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Derivative expansions in quantum electrodynamics

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A new method for calculating the covariant derivative expansions is presented, particularly in Abelian gauge theories, which can be used to find derivative expansions around nonvanishing gauge field-strength tensors. We apply this method to find the $O((\partial_\lambda F_{\mu\nu})^2)$ terms of electron determinants in quantum electrodynamics.

Recently there has been much effort to find a systematic way to calculate the derivative expansions, which play an important role in finding low-energy effective-field theories.[1-6] In this expansion the highly nonlocal quantity, the effective action, can be written as an infinite sum of local functions of basic fields and their derivatives. In this paper we present a new method for calculating derivative expansions, which is applicable to curved space-time. Although our method can be generalized to higher loops, we will limit our study to one-loop amplitudes.

For the one-loop effective action we use the heat-kernel representation. A crucial point in our method is to represent the heat kernel by a flat normal-coordinate system (FNCS).[7] In a FNCS differential operators on a vector bundle are expressed by a normal coordinate X and fiber frames obtained by the parallel transportation from a base point x. In a FNCS a differential operator behaves like a covariant function as for x and a differential operator in a flat space-time as for X. In Ref. 7 we used a FNCS to calculate asymptotic expansions for the heat kernel of general minimal operators. The algorithm developed in Ref. 7 can be easily generalized to derivative expansions.

In this paper we illustrate our method by calculating derivative expansions of electron determinants in QED. In Abelian gauge theories like QED, our method has another advantage. We can expand the effective action around nonvanishing field strength. The previous methods could not handle this case. In our method we can expand the effective Lagrangian in the form

$$\mathcal{L}_{\text{eff}}=\mathcal{L}_0(F_{\tau\rho})+\partial_\lambda F_{\mu\nu}\partial_\gamma F_{\alpha\beta}\mathcal{L}_1^{\mu\nu\lambda\alpha\beta\gamma}(F_{\tau\rho})+\cdots\ , \tag{1}$$

where the $\mathcal{L}_k$'s denote some local functions of the field-strength tensor $F_{\mu\nu}$, and the background gauge is assumed. $\mathcal{L}_0$ was first calculated by Schwinger.[8] Our aim is to calculate the next term $\mathcal{L}_1$.

The one-loop effective action for the photon field is given by the electron determinant

$$iW^{(1)}=\ln\det(-i\gamma_\mu D^\mu-m)=\tfrac{1}{2}\ln\det(-D^2-\tfrac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}+m^2)\ , \tag{2}$$

where $D^\mu=\partial^\mu-ieA^\mu$, $F^{\mu\nu}=i[D^\mu,D^\nu]$, $\sigma^{\mu\nu}=\frac{1}{2}i[\gamma^\mu,\gamma^\nu]$, and m is the electron mass. We use the metric $\eta^{\mu\nu}=(1,1,1,-1)$ and our Dirac matrices satisfy $\{\gamma^\mu,\gamma^\nu\}=-2\eta^{\mu\nu}$. In the proper-time heat-kernel method $W^{(1)}$ can be written as

$$\begin{aligned} iW^{(1)}&=-\tfrac{1}{2}\int_\zeta^\infty\frac{d\tau}{\tau}\,\mathrm{Tr}\exp[-\tau(-D^2+m^2-\sigma\cdot F/2)]\\ &=-\tfrac{1}{2}\int_\zeta^\infty\frac{d\tau}{\tau}\int d^4x\,\mathrm{tr}\{e^{-\tau m^2}\langle x|\exp[-\tau(-D^2-\sigma\cdot F/2)]|x\rangle\}\ , \end{aligned}\tag{3}$$

where tr denotes the trace over the Dirac indices and ζ the proper-time cutoff.

Now we briefly review a FNCS for general gauge theories in flat space-time.[7] First, we choose a base point x in space-time and express position variable y as $y=X+x$, where X can be regarded as a normal coordinate for y. Consider a differential operator of the form $M(D_\mu,\phi)$, where ϕ denotes an arbitrary tensor field. Let us define $\bar{D}_\mu(X)$ and $\bar{\phi}(X)$ such that

$$\begin{aligned}\bar{D}_\mu(X)&\equiv T(x,y)D_\mu(y)T(y,x)\,,\\ \bar{\phi}(X)&\equiv T(x,y)\phi(y)T(y,x)\,,\end{aligned} \tag{4}$$

where $T(y,x)$ is a parallel-transportation matrix from x to y satisfying

$$X\cdot\vec{D}(y)T(y,x)=0=T(y,x)X\cdot\overleftarrow{D}(y)$$

and $T(x,x)=1$. For most cases we can write

$$M(D_\mu(y),\phi(y))=T(y,x)\bar{M}T(x,y)\,,$$

where $\bar{M}=M(\bar{D}_\mu,\bar{\phi})$. This implies

$$\begin{aligned}\langle y|e^{-\tau M}|x\rangle&=T(y,x)\langle X|e^{-\tau\bar{M}}|0\rangle\,,\\ \langle x|e^{-\tau M}|x\rangle&=\langle 0|e^{-\tau\bar{M}}|0\rangle\,.\end{aligned} \tag{5}$$

The next step is to expand $\bar{D}^\mu$ and $\bar{\phi}$ in terms of X. It is easy to find

$$\begin{aligned}\bar{D}^\mu&=\bar{\partial}^\mu-\sum i\frac{n-1}{n!}X_{\alpha_1\cdots\alpha_n}F^{\mu\alpha_1;\cdots\alpha_n}\,,\\ \bar{\phi}&=\sum\frac{1}{n!}X_{\alpha_1\cdots\alpha_n}\phi^{;\alpha_1\cdots\alpha_n}\,,\end{aligned} \tag{6}$$

where $\bar{\partial}^\mu=\partial/\partial X_\mu$, $X_{\alpha_1\cdots\alpha_n}=X_{\alpha_1}\cdots X_{\alpha_n}$, and the semicolons denote covariant differentiations. Therefore, the expansion of $\bar{M}$ has the general form

$$\bar{M}=\sum a_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_n}(x)X^{\alpha_1\cdots\alpha_n}\bar{\partial}^{\beta_1\cdots\beta_n}\,, \tag{7}$$

where $\bar{\partial}_{\alpha_1\cdots\alpha_n}=\bar{\partial}_{\alpha_1}\cdots\bar{\partial}_{\alpha_n}$ and $a_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_n}(x)$ is a gauge-covariant tensor field. This representation of $\bar{M}$ is useful to evaluate the heat kernel and the derivative expansions.

Now we return to QED. From Eq. (5) we can write

$$\langle x|\exp[-\tau(-D^2-\sigma\cdot F/2)]|x\rangle=\langle 0|\exp[-\tau(-\bar{D}^2-\sigma\cdot\bar{F}/2)]|0\rangle\,. \tag{8}$$

First, we should expand $-\bar{D}^2-\sigma\cdot\bar{F}/2$ in the form of (7). Using Eq. (6), we find

$$-\bar{D}^2-\sigma\cdot\bar{F}/2=-(\mathcal{D}^2+\tfrac{1}{2}\sigma\cdot F+\epsilon_1+\epsilon_2+\cdots)\,, \tag{9}$$

where

$$\begin{aligned}\epsilon_1&=-iF_{\alpha\beta;\gamma}(\tfrac{2}{3}X^{\alpha\gamma}\mathcal{D}^\beta+\tfrac{1}{3}X^\alpha\eta^{\gamma\beta}+\tfrac{1}{2}i\sigma^{\alpha\beta}X^\gamma)\,,\\ \epsilon_2&=-\tfrac{1}{4}iF_{\alpha\beta;\gamma\delta}(X^{\alpha\gamma\delta}\mathcal{D}^\beta+X^{\alpha\gamma}\eta^{\delta\beta}+i\sigma^{\alpha\beta}X^{\gamma\delta})\\ &\quad-\tfrac{1}{9}F_{\alpha\mu;\beta}F_\gamma{}^\mu{}_{;\delta}X^{\alpha\beta\gamma\delta}\,,\end{aligned}$$

with $\mathcal{D}_\mu=\bar{\partial}_\mu-\frac{1}{2}iX^\tau F_{\tau\mu}$. Since we are interested in the expansion around nontrivial $F_{\mu\nu}$, in Eq. (9), $\epsilon\equiv-(\epsilon_1+\epsilon_2+\cdots)$ is regarded as a perturbation and we evaluate Eq. (8) by expanding with ϵ. For this purpose we will use

$$e^{-\tau(H+\epsilon)}=\left[1-\int_0^\tau d\tau_1\epsilon(\tau_1)+\int_0^\tau d\tau_2\int_0^{\tau_2}d\tau_1\epsilon(\tau_1)\epsilon(\tau_2)+\cdots\right]e^{-\tau H}\,, \tag{10}$$

where $\epsilon(\tau)=e^{-\tau H}\epsilon e^{\tau H}$. In our case $H=-(\mathcal{D}^2+\sigma\cdot F/2)$ and we can find

$$\begin{aligned}&\mathcal{D}^\alpha(\tau)=(e^{2i\tau F}\mathcal{D})^\alpha,\ X^\alpha(\tau)=[(1-e^{2i\tau F})F^{-1}\mathcal{D}]^\alpha+X^\alpha\,,\\ &\sigma_{\mu\nu}(\tau)=e^{\tau\sigma\cdot F/2}\sigma_{\mu\nu}e^{-\tau\sigma\cdot F/2}\,,\end{aligned} \tag{11}$$

where F means $F_{\mu\nu}$ as a matrix.

Fortunately an analytic expression for $\langle X|\exp(-\tau H)|0\rangle$ is known in the Abelian case:[8]

$$\langle X|\exp(-\tau H)|0\rangle=\frac{i}{(4\pi)^2\tau^2}\exp[-\tfrac{1}{2}\operatorname{tr}\ln(\tau F)^{-1}\sin(\tau F)]\exp[-\tfrac{1}{4}X\cdot F\cdot\cot(\tau F)\cdot X]\exp(\tfrac{1}{2}\tau\sigma\cdot F)\,. \tag{12}$$

Equations (10) and (12) can be combined to yield

$$\begin{aligned}\langle 0|\exp[-\tau(H+\epsilon)]|0\rangle&=\frac{i}{(4\pi)^2\tau^2}\exp[-\tfrac{1}{2}\operatorname{tr}\ln(\tau F)^{-1}\sin(\tau F)]\\ &\quad\times\left[\left[1+\int_0^\tau d\tau_1\epsilon_2(\tau_1)+\int_0^\tau d\tau_2\int_0^{\tau_2}d\tau_1\epsilon_1(\tau_1)\epsilon_1(\tau_2)+\cdots\right]e^{f(X)}\right]_{X=0}e^{\tau\sigma\cdot F/2}\,,\end{aligned} \tag{13}$$

where $f(X)=-\frac{1}{4}X\cdot F\cdot\cot(\tau F)\cdot X$. To evaluate the various quantities in Eq. (13) at $X=0$, we use the formulas

$$\begin{aligned}I(\tau_1,\tau_2)&\equiv X(\tau_1)X(\tau_2)e^{f(X)}|_{X=0}=-2\exp[i(\tau_1-\tau_2)F]\frac{\sin(\tau_1F)}{F}\frac{\sin[(\tau_2-\tau)F]}{\sin(\tau F)}\,,\\ J(\tau_1,\tau_2)&\equiv X(\tau_1)\mathcal{D}(\tau_2)e^{f(X)}|_{X=0}=-\exp[i(\tau_1-2\tau_2+\tau)F]\frac{\sin(\tau_1F)}{\sin(\tau F)}\,,\\ K(\tau_1,\tau_2)&\equiv\mathcal{D}(\tau_1)X(\tau_2)e^{f(X)}|_{X=0}=-\exp[i(2\tau_1-\tau_2)F]\frac{\sin[(\tau_2-\tau)F]}{\sin(\tau F)}\,,\\ L(\tau_1,\tau_2)&\equiv\mathcal{D}(\tau_1)\mathcal{D}(\tau_2)e^{f(X)}|_{X=0}=-\tfrac{1}{2}\exp[i(2\tau_1-2\tau_2+\tau)F]\frac{F}{\sin(\tau F)}\,.\end{aligned} \tag{14}$$

From Eqs. (9) and (14) we obtain

$$\epsilon_2(\tau_1)e^{f(X)}|_{X=0} = -\tfrac{1}{4} iF_{\alpha\beta;\gamma\delta}(2I_1^{\alpha\dot\gamma}J_1^{\dot\delta\beta}+I_1^{\gamma\delta}J_1^{\alpha\beta}+I_1^{\alpha\gamma}\eta^{\delta\beta}+iI_1^{\gamma\delta}\sigma_1^{\alpha\beta}) - \tfrac{1}{3} F_{\alpha\mu;\beta}F_{\gamma}{}^{\mu}{}_{;\delta}I_1^{\dot\alpha\dot\beta}I_1^{\dot\gamma\dot\delta} ,$$

$$\begin{aligned}\epsilon_1(\tau_1)\epsilon_1(\tau_2)e^{f(X)}|_{X=0} = -F_{\alpha\beta;\gamma}F_{\mu\nu;\lambda}\{ &\tfrac{8}{9} [I_1^{\alpha\gamma}(K_{12}^{\beta\dot\mu}J_2^{\dot\lambda\nu}+\tfrac{1}{2} L_{12}^{\beta\nu}I_2^{\mu\lambda})+J_1^{\alpha\beta}I_{12}^{\gamma\dot\mu}J_2^{\dot\lambda\nu}+J_1^{\dot\alpha\beta}J_{12}^{\dot\gamma\nu}I_2^{\mu\lambda} \\ &+I_{12}^{\alpha\dot\mu}(J_1^{\gamma\beta}J_2^{\dot\lambda\nu}+I_{12}^{\gamma\dot\lambda}L_{12}^{\beta\nu})+I_{12}^{\alpha\dot\lambda}J_{12}^{\gamma\nu}K_{12}^{\beta\dot\mu}+I_{12}^{\gamma\dot\mu}J_{12}^{\alpha\nu}K_{12}^{\beta\dot\lambda}] \\ &+\tfrac{2}{9}\eta^{\lambda\nu}(I_1^{\alpha\gamma}K_{12}^{\beta\mu}+2J_1^{\dot\alpha\beta}I_{12}^{\dot\gamma\mu})+\tfrac{2}{9}\eta^{\gamma\beta}(2I_{12}^{\alpha\dot\mu}J_2^{\dot\lambda\nu}+J_{12}^{\alpha\nu}I_2^{\mu\lambda}) \\ &+\tfrac{1}{3} i\sigma_2^{\mu\nu}(I_1^{\alpha\gamma}K_{12}^{\beta\lambda}+2J_1^{\dot\alpha\beta}I_{12}^{\dot\gamma\lambda})+\tfrac{1}{3} i\sigma_1^{\alpha\beta}(2I_{12}^{\gamma\dot\mu}J_2^{\dot\lambda\nu}+J_{12}^{\gamma\nu}I_2^{\mu\lambda}) \\ &+\tfrac{1}{9}\eta^{\gamma\beta}\eta^{\lambda\nu}I_{12}^{\alpha\mu}-\tfrac{1}{4}\sigma_1^{\alpha\beta}\sigma_2^{\mu\nu}I_{12}^{\gamma\lambda}+\tfrac{1}{6} i\sigma_1^{\alpha\beta}\eta^{\lambda\nu}I_{12}^{\gamma\mu}+\tfrac{1}{6} i\eta^{\gamma\beta}\sigma_2^{\mu\nu}I_{12}^{\alpha\lambda}\} ,\end{aligned} \tag{15}$$

where the dotted indices must be symmetrized, $\sigma_1^{\alpha\beta}=\sigma^{\alpha\beta}(\tau_1)$, $\sigma_2=\sigma^{\alpha\beta}(\tau_2)$, $I_1=I(\tau_1,\tau_1)$, $I_2=I(\tau_2,\tau_2)$, $I_{12}=I(\tau_1,\tau_2)$, and so on.

Next we must integrate Eq. (15) with respect to τ_1 and τ_2. In weak-field cases where $F_{\mu\nu}/m^2\simeq 0$, these integrations are simple and the result agrees with the previous one.[9] For general cases it is necessary to investigate some properties of the field-strength tensor $F_{\mu\nu}$.[8] $F_{\mu\nu}$, regarded as a matrix, satisfies $F^4+2\mathcal{F}F^2-\mathcal{G}^2=0$, where $\mathcal{F}=\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=(B^2-E^2)/2$, $\mathcal{G}=\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}=\mathbf{E}\cdot\mathbf{B}$, and ${}^*F^{\mu\nu}=-\frac{1}{2}\epsilon^{\mu\nu\lambda\tau}F_{\lambda\tau}$. The eigenvalues of F are $f_1\equiv\frac{1}{2}i(X+X^*)$, $f_2\equiv\frac{1}{2}i(X-X^*)$, and their negative values, where $X=[2(\mathcal{F}+i\mathcal{G})]^{1/2}$. Note that $\pm X$ and $\pm X^*$ are nothing but the four eigenvalues of $\sigma\cdot F/2$. In integrating Eq. (15), we need

$$\exp(i\tau F)=\frac{f_1^2-F^2}{f_1^2-f_2^2}\left[\cos(\tau f_2)+i\,\frac{\sin(\tau f_2)}{f_2}F\right]+(f_1\leftrightarrow f_2) . \tag{16}$$

A similar equation holds for $\sin(\tau F)$. Inserting these results into Eq. (14), we are able to integrate Eq. (15). On the other hand, as shown in Eq. (3), we should evaluate the trace of several quantities over Dirac indices. In our case, we need

$$\exp(\tfrac{1}{2}\tau\sigma\cdot F)=\frac{X^{*2}-\Omega^2}{X^{*2}-X^2}\left[\cosh(\tau X)+\frac{\sinh(\tau X)}{X}\Omega\right]+(X\leftrightarrow X^*) , \tag{17}$$

where $\Omega=\sigma\cdot F/2$. Note that $\Omega^2=2(\mathcal{F}+i\gamma_5\mathcal{G})$ with $\gamma_5\equiv i\gamma^1\gamma^2\gamma^3\gamma^4$.

Using Eqs. (3) and (13)–(17), it is straightforward to calculate $\mathcal{L}_0$ and $\mathcal{L}_1$ in Eq. (1). As for the integrations over τ_1 and τ_2 in Eq. (13), we can find these in closed forms and, after evaluating the trace, the results have the tensor structure

$$\begin{aligned}\mathrm{tr}\langle x|\exp[-\tau(-D^2-\sigma\cdot F/2)]|x\rangle = &\frac{\mathcal{G}}{4\pi^2}\cot(\tau f_1)\cot(\tau f_2) \\ &\times[1+(F^iF_\mu F^jF_\nu F^kY_1^{ijk}+F^iF_\nu F^jF_\mu Y_2^{ijk}+F^i\langle F_\mu{}^*F_\nu F^j\rangle Y_3^{ij})^{\mu\nu}+\cdots] ,\end{aligned} \tag{18}$$

where we have discarded total divergence terms, F_λ is the matrix form of the tensor $F_{\mu\nu;\lambda}$, and $\langle\cdots\rangle$ denotes the trace over the Lorentz indices. In Eq. (18), Y_n^{ijk} and Y_m^{ij} $(i,j,k=0,1,2,3)$ are some functions of $\mathcal{F}$, $\mathcal{G}$, and τ, and summation over i, j, and k is assumed. To reduce each term into this form, we used the Bianchi identity and the relation $F^4+2\mathcal{F}F^2-\mathcal{G}^2=0$. Since the effective action is invariant under $F_{\mu\nu}\to -F_{\mu\nu}$, Y_1 and Y_2 vanish when $i+j+k$ is odd, and Y_3 also vanishes for odd $i+j$. In this paper we do not pursue the complete calculations, which will be considered elsewhere.

Instead we specialize to the simpler cases: $F(x)=\phi(x)\mathbb{F}$ and $\mathcal{G}=0$, where $\mathbb{F}$ is a constant tensor and ϕ a scalar field. When $\mathcal{G}=0$, one of f_1 and f_2 is zero. Let $f_2=0$ and $f_1=f$. Note that $f^2=-2\mathcal{F}$ and $F^3=f^2F$. In these cases our calculations are relatively simple, but still need some tools which can do symbolic calculations. We used REDUCE and found, for the higher-derivative part,

$$iW_{\mathrm{hd}}^{(1)}=\frac{i}{2}\int d^4x\,\phi^{-2}(\partial^\mu\phi)(\partial_\mu\phi)\int_0^\infty\frac{ds}{s}e^{-\tau m^2}\frac{\tau t}{(4\pi)^2\tau^2}\left[\frac{tY^4+3Y^3-4tY^2-3Y+3t}{Y^4}\right] , \tag{19}$$

where $t=-i\tau f$ and $Y=\tanh(t)$.

Now we study the consequences of the Bianchi identiy. In our cases, we may write, without loss of generality, the field-strength tensors as

$$\mathbf{E}=E_0\phi\hat{\mathbf{y}},\ \ \mathbf{B}=B_0\phi\hat{\mathbf{z}} , \tag{20}$$

where E_0 and B_0 denote constants. Solving the Bianchi identity $\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mathbf{E}=-\partial_t\mathbf{B}$, we find

$$\partial_z\phi=0\ \text{ and }\ E_0\partial_x\phi=-B_0\partial_t\phi . \tag{21}$$

Equation (19) holds for any field satisfying Eq. (21). Note that $f^2=(E_0^2-B_0^2)\phi^2=E^2-B^2$. Equation (21) has the two interesting solutions: (i) $\mathbf{B}=B(x,y)\hat{\mathbf{z}}$, $\mathbf{E}=\mathbf{0}$, and (ii) $\mathbf{E}=E(x,t)\hat{\mathbf{y}}$, $\mathbf{B}=\mathbf{0}$.

Now we consider the effective Lagrangian for the case (i). A general case is obtained by replacing B with $-if$. From Eq. (19) we find

$$\mathcal{L}_{\rm reg}^{(1)}=\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}(e^{-\tau m^2}-e^{-\tau\Lambda^2})\frac{t}{(4\pi)^2\tau^2}\left[-\frac{4}{Y}+\frac{4}{t}+\tau\frac{tY^4+3Y^3-4tY^2-3Y+3t}{Y^4}\frac{(\partial B)^2}{B^2}+\cdots\right],$$

$$=\frac{1}{2}\frac{B^2}{(4\pi)^2}\frac{4}{3}\ln(m^2/\Lambda^2)+\frac{1}{2}B^2A_1(|B/m^2|)+\frac{1}{2m^2}\partial_\mu B\,\partial^\mu B A_2(|B/m^2|)+\cdots, \tag{22}$$

where we converted the proper-time cutoff into a Pauli-Villars regulator mass, and

$$A_1(b)=\frac{1}{(4\pi)^2}\int_0^\infty \frac{dt}{t^2}e^{-t/b}\left[-\frac{4}{Y}+\frac{4}{t}-\frac{4}{3}t\right],$$

$$A_2(b)=\frac{1}{(4\pi)^2}\frac{1}{b}\int_0^\infty \frac{dt}{t}e^{-t/b}\left[\frac{tY^4+3Y^3-4tY^2-3Y+3t}{Y^4}\right].$$

Note that $A_1(b)$ and $A_2(b)$ are positive quantities. Here we use on-shell renormalization. A one-loop counterterm in this renormalization scheme and regularization method is known as[10]

$$\mathcal{L}_{\rm coun}^{(1)}=-\frac{1}{2}\frac{B^2}{(4\pi)^2}\frac{4}{3}\ln(m^2/\Lambda^2).$$

Therefore the renormalized effective Lagrangian density becomes

$$\mathcal{L}_{\rm ren}=-\frac{1}{2}B^2\left[\frac{1}{e^2}-A_1(|B/m^2|)+\cdots\right]+\frac{1}{2m^2}\partial_\mu B\partial^\mu B[A_2(|B/m^2|)+\cdots]+\cdots. \tag{23}$$

For the weak field ($b\to 0$):

$$A_1(b)=\frac{1}{(4\pi)^2}\frac{4}{45}b^2(1-\frac{4}{7}b^2+\cdots),\quad A_2(b)=\frac{1}{(4\pi)^2}\frac{4}{15}(1-\frac{20}{21}b^2+\cdots). \tag{24}$$

For the strong field ($b\to\infty$):

$$A_1(b)=\frac{4}{3}\frac{1}{(4\pi)^2}\ln(b)+O(1),\quad A_2(b)=\frac{1}{(4\pi)^2}\frac{1}{b}\int_0^\infty\frac{dt}{t}\left[\frac{tY^4+3Y^3-4tY^2-3Y+3t}{Y^4}\right]+O\left[\frac{1}{b^2}\right]. \tag{25}$$

From Eq. (25) we can see that the higher-derivative corrections are suppressed at the strong-field limit. Indeed for large b, $A_2(b)\approx Cb^{-1}$ for some constant C. Our result given in Eq. (23) is the generalization of the effective Lagrangian for constant electromagnetic fields first obtained by Schwinger.[8] In non-Abelian cases it is difficult to evaluate the similar quantities except for the field configurations restricted to the Abelian sector. Our derivative expansions will be useful to study effective actions for field configurations which are slowly varying in space-time around nonvanishing background fields.

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