Temperature inversion symmetry in the Casimir effect

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The finite-temperature Casimir effect between parallel plates has a simple symmetry under temperature inversion. With symmetric boundary conditions the ordinary, zero-temperature Casimir energy is directly related to the Stefan-Boltzmann energy of thermal radiation. This symmetry holds for both boson and fermion fields.

Quantum fluctuations in the presence of confining boundaries give a volume-dependent vacuum energy. This is the effect first found by Casimir[1] in the simplest geometry of an electromagnetic field between two parallel plates.

Because of the special geometry of this particular case, there is a simple symmetry in the case of finite temperatures.[2] It can be used to relate directly the zero-temperature Casimir energy to the energy density of free blackbody radiation in the absence of any confining obstacles as also recently pointed out by Ford.[3] In a somewhat different context, it was previously used as a starting point for the investigation of symmetry breaking in quantum field theories in spacetimes with at least one compactified spatial dimension.[4]

By its very existence, the Casimir energy is intimately related to the boundary conditions of the confined field. We will show that it is only for boson fields with symmetric and fermion fields with antisymmetric boundary conditions that the symmetry obtains. It is then a result of the invariance under the exchange of spatial and imaginary-time directions in the partition function. Ignoring the boundary conditions, the whole symmetry boils down to the physical equivalence between the free energy of the finite-temperature Casimir effect and the free energy of blackbody radiation between parallel plates.

This temperature inversion symmetry can be seen already in the generic paper by Brown and Maclay[5] who calculated the finite-temperature Casimir energy of an electromagnetic field between two parallel plates. Rather than the free energy per unit plate area $F(T,L)$ where L is the plate separation and T the temperature, it is convenient to introduce the dimensionless function $f=L^3F$ which can only be a function of the dimensionless variable $\xi=LT$ in units where Boltzmann's constant $k=1$. They found that it can be written as a sum of three terms, $f(\xi)=f_0+f_\infty(\xi)+f_T(\xi)$, in a slightly different notation. Here $f_0=-\pi^2/720$ is the conventional Casimir energy and $f_\infty(\xi)=-(\pi^2/45)\xi^4$ is the Stefan-Boltzmann expression for the free energy of blackbody radiation. The nontrivial temperature dependence is contained in the function

$$f_T(\xi)=-\frac{1}{4\pi^2}\sum_{m,n=1}^{\infty}\frac{(2\xi)^4}{[m^2+(2\xi n)^2]^2} . \tag{1}$$

Brown and Maclay noticed that this function has the property

$$f(\xi)=(2\xi)^4 f\left[\frac{1}{4\xi}\right] \tag{2}$$

which is a relation between the functional values at low and high temperatures for a given plate separation L.

Now it can easily be seen that the three contributions to the free energy can be combined into the compact expression

$$f(\xi)=-\frac{1}{16\pi^2}\sum_{m,n}{}'\frac{(2\xi)^4}{[m^2+(2\xi n)^2]^2} , \tag{3}$$

where the sum m,n extends over all positive and negative integers except for $m=n=0$. It is obvious that this new function has the same symmetry under temperature inversion as Brown and Maclay found in the partial free energy $f_T(\xi)$. In fact, from the thermal radiation free energy $f_\infty=-(\pi^2/45)\xi^4$ we obtain, in the zero-temperature limit,

$$f_0(\xi)=(2\xi)^4 f_\infty\left[\frac{1}{4\xi}\right]=-\frac{\pi^2}{720} \tag{4}$$

which is just the original Casimir energy.

Recently, the Casimir free energy of massless fermions confined between two parallel plates has been calculated.[6] The fluctuating Dirac field satisfies the MIT boundary condition $i\not{n}\psi=\psi$ on the plates. One then finds that the dimensionless free energy can be written as the double sum

$$f(\xi)=-\frac{1}{8\pi^2}\sum_{m,n}{}'(-1)^{m+n}\frac{(2\xi)^4}{[m^2+(2\xi n)^2]^2} \tag{5}$$

which again can be seen to satisfy the functional relationship (2) implying symmetry under temperature inversion.

At very high temperatures where the presence of the confining plates is no longer felt, the fermionic free energy attains the well-known value $\frac{7}{4}$ of the corresponding free energy of thermal photon radiation. The temperature inversion symmetry then gives the same ratio between the Casimir energies of the two fields between parallel plates at zero temperature. This agrees with a previous result by Johnson.[7]

The origin of the symmetry is most easily seen after a Wick rotation to imaginary time $\tau=it$. Boson fields at finite temperature $\beta=1/T$ will then be periodic in the τ direction with period β. Symmetric boundary conditions on the plates then also imply periodicity in the z direction normal to the plates with period $2L$. Since the fields move unconstrained in the two transverse directions, we will then have a symmetry in the exchange of β with $2L$; i.e., the scaled free energy will be just a function of the dimensionless variable $2L/\beta=2\xi$.

Fermions will be antisymmetric in the τ direction with the same period β. Using the MIT boundary condition, one has also effectively antisymmetry in the z direction with period $2L$. The same temperature inversion symmetry then follows as is most simply seen using functional methods.[2] The full partition function is then just given by the determinant of the Euclideanized Dirac operator which is symmetric under the exchange $\beta\leftrightarrow 2L$.

The double sums in free energies (3) and (5) are both of the form

$$Z_{uv}(x)=\sum_{m,n}{}' e^{2\pi i(mu+nv)}\frac{1}{(m^2x^2+n^2)^2} \tag{6}$$

which define Epstein zeta functions[8] for different choices of indices u and v. One can then write the electromagnetic free energy (3) in the more compact form

$$f(\xi)=-\frac{1}{16\pi^2}Z_{00}\left[\frac{1}{2\xi}\right] \tag{7}$$

while the Dirac free energy (5) becomes

$$f(\xi)=\frac{1}{8\pi^2}Z_{\frac{1}{2}\frac{1}{2}}\left[\frac{1}{2\xi}\right]. \tag{8}$$

Both of these have temperature inversion symmetry.

It should also result for other massless fields with symmetric boundary conditions. For instance, Tadaki and Takagi[9] have calculated the Casimir free energies of scalar fields between parallel plates. Adding together the contribution of two such fields, one satisfying the Dirichlet boundary condition $\phi=0$ and the other satisfying the Neumann boundary condition $\partial_z\phi=0$ on both plates, one obtains[10] exactly the electromagnetic result (7). On the other hand, a scalar field which satisfies a Dirchlet condition on one plate and a Neumann condition on the other plate has a free energy given as

$$f(\xi)=-\frac{1}{32\pi^2}Z_{0\frac{1}{2}}\left[\frac{1}{2\xi}\right]. \tag{9}$$

As expected, this is not symmetric under temperature inversion.

Instead of the MIT boundary condition for fermions which resulted in the symmetric free energy (8), one can also consider other boundary conditions. From the requirements of supersymmetry Igarashi[11] has derived a new set of boundary conditions for a Majorana field between two plates which can be seen to require the field to by symmetric in the z direction with period $2L$. One would then not expect symmetry under temperature inversion in the corresponding free energy. An explicit calculation[10] gives in fact

$$f(\xi)=\frac{1}{16\pi^2}Z_{\frac{1}{2}0}\left[\frac{1}{2\xi}\right] \tag{10}$$

which clearly reveals this asymmetry.

The symmetry obviously shows up also in higher dimensions. In a D-dimensional spacetime with $D>2$ and still only one compactified spatial dimension, it can again be expressed in a functional way as in (2) when the exponent 4 is changed to D. When $D=2$, i.e., in one space dimension, the symmetry manifests itself most directly in the partition function and not in the free energy. In fact, it is now enlarged to a much higher symmetry due to the conformal invariance of two-dimensional, massless quantum field theories. This higher symmetry is modular invariance and plays an important role in quantum theories of strings[12] and critical phenomena in two dimensions.[13]

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