(2g+N)$-string vertex by connecting $2g$ legs with propagators does not depend on the way it has been constructed from three-string vertices and propagators. One way to prove this may be the explicit calculation. However, this can be also understood as follows: Since the BRST invariance holds, we can prove the no-ghost theorem using the results given by Freeman and Olive.[45,46] Then, into each handle operator we can insert the projection operator onto the physical states. After inserting the projection operator, we can use the duality properties of the tree amplitude to show that the multiloop vertex does not depend on whether we use the tree vertex of the peripheral type or not.

In the operator formalism the factorization property is manifest by the way of construction which is not easy to see in the algebraic geometry approach. The modular invariance of the multiloop amplitude, however, has to be shown separately. Our idea to achieve this is to complete the above discussed identifications of the vector of Riemann constants and of the $g/2$ differentials. Then, we can apply the results on the modular invariance obtained from the geometrical approach to show the modular invariance. As for the one-loop case, we have shown the complete agreement with the well-known result and thus the modular invariance is manifest. The higher-loop case is now under investigation.

The Feynman-like diagram approach does not determine the integration region over the moduli, i.e., the fixed points and the multipliers associated with the generators $T_\nu(z)$ of the Schottky group corresponding to the g-loop amplitude, and the Koba-Nielsen points associated with the external legs. We have to determine the integration domain by some other criterium so that it covers the fundamental region only once.

Note added. After finishing this paper we became aware of the work of Cristofano, Mosto, Nicodemi, and Pettorino[47] and DiVecchia *et al.*[48] where also the differential $S_g(z)$ was calculated.

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APPENDIX A

We shall prove that the quantities Δ and $S_1(z)$ in Eqs. (3.11) and (3.12) can be identified with the Riemann constant and the half differential, respectively.

Identification of the Riemann constant. The vector of Riemann constants[41] is written in general as

$$\begin{aligned}\tilde{\Delta}_\mu = {} & \tfrac{1}{2}B_{\mu\mu}+\phi_\mu(z_1,z_0) \\ & -\sum_\nu \oint_{a_\nu} d\phi_\nu(z)\phi_\mu(z,z_0) \quad (\mathrm{mod}J)\ ,\end{aligned} \tag{A1}$$

where z_1 is the starting point of the integration around the a cycle a_μ, and $(\mathrm{mod}J)$ means that $\tilde{\Delta}_\mu$ is an element in the Jacobian; i.e., it is defined up to the lattice vector $m_\mu+\sum B_{\mu\nu}n_\nu$, where m_μ and n_μ are integers. Note that the $\tilde{\Delta}_\mu$ have no dependence on z_1 and negative sign compared to the standard definition. For the one-loop case we can easily evaluate this quantity:

$$\begin{aligned}\tilde{\Delta} = {} & \frac{B}{2}+\phi(z_1,z_0) \\ & -\oint_a \frac{dz}{2\pi i}\left[\frac{1}{z-\eta}-\frac{1}{z-\xi}\right]\left[\frac{1}{2\pi i}\ln(z,\eta,z_0,\xi)\right] \\ = {} & \frac{B}{2}-\frac{1}{2\pi i}\ln(-1)=\tfrac{1}{2}(B+1) \quad (\mathrm{mod}J)\ ,\end{aligned} \tag{A2}$$

where we used the formulas

$$\oint \frac{dz}{2\pi i}\frac{1}{z-a}\ln(z-b)=-\ln\left[\frac{a-b}{a-z_1}\right]\ , \tag{A3}$$

when the contour encircles only the point b, and

$$\oint \frac{dz}{2\pi i}\frac{1}{z-a}\ln(z-b)=-\ln(a-b)\ , \tag{A4}$$

when the contour encircles both points a and b. Then, we can identify our Δ in Eq. (3.11) with the Riemann constant $\tilde{\Delta}$ for the one-loop case.

Identification of the half differential. It is easily shown that the function $S_1(z)=1/(z-\xi)$ has the following properties.

(1) $S_1(z)$ is a half differential on the covering space of the Riemann surface, i.e., for an arbitrary projective transformation $g(z)$ we get

$$\frac{1}{g(z)-\xi}\sqrt{\partial g(z)}=\frac{1}{z-g^{-1}(\xi)}\sqrt{\partial g^{-1}(\xi)}\ . \tag{A5}$$

(2) $S_1(z)$ has no zeros and no poles in the fundamental region.

(3) $S_1(z)$ is single valued around the a cycle and multivalued around the b cycle:

$$\frac{1}{T(z)-\xi}\sqrt{\partial T(z)}=\frac{1}{z-\xi}\exp(2\pi i\Delta)\ . \tag{A6}$$

Conversely, these properties determine $S_1(z)$ up to a constant, and it may be identified with the half differential:[40]

$$\tilde{S}_1(z)=\exp\left[-\oint_a d\phi(y)\ln E(y,z)\right]\ , \tag{A7}$$

where the contour is along the a cycle. In fact, following this definition, by taking the contour around the fixed point ξ in the Schottky parametrization, we can explicitly show that

$$\tilde{S}_1(z)=f(K)\frac{1}{z-\xi},\quad f(K)=\prod_{n=1}^{\infty}(1-K^n)^2\ , \tag{A8}$$

where K is the multiplier of the projective transformation $T(z)$. Hence, in the one-loop case, our $S_1(z)$ is equal to the half differential $\tilde{S}_1(z)$ up to a constant factor $f(K)$.

APPENDIX B

Two-loop handle operator. Here, we give the result of the two-loop calculation. The two-loop handle operator $\Omega^{2\ \mathrm{loop}}$ can be constructed by joining two handle operators given in Eq. (3.10):

$$\Omega^{2\ \mathrm{loop}}_{AC}=\Omega^{(1)}_{AE}1_{EF}\Omega^{(2)}_{FC}\ , \tag{B1}$$

where the superscripts (1) and (2) of the one-loop handle operators distinguish the loops. We also distinguish the geometrical objects by a superscript (μ) or by a subscript μ $(\mu=1,2)$ and therefore the handle operator for the μth handle is

$$\begin{aligned}\Omega^{(\mu)}_{\mathrm{ghost},\,AC} = {}_C\Big\langle q=0\Big| & \delta(\alpha^A_{\sigma,0}-\alpha^C_{\sigma,0}+Q)(\det\mathbf{C}^{(\mu)})^{-1} \\ & \times :\vartheta\begin{bmatrix}0\\0\end{bmatrix}\left[B^{(\mu)}\middle|\oint_{C_0}\frac{dz}{2\pi i}\left\{\partial\sigma^{(C)}(z)\phi^{(\mu)}(z,z_0)+\partial_z\left[\sigma^{(A)}\left[\frac{1}{z}\right]+Q\ln\left[\frac{1}{z}\right]\right]\phi^{(\mu)}(\Gamma(z),z_0)\right\}+Q\Delta^{(\mu)}\right] \\ & \times\exp\left[\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\left\{\frac{1}{2}\partial\sigma^{(C)}(x)\partial\sigma^{(C)}(y)\ln\left[\frac{E^{(\mu)}(x,y)}{x-y}\right]\right.\right. \\ & \qquad +\frac{1}{2}\partial_x\left[\sigma^{(A)}\left[\frac{1}{x}\right]+Q\ln\left[\frac{1}{x}\right]\right]\partial_y\left[\sigma^{(A)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right] \\ & \qquad \times\ln\left[\frac{xy}{x-y}E^{(\mu)}(\Gamma(x),\Gamma(y))\right]\end{aligned}$$

$$+\partial\sigma^{(C)}(x)\partial_y\left[\sigma^{(A)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\ln yE^{(\mu)}(x,\Gamma(y))\Bigg\}$$

$$-Q\oint_{C_0}\frac{dz}{2\pi i}\left\{\partial_z\left[\sigma^{(A)}\left[\frac{1}{z}\right]+Q\ln\left[\frac{1}{z}\right]\right]\right.$$

$$\left.\times\ln(1-z\xi_\mu)+\partial\sigma^{(C)}(z)\ln(z-\xi_\mu)\right\}\Bigg]:\Bigg|q=0\Big\rangle_A\ . \tag{B2}$$

The generator of the Schottky group for each handle operator is given by the projective transformation denoted by $T_\mu(z)$, and the two fixed points of each $T_\mu(z)$ are denoted by ξ_μ and η_μ, respectively. Then the two-loop handle operator is obtained by performing the Gaussian integration appearing in the unit operator in the same way as in the case of the matter part.[31] The result is

$$\Omega^{2\,\text{loop}}_{\text{ghost},\,AC}={}_C\Big\langle q=0\Big|\delta(\alpha^A_{\sigma,0}-\alpha^C_{\sigma,0}+2Q)(\det\mathbf{C}^{2\,\text{loop}})^{-1}$$

$$\times:\vartheta\begin{bmatrix}0\\0\end{bmatrix}\left[B_{\mu\nu}\Bigg|\oint_{C_0}\frac{dz}{2\pi i}\left\{\partial\sigma^{(C)}(z)\phi_\nu(z,z_0)+\partial_z\left[\sigma^{(A)}\left[\frac{1}{z}\right]+Q\ln\left[\frac{1}{z}\right]\right]\phi_\nu(\Gamma(z),z_0)\right\}+Q\Delta_\nu\right]$$

$$\times\exp\left[\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\left\{\tfrac{1}{2}\partial\sigma^{(C)}(x)\partial\sigma^{(C)}(y)\ln\left[\frac{E(x,y)}{x-y}\right]\right.\right.$$

$$+\frac{1}{2}\partial_x\left[\sigma^{(A)}\left[\frac{1}{x}\right]+Q\ln\left[\frac{1}{x}\right]\right]$$

$$\times\partial_y\left[\sigma^{(A)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\ln\left[\frac{xy}{x-y}E(\Gamma(x),\Gamma_{(y)})\right]$$

$$\left.\times\partial\sigma^{(C)}(x)\partial_y\left[\sigma^{(A)}\left[\frac{1}{y}\right]+Q\ln\left[\frac{1}{y}\right]\right]\ln yE(x,\Gamma(y))\right\}$$

$$\left.+Q\oint_{C_0}\frac{dz}{2\pi i}\left\{\partial_z\left[\sigma^{(A)}\left[\frac{1}{z}\right]+Q\ln\left[\frac{1}{z}\right]\right]\ln\frac{1}{z^2}S_2\left[\frac{1}{z}\right]+\partial\sigma^{(C)}(z)\ln S_2(z)\right\}\right]:\Bigg|q=0\Big\rangle_A\ . \tag{B3}$$

The quantities $\det\mathbf{C}^{2\,\text{loop}}$, $\phi_\nu(z,z_0)$ $(\nu=1,2)$, and $E^{2\,\text{loop}}(x,y)$ are the determinant factor, the first Abelian integrals, and the prime form for two loop, respectively. They are the same as the ones which appeared in the calculation of the matter part.[31]

The quantities Δ_ν and $S_2(z)$ for the two-loop handle operator are

$$\Delta_1=\tfrac{1}{2}B_{11}+\tfrac{1}{2}+\phi_1(z_0,\Gamma(0))+\frac{1}{2\pi i}\ln\left[\left[\frac{\xi_1-\xi_2}{\eta_1-\xi_2}\right]\prod_{\alpha_{22}}\left[\frac{T_{\alpha_{22}}(\xi_1)-\eta_1}{T_{\alpha_{22}}(\eta_1)-\eta_1}\frac{T_{\alpha_{22}}(\xi_1)-\xi_1}{T_{\alpha_{22}}(\eta_1)-\xi_1}\right]^{1/2}\prod_{\alpha_{12}}\left[\frac{T_{\alpha_{12}}(\xi_1)-\xi_2}{T_{\alpha_{12}}(\eta_1)-\xi_2}\right]\right],$$

$$\Delta_2=\tfrac{1}{2}B_{22}+\tfrac{1}{2}+\phi_2(z_0,\Gamma(0))+\frac{1}{2\pi i}\ln\left[\left[\frac{\xi_2-\xi_1}{\eta_2-\xi_1}\right]\prod_{\alpha_{11}}\left[\frac{T_{\alpha_{11}}(\xi_2)-\eta_2}{T_{\alpha_{11}}(\eta_2)-\eta_2}\frac{T_{\alpha_{11}}(\xi_2)-\xi_2}{T_{\alpha_{11}}(\eta_2)-\xi_2}\right]^{1/2}\prod_{\alpha_{21}}\left[\frac{T_{\alpha_{21}}(\xi_2)-\xi_1}{T_{\alpha_{21}}(\eta_2)-\xi_1}\right]\right], \tag{B4}$$

and

$$(S_2(z))^2=\left[\frac{\xi_2-\xi_1}{(z-\xi_1)(z-\xi_2)}\right]^2\prod_{\alpha_{11}}(T_{\alpha_{11}}(z),z,T_{\alpha_{11}}(\xi_2),\xi_2)\prod_{\alpha_{12}}(T_{\alpha_{12}}(z),z,T_{\alpha_{12}}(\xi_1),\xi_2)$$

$$\times\prod_{\alpha_{21}}(T_{\alpha_{21}}(z),z,T_{\alpha_{21}}(\xi_2),\xi_1)\prod_{\alpha_{22}}(T_{\alpha_{22}}(z),z,T_{\alpha_{22}}(\xi_1),\xi_1)\ . \tag{B5}$$

The suffix α_{ij} has to be understood such that the matrix $T_{\alpha_{ij}}$ is a product starting with T_i and ending with T_j: for example,

$$T_{\alpha_{11}}=T_1^{m_1}T_2^{l_1}T_1^{m_2}T_2^{l_2}\cdots T_2^{l_{N-1}}T_1^{m_N} \text{ with } m_1,m_N,l_1\neq 0 ,$$

or

$$T_{\alpha_{11}}=T_1^{m} \text{ with } m\neq 0 . \tag{B6}$$

The suffix α_{22} has to be understood in the corresponding way.

In general, we have a factor

$$\exp(Q^2\times\text{const})$$

in Eq. (B3). However, this constant can be absorbed into the normalization factor of the differential $S_2(z)$. By imposing such a factor not to appear in Eq. (B3), the normalization of the differential $S_2(z)$ in Eq. (B5) has been determined.

APPENDIX C

Evaluation of the matter part. When the external string is saturated by the tachyon state, the string-emission operator is reduced to the tachyon vertex operator, $V^{(r)}_{\text{matter}}=:\exp[p_rX(z_r)]:$, where p_r is the momentum of the rth tachyon. Therefore, the g-loop N-tachyon amplitude is given by

$$(\mathbf{V}^{g\text{ loop},N}|\prod_{r=2g+1}^{2g+N}(|p_r\rangle_r\otimes|1;\text{ghost}\rangle_r)=\int dM^{g\text{ loop},N}\left\langle\Omega^{g\text{ loop}}_{\text{matter}}\right|\prod_{r=2g+1}^{2g+N}:\exp[p_rX(z_r)]:\left|0\right\rangle . \tag{C1}$$

To evaluate the integrand of the right-hand side, we insert the unit operator of the matter part[31] between $\langle\Omega^{g\text{ loop}}_{\text{matter}}|$ and $\prod:\exp[p_rX(z_r)]:|0\rangle$. Then, the integrand can be expressed as

$$\begin{aligned}\left\langle\Omega^{g\text{ loop}}_{\text{matter}}\right|&\prod_{r=2g+1}^{2g+N}:\exp[p_rX(z_r)]:\left|0\right\rangle\\ &=(\det\mathbf{C}^{g\text{ loop}})^{-1}\prod_{s<t}(z_s-z_t)^{p_sp_t}\\ &\quad\times\left\langle 0\right|\int\prod_{\nu=1}^{g}dk_\nu\prod_{n=1}^{\infty}d\tilde{a}^*_n d\tilde{a}_n\exp\left\{i\pi k_\mu B^{g\text{ loop}}_{\mu\nu}k_\nu+2\pi i k_\nu\oint_{C_0}\frac{dz}{2\pi i}\partial\hat{X}(z)\phi^{g\text{ loop}}_\nu(z,z_0)\right.\\ &\qquad+\frac{1}{2}\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\partial\hat{X}(x)\partial\hat{X}(y)\ln\left[\frac{E^{g\text{ loop}}(x,y)}{x-y}\right]\\ &\qquad-\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\partial_x\left[\hat{X}\left[\frac{1}{x}\right]\right]\ln(1-xy)\partial\hat{X}(y)\\ &\qquad\left.+\oint_{C_0}\frac{dw}{2\pi i}\partial_w\left[\hat{X}\left[\frac{1}{w}\right]\right]\sum_s\ln(1-wz_s)p_s\right\}\left|\sum_t p_t\right\rangle ,\end{aligned} \tag{C2}$$

and after the integration over the nonzero modes $d\tilde{a}^*_n d\tilde{a}_n$, this becomes

$$\begin{aligned}(\det\mathbf{C}^{g\text{ loop}})^{-1}\prod_{s<t}(z_s-z_t)^{p_sp_t}\int\prod_{\nu=1}^{g}dk_\nu\exp&\left[i\pi k_\mu B^{g\text{ loop}}_{\mu\nu}k_\nu-2\pi i k_\nu\sum_s p_s\phi^{g\text{ loop}}_\nu(z_s,z_0)\right.\\ &\left.+\sum_{s\le t}p_sp_t\ln\left[\frac{E^{g\text{ loop}}(z_s,z_t)}{z_s-z_t}\right]\right] .\end{aligned} \tag{C3}$$

The integration over the zero mode dk_v leads us to the final form

$$\left\langle \Omega^{g\ \text{loop}}_{\text{matter}} \left| \prod_{s=2g+1}^{2g+N} :\exp[p_s X(z_s)]: \right| 0 \right\rangle$$
$$= (\det \mathbf{C}^{g\ \text{loop}})^{-1} (\det B^{g\ \text{loop}})^{-1/2}$$
$$\times \prod_{2g+1 \le s < t \le 2g+N} (z_s - z_t)^{p_s p_t} \exp\left[\frac{1}{2} \sum_{s,t=2g+1}^{2g+N} p_s p_t \ln\left[\frac{E^{g\ \text{loop}}(z_s,z_t)}{z_s - z_t} \right] + \left[\sum_s p_s \phi^{g\ \text{loop}}_\mu (z_s, z_0) \right] (2B_{\mu\nu})^{-1} \left[\sum_s p_s \phi^{g\ \text{loop}}_\nu (z_s,z_0) \right] \right] . \quad \text{(C4)}$$

In case of the one-loop diagram ($g=1$), this leads to

$$\left\langle \Omega^{1\ \text{loop}}_{\text{matter}} \left| \prod_{s=3}^{N+2} :\exp\{p_s X(z_s)\}: \right| 0 \right\rangle = \frac{1}{\prod_{r=3}^{N+2} z_r} \prod_{m=1}^{\infty} \left[\frac{1}{1-K^m} \right]^d \left[\frac{1}{\ln K} \right]^{d/2} \exp\left[\sum_{r<s} p_r p_s \ln \psi_{rs} \right] , \quad \text{(C5)}$$

with

$$\psi_{rs} = \frac{1}{\sqrt{z_r z_s}} \exp\left[\frac{[\ln(z_s/z_r)]^2}{2\ln K} \right] E(z_r, z_s) . \quad \text{(C6)}$$

In deriving these equations we have repeatedly used the momentum conservation, $\sum_s p_s = 0$.

The formula above is for the open string. When we consider the case of the closed string, we have to include the contribution from the right movers before performing the integration over the zero modes. Then, our formula produces the well-known result of the matter part of the closed string as shown in Ref. 31. Then, with the formula for the one-loop closed string, using the result of the measure Eq. (4.24), we get the modular-invariant one-loop N-tachyon amplitudes.

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