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Topologically nontrivial solutions to Yang-Mills equations with axisymmetric external sources

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We present a new set of solutions to Yang-Mills equations with axially symmetric external charge sources. Our solutions for the gauge fields are not explicitly axisymmetric, but the noninvariance of the fields under a rotation about the symmetry axis can be compensated by a gauge transformation about a symmetry axis in gauge space. All gauge-invariant quantities are therefore axisymmetric. Our solutions are characterized by a gauge-invariant integer winding number n, and all winding numbers are allowed. We prove that the total gauge-invariant charge of the system (source plus gauge fields) vanishes identically in our solutions for $n \neq 0$, even if the source has net charge. We explicitly solve the equations of motion for a spherical shell of charge. The solution depends on the gauge coupling g, the total charge of the shell Q_S, and the topological number n. We use perturbative methods to obtain the solution in closed form for $\bar{\alpha} \equiv g^2 Q_S/(4\pi) \ll 1$. We show analytically that in this limit the energy $\mathcal{E}_n$ of the system satisfies the bound $\mathcal{E}_n \leq [g^2 Q_S^2/(8\pi a)] \times 1/(2n+1)$, where a is the radius of the shell. Using relaxation methods to find the exact solution to the equations of motion numerically for arbitrary $\bar{\alpha}$, we establish that this bound is satisfied for all g, Q_S, and n.

I. INTRODUCTION AND SUMMARY

In this paper we present a new set of solutions to Yang-Mills equations with axially symmetric external charge sources. We work exclusively in the SU(2) gauge theory.[1] Our solutions for the gauge fields are *not* explicitly symmetric under rotations about the symmetry axis, but their noninvariance can be compensated by a gauge transformation about a symmetry axis in gauge space. All gauge-invariant quantities are therefore axially symmetric. Our solutions carry a topological quantum number, which is the winding number of a vector that describes the orientation of the gauge fields about the symmetry axis in the internal gauge space. We find an infinite tower of solutions, corresponding to all possible values for the integer winding number. Our solution with zero winding number is the explicitly axisymmetric solution discovered several years ago by Sikivie and Weiss.[2]

The methods we use to find these new solutions are due to Jackiw, Jacobs, and Rebbi[3] (JJR), who found a solution to the Yang-Mills equations for a spherically symmetric source. Their solution carries a topological quantum number of unity. As in our case, the topological quantum number of their solution is the winding number of a vector describing the orientation of the gauge fields in the internal space. However, in their case, the vector covers the sphere, and can do so only once, while in our solutions, the corresponding vector covers the circle, and all winding numbers are allowed.

We now summarize the organization and main results of this paper. We begin by establishing some basic properties of the Yang-Mills equations for external charges, in Sec. II. In Sec. III we review the methods developed by JJR for finding perturbative solutions to the Yang-Mills equations for small source strengths. We review their methods in some detail, in order to facilitate our new application. In Sec. IV we use this perturbative approach to find a simple ansatz for the gauge fields $A^{a\mu}$ induced by an axially symmetric charge density. In this ansatz, the coordinate and gauge indices of the gauge fields are factorized into a product of unit vectors in the two spaces. We identify the topological quantum number of these solutions as corresponding to the winding number of a vector describing the gauge index of $A^{a\mu}$. We then verify that the ansatz solves the Yang-Mills equations for sources of *arbitrary* strength. The coordinate and gauge indices of the potential are completely factored out of the Yang-Mills equations, which are reduced to a set of only two (coupled, nonlinear) partial-differential equations, for two axisymmetric scalar functions.

We obtain an analytical expression for the leading behavior of the solution to these equations at large distances from the source. Using this result, we prove that the total gauge-invariant charge of the system (source plus gauge fields) vanishes identically, even if the source carries a net charge. These solutions thus exhibit complete screening of the gauge charge.[4] We also devote particular attention to establishing the fact that the winding number of our solutions is a gauge-invariant quantity.

To illustrate the properties of our solutions, we explicitly solve the equations of motion for a spherical shell of charge, in Sec. V (Ref. 5). The solution depends on the gauge coupling g, the total charge of the shell Q_S, and the winding number n. We use the perturbative analysis of Sec. III to find the solution to the equations of motion in closed form for $\bar{\alpha} \equiv g^2 Q_S/(4\pi) \ll 1$. We find an analytical expression for the energy $\mathcal{E}_n$ of the system in the limit $\bar{\alpha} \ll 1$. Using this result, we derive an upper bound for the energy in this limit, $\mathcal{E}_n \le [g^2 Q_S^2/(8\pi a)] \times 1/(2n+1)$, where a is the radius of the shell. We then use relaxation methods to find the exact solution to the equations of motion numerically for arbitrary $\bar{\alpha}$. We find that the above bound is satisfied for *all* g, Q_S, and n. We also compare our results to the energy of a shell of charge in the ansatz due to JJR (which they computed numerically in Ref. 3), and in the explicitly axisymmetric solution of Sikivie and Weiss [(corresponding to our $n=0$) (Ref. 6)].

Finally, in Sec. VI we draw some conclusions about our results and outline some prospects for future work.

II. YANG-MILLS EQUATIONS FOR EXTERNAL SOURCES

The Lagrangian for an SU(2) gauge field $A^{a\mu}$ coupled to an external source $j^{a\mu}$ is[7]

$$\mathcal{L} = -\frac{1}{4g^2} F^{a\mu\nu} F^a_{\mu\nu} - j^{a\mu} A^a_\mu \ , \tag{1a}$$

where the gauge coupling g is made explicit in the first term in Eq. (1a) by using the following form for the field-strength tensor $F^{a\mu\nu}$:

$$F^{a\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + \epsilon^{acb} A^{b\mu} A^{c\nu} \ . \tag{1b}$$

The Yang-Mills equations are

$$\partial_\mu F^{a\mu\nu} + \epsilon^{abc} A^b_\mu F^{c\mu\nu} = g^2 j^{a\nu} \ . \tag{1c}$$

In explicitly covariant form, Eq. (1c) reads

$$\mathcal{D}^{ab}_\mu F^{b\mu\nu} = g^2 j^{a\nu} \ , \tag{1d}$$

where

$$\mathcal{D}^{ab}_\mu \equiv \delta^{ab} \partial_\mu + \epsilon^{acb} A^c_\mu \ . \tag{1e}$$

Antisymmetry of the tensor $F^{a\mu\nu}$ implies a consistency condition on the external current, which must be covariantly conserved:[7]

$$\mathcal{D}^{ab}_\mu j^{b\mu} = 0 \ . \tag{1f}$$

We are interested in an external source $j^{a\mu}$ that describes a static charge density:

$$j^{a\mu}(\mathbf{r},t) \equiv Q^a(\mathbf{r}) \delta^{\mu 0} \ . \tag{2a}$$

In this case, Eq. (1f) implies that A^{a0} and Q^a must be parallel vectors in the internal gauge space

$$\epsilon^{abc} A^{b0} Q^c(\mathbf{r}) = 0 \ . \tag{2b}$$

We express the charge density $Q^a(\mathbf{r})$ in terms of its magnitude $Q(\mathbf{r})$, and a unit vector $\hat{q}^a(\mathbf{r})$ in the internal gauge space, which may depend on position:

$$Q^a(\mathbf{r}) \equiv \hat{q}(\mathbf{r})^a Q(\mathbf{r}), \quad \hat{q}(\mathbf{r})^a \hat{q}(\mathbf{r})^a \equiv 1 \ . \tag{2c}$$

The Yang-Mills equations are gauge covariant under a simultaneous transformation of the gauge fields and the external charge density Q^a (Ref. 7). Thus, given a solution $A^{a\mu}$ to Eq. (1c) for a charge density $Q^a = \hat{q}^a Q$, a gauge-equivalent solution can be obtained from the transformation

$$\begin{aligned} &\hat{q}^a \sigma^a \to U \hat{q}^a \sigma^a U^{-1} \ , \\ &A^{a\mu}\sigma^a \to U A^{a\mu} \sigma^a U^{-1} + i2U\partial^\mu U^{-1} \ , \\ &U(\mathbf{r},t) \equiv e^{-i\theta^a(\mathbf{r},t)\sigma^a/2} \ , \end{aligned} \tag{3}$$

where σ^a are the Pauli matrices. We must therefore be careful to characterize the external source Q^a in a gauge-invariant manner. In particular, the integral of $Q^a(\mathbf{r})$ is *not* a gauge-invariant measure of the total charge of the source.[8] We would also like to have a gauge-invariant measure of the charge induced by the gauge fields. Since gauge-invariant quantities can be obtained by taking the trace over a product of gauge-covariant terms, a gauge-invariant, conserved current can be immediately obtained from $F^{a\mu\nu}$ and $\hat{q}^a$ (Ref. 8):

$$g^2 J^\mu_T \equiv \partial_\rho (F^{a\rho\mu} \hat{q}^a) \ . \tag{4a}$$

In the case of the static charge density of Eq. (2a), J_T can be further separated into two gauge-invariant terms which are separately conserved:

$$J^\mu_T \equiv J^\mu_S + J^\mu_F, \quad \partial_\mu J^\mu_S = \partial_\mu J^\mu_F = 0 \ , \tag{4b}$$

where

$$J^\mu_S = j^{a\mu} \hat{q}^a = Q(\mathbf{r}) \delta^{\mu 0} \ , \tag{4c}$$

and

$$g^2 J^\mu_F = F^{a\rho\mu} \mathcal{D}^{ab}_\rho \hat{q}^b \ . \tag{4d}$$

J^μ_S clearly characterizes the external source, independently of the gauge fields.[9] We can therefore interpret[8] J^μ_F as the current density induced by the gauge fields and J^μ_T as the current density of the whole system (source plus gauge fields). For later use, we here record the expression for the conserved gauge-invariant total charge Q_S of the external source,

$$Q_S = \int d^3 r Q(\mathbf{r}) \ , \tag{5}$$

and the total charge Q_T of the source-field system:

$$g^2 Q_T = \oint_{r\to\infty} d\mathbf{S}^i F^{ai0}(\mathbf{r}) \hat{q}^a(\mathbf{r}) \ . \tag{6}$$

We note that the gauge-invariant characterization of an axisymmetric charge density is only that its magnitude is independent of the angle about the symmetry axis [e.g., $Q(\mathbf{r}) = Q(\rho, z)$]. The vector $\hat{q}^a(\mathbf{r})$ describing the orientation of the charge in internal-symmetry space need *not* be axisymmetric.

III. PERTURBATIVE SOLUTION TO THE YANG-MILLS EQUATIONS

We now review the techniques developed by JJR (Ref. 3) for solving the Yang-Mills equations perturbatively, for small source strengths $Q(\mathbf{r})$. To do this, it is convenient to write Eq. (1c) in terms of the usual electric and magnetic fields:

$$\nabla\cdot\mathbf{E}^a-\epsilon^{abc}\mathbf{A}^b\cdot\mathbf{E}^c=g^2\hat{q}^aQ(\mathbf{r})\ , \tag{7a}$$

$$\nabla\times\mathbf{B}^a-\epsilon^{abc}\mathbf{A}^b\times\mathbf{B}^c-\epsilon^{abc}A^{b0}\mathbf{E}^c=0\ , \tag{7b}$$

where

$$\mathbf{E}^a\equiv-\nabla A^{a0}+\epsilon^{abc}\mathbf{A}^bA^{c0}\ , \tag{7c}$$

$$\mathbf{B}^a\equiv\nabla\times\mathbf{A}^a-\tfrac{1}{2}\epsilon^{abc}\mathbf{A}^b\times\mathbf{A}^c\ , \tag{7d}$$

and where we have assumed for convenience that we are working in a gauge in which the gauge fields $A^{a\mu}$ are explicitly time independent.

A perturbative solution that correctly accounts for powers of Q in Eqs. (7) may be found for A^{a0} of $O(Q)$, and $\mathbf{A}^a$ of $O(Q^2)$. In this case, the leading contribution from the gauge fields to Eq. (7a) comes from $-\nabla^2(A^{a0})$, which is of the same order as the source, and the leading terms in Eq. (7b) for $\mathbf{A}^a$ are of $O(Q^2)$. If the unit vector $\hat{q}^a(\mathbf{r})$ describing the orientation of the external charge in gauge space is position independent (e.g., $\hat{q}^a=\delta^{a3}$), then this perturbative solution does in fact solve the Yang-Mills equations to leading order, yielding the Abelian Coulomb solution (which, of course, solves the Yang-Mills equations to all orders in Q; see Mandula, Ref. 1).

However, this is not the most general possibility. To find a more general perturbative solution, we have to be sure to distinguish between gauge-equivalent solutions. Following JJR, we do this by first working in a "gauge frame" in which $\hat{q}^a$ is a constant:

$$\hat{q}^a(\mathbf{r})\equiv\delta^{a3}\ . \tag{8a}$$

A more general perturbative solution, that correctly accounts for powers of Q in Eqs. (7) to lowest order, is

$$A^{a0}=O(Q)\delta^{a3}\ , \tag{8b}$$

$$\mathbf{A}^a\sigma^a=-i2U^{-1}\nabla U+O(Q^2),\quad U\equiv e^{-i\theta^a(\mathbf{r})\sigma^a/2}\ , \tag{8c}$$

since the pure gauge term in Eq. (8c) makes a vanishing contribution to the magnetic field $\mathbf{B}^a$, and thus only contributes terms to Eqs. (7) that are, at most, of the same order as the leading terms that we have already identified. The perturbative solution of Eqs. (8) can yield solutions of a truly non-Abelian character.

However, not all choices for the pure gauge term in Eq. (8c) result in a consistent perturbative solution. To further analyze this situation, it is convenient to first transform away the pure gauge term in Eq. (8c), passing to a new "gauge frame" in which the charge vector and gauge fields assume the forms

$$\hat{q}'^a(\mathbf{r})\sigma^a\equiv U\sigma^3U^{-1}\ , \tag{9a}$$

$$A'^{a0}\sigma^a\equiv UA^{30}\sigma^3U^{-1}=O(Q)\ , \tag{9b}$$

$$\mathbf{A}'^a\sigma^a\equiv U\,\mathbf{A}^a\sigma^aU^{-1}-2iU\nabla U^{-1}=O(Q^2)\ , \tag{9c}$$

where the prime serves to identify quantities that are obtained by the gauge transformation from Eqs. (8). It should be emphasized that this perturbative solution is, in general, *not* equivalent to the Coulomb solution described below Eqs. (7), despite the fact that the potentials in Eqs. (9) are of the same order in Q. This should be clear from the presence of the pure-gauge term in Eq. (8c), or from the fact that the direction of the charge density in the internal-symmetry space acquires a nontrivial dependence on coordinates in the gauge frame of Eqs. (9).

In the gauge frame of Eqs. (9), the electric and magnetic fields to lowest order in Q are given by

$$\mathbf{E}'^a=-\nabla A'^{a0},\quad \mathbf{B}'^a=\nabla\times\mathbf{A}'^a\ . \tag{10a}$$

Gauss's law, Eq. (7a), then reduces to

$$-\nabla^2(A'^{a0})=g^2\hat{q}'^a(\mathbf{r})Q(\mathbf{r})\ , \tag{10b}$$

and Ampere's law, Eq. (7b), reduces to

$$\nabla\times\mathbf{B}'^a=\epsilon^{abc}A'^{b0}\mathbf{E}'^c\ . \tag{10c}$$

Note that Ampere's law is only consistent if A'^{a0} satisfies Eq. (2b) for covariant conservation of the external current, as follows from the fact that the right-hand side of Eq. (10c) must be divergenceless. A'^{a0} must therefore take the form[10]

$$A'^{a0}(\mathbf{r})\equiv\hat{q}'^a(\mathbf{r})\Phi(\mathbf{r})\ . \tag{10d}$$

We now see that the perturbative solution of Eqs. (9) must meet another consistency condition, implied by Gauss's law to leading order in Q. Expanding the Laplacian in Eq. (10b), we find

$$\hat{q}'^a\nabla^2\Phi+2\nabla\hat{q}'^a\cdot\nabla\Phi+\Phi\nabla^2\hat{q}'^a=-g^2\hat{q}'^aQ(\mathbf{r})\ . \tag{11}$$

Since the two sides of Eq. (11) must be parallel in gauge space, we see that the derivative operators acting on $\hat{q}'^a(\mathbf{r})$ must result in vectors parallel to $\hat{q}'^a$. Clearly, not all choices for $\hat{q}'^a$ will meet this condition. JJR found an ansatz which meets this consistency condition for a spherically symmetric external charge, $Q(\mathbf{r})=Q(r)$. Their ansatz is $\hat{q}'^a(\mathbf{r})=\hat{\mathbf{r}}^a$, $\Phi(\mathbf{r})=\Phi(r)$. The structure of $\mathbf{A}^a$ in coordinate and gauge space is then determined by Eq. (10c). Their solution carries a topological quantum number of unity, which is the winding number of the vector $\hat{\mathbf{r}}^a$ over the sphere in gauge space. It turns out that potentials of the same structure as this perturbative solution actually satisfy the Yang-Mills equations for *arbitrary* Q, as verified by direct substitution into the full equations [Eqs. (7)].

IV. SOLUTIONS FOR AXISYMMETRIC SOURCES

Following the perturbative methods due to JJR described in the previous section, we have found a new set of solutions for axially symmetric external charges:

$$Q^a(\mathbf{r})=\hat{q}^a(\mathbf{r})Q(\rho,z)\ , \tag{12}$$

where, from here on, we work exclusively in the gauge frame of Eqs. (9), dropping the prime.

An ansatz for $\hat{q}^a$ and A^{a0} which meets the consistency conditions implied by current conservation [Eq. (2b)], and the lowest-order Gauss's law [Eqs. (10b) or (11)], for this axisymmetric source is

$$\hat{q}^a(\mathbf{r})\equiv\hat{q}^a_n(\phi)\equiv\cos(n\phi)\delta^{a1}+\sin(n\phi)\delta^{a2}\ , \tag{13a}$$

$$A^{a0}(\mathbf{r})\equiv\hat{q}^a_n(\phi)\Phi_n(\rho,z)\ , \tag{13b}$$

where n is an integer, $n=0,\pm1,\pm2,\ldots$. We note that we have chosen the coefficient function Φ_n of the symmetry vector $\hat{q}^a$ in A^{a0} to be independent of the azimuthal angle ϕ, in order that gauge-invariant quantities constructed from the potentials (such as the energy density) will be axially symmetric. Requiring $\hat{q}^a$ to be a function only of ϕ then eliminates the second term in the consistency condition of Eq. (11), and the final form for $\hat{q}^a$ then emerges from the requirement that $\nabla^2\hat{q}^a(\phi)\propto\hat{q}^a(\phi)$ [subject to the condition that $\hat{q}^a(\phi)$ be a single-valued function]. The index n is clearly a winding number, the number of times the circle spanned by the vector $\hat{q}^a_n(\phi)$ is covered, when the azimuth $\phi\in[0,2\pi]$ is covered once.

In general, the winding number (or Kronecker index) $\omega[v]$ of a two-component unit vector $v^i(\phi)$, $i=1,2$ ($v^iv^i\equiv 1$), can be calculated from[11]

$$\omega[v]=\frac{1}{2\pi}\int_0^{2\pi}d\phi\,\epsilon^{ij}v^i\frac{dv^j}{d\phi}\ , \tag{14}$$

where $\epsilon^{12}=-\epsilon^{21}=1$. We verify that $\omega[\hat{q}_n]=n$ [note that $\hat{q}^i$, $i=1,2$ in Eq. (14) refers to the two nonvanishing components of the charge vector in the gauge of Eqs. (13)]. However, in order for the winding number of our solution to be a meaningful quantity, it must be gauge invariant. We will prove that this is, in fact, the case, after we develop some further properties of our ansatz.

To get the form of the three-vector potential $\mathbf{A}^a$ to lowest order in Q, we solve Eq. (10c), using Eq. (10a) for $\mathbf{E}^a$ and $\mathbf{B}^a$. Choosing a divergenceless form for $\mathbf{A}^a$, we have [to $O(Q^2)$] $\nabla^2\mathbf{A}^a=n\rho^{-1}\Phi_n^2(\rho,z)\delta^{a3}\hat{\boldsymbol{\phi}}$, to which the solution is of the form

$$\mathbf{A}^a(\mathbf{r})=\delta^{a3}A_n(\rho,z)\hat{\boldsymbol{\phi}}\ . \tag{15}$$

It turns out that an ansatz having the structure of Eqs. (13) and (15) actually solves the full Yang-Mills equations [Eqs. (7)] for axisymmetric sources of *arbitrary* strength, as can be readily verified by direct substitution. For a source of arbitrary strength, the electric and magnetic fields [Eqs. (7a) and (7b)] are given by

$$\mathbf{E}^a_n=-\hat{q}^a_n\nabla\Phi_n(\rho,z)-\hat{\phi}^a_n\Phi_n\left[A_n(\rho,z)-\frac{n}{\rho}\right]\hat{\boldsymbol{\phi}}\ , \tag{16a}$$

$$\mathbf{B}^a_n=\nabla\times[A_n(\rho,z)\hat{\boldsymbol{\phi}}]\delta^{a3} \tag{16b}$$

[where $\hat{\phi}^a_n\equiv-\sin(n\phi)\delta^{a1}+\cos(n\phi)\delta^{a2}$], while Gauss's law [Eq. (7c)] reduces to

$$-\nabla^2\Phi_n(\rho,z)+\left[A_n-\frac{n}{\rho}\right]^2\Phi_n=g^2Q(\rho,z)\ , \tag{16c}$$

and Ampere's law [Eq. (7d)] reduces to

$$-\nabla^2A_n(\rho,z)+\frac{1}{\rho^2}A_n-\Phi_n^2\left[A_n-\frac{n}{\rho}\right]=0\ . \tag{16d}$$

We note that our solution for $n=0$ is the explicitly axisymmetric solution discovered by Sikivie and Weiss (their so-called "magnetic dipole solution").[2] We also note that

$$\begin{aligned}\Phi_{-n}(\rho,z)&=\Phi_n(\rho,z)\quad(n\neq0)\ ,\\ A_{-n}(\rho,z)&=-A_n(\rho,z)\quad(n\neq0)\ .\end{aligned} \tag{17}$$

This implies, in particular, that the total energy $\mathcal{E}_n$ of the system,

$$\mathcal{E}_n=\frac{1}{2g^2}\int d^3r(\mathbf{E}^{a2}_n+\mathbf{B}^{a2}_n)\ , \tag{18}$$

is invariant under $n\to-n$.

The ansatz of Eqs. (13) and (15) thus factorizes the coordinate and gauge indices of the gauge fields into simple unit vectors in the two spaces, which then factor out of the equations of motion. Although gauge-variant quantities such as $A^{a\mu}_n$, $\mathbf{E}^a_n$, and $\mathbf{B}^a_n$ are not explicitly axisymmetric, their noninvariance under a rotation about the symmetry axis $\hat{\mathbf{z}}$ in coordinate space can be compensated by a gauge transformation about the symmetry axis δ^{a3} in gauge space.[12] For example,

$$\begin{aligned}A^{a0}(\phi)\sigma^a&=U_\Delta A^{a0}(\phi-\Delta)\sigma^aU_\Delta^{-1}\ ,\\ \mathbf{A}^{ai}(\phi)\sigma^a&=U_\Delta R^{ij}_\Delta\mathbf{A}^{aj}(\phi-\Delta)\sigma^aU_\Delta^{-1}\ ,\end{aligned} \tag{19a}$$

where

$$U_\Delta\equiv e^{-i\Delta\sigma^3/2}\ , \tag{19b}$$

and where R^{ij}_Δ is the rotation matrix for a rotation by an angle Δ about the three-axis in coordinate space. All gauge-invariant quantities, such as the energy density, are therefore axially symmetric.

For a localized source, the leading behavior of Φ_n and A_n in the limit $r\to\infty$ can be obtained analytically. In order for the total energy of the system to be finite in the infrared, Φ_n and A_n must both tend to zero in this limit, so we attempt to solve Eqs. (16c) and (16d) to leading order in $1/r$ by keeping only terms linear in Φ_n and A_n. The equations then become decoupled, and the leading behavior of the solution is readily obtained in spherical coordinates:

$$\Phi_n(r\to\infty,\theta)\sim\frac{\sin^{|n|}\theta}{r^{|n|+1}}\ ,\quad A_n(r\to\infty,\theta)\sim\frac{\sin\theta}{r^2}\ . \tag{20}$$

We then verify that the nonlinear terms in Eqs. (16c) and (16d) can truly be neglected to leading order in $1/r$, which justifies our derivation of Eq. (20).

We can use Eq. (20) to establish a very interesting property of the total charge Q_T [Eq. (6)] of the source-field system in these solutions. In general, we expect Q_T to be a function of the source charge $Q(\rho,z)$ (and of the winding number n). However, since $F^{ai0}(r\to\infty)$ for our solutions with $n\neq0$ goes to zero faster than $1/r^2$, we find that Q_T vanishes *identically:*

$$Q_T\{n,Q(\rho,z)\}=0\quad\text{for all } Q(\rho,z)\quad(n\neq0)\ , \tag{21}$$

independent of the charge density $Q(\rho,z)$ of the source. In particular, we note that if the source carries net charge (i.e., $Q_S\neq0$), it gets completely screened by the charge induced by the gauge fields (i.e., $Q_F=-Q_S$), and the total charge of the source-field system is zero.[4] We note that Eq. (20) also shows that for $n=0$ a charged source is *not* completely screened. Explicit numerical calculations show that the source is partially screened, with $Q_T\{n=0\}\leq Q_S$ (Ref. 6).

We can also use the result of Eq. (20) to recast Eq. (14) for the winding number $\omega[\hat{q}_n]$ into a manifestly gauge-invariant form. Working in the gauge frame of Eqs. (13) and (15), we have

$$\omega[\hat{q}] \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \epsilon^{ij}\hat{q}^i \frac{d\hat{q}^j}{d\phi}$$
$$= \frac{1}{2\pi}\oint d\phi\, \rho\epsilon^{3bc}\hat{q}^b \nabla_\varphi \hat{q}^c$$
$$= -\frac{1}{2\pi}\oint_{\rho\to\infty} d\phi\, \rho\epsilon^{3bc}\hat{q}^b \mathcal{D}_\varphi^{cd}\hat{q}^d , \tag{22}$$

where, in the first two lines above, we have used the fact that the two- and three-dimensional Levi-Civita tensors are related by $\epsilon^{ij}=\epsilon^{3ij}$ (we then replaced the indices $i,j=1,2$ by $b,c=1,2,3$), and where $\nabla_\varphi \equiv \hat{\boldsymbol{\phi}}\cdot\nabla$. To arrive at Eq. (22), we then used the fact that $\mathbf{A}^a$ vanishes at infinity faster that $1/\rho$, in order to introduce the covariant derivative in the third line (note the limit on the integral).

We recognize the expression in the integrand in Eq. (22) as the component along the third direction in gauge space of the commutator of $\hat{q}$ and $\mathcal{D}_\varphi\hat{q}$. We define

$$C^a \equiv \epsilon^{abc}\hat{q}^b \mathcal{D}_\varphi^{cd}\hat{q}^d, \quad \hat{C}^a \equiv \frac{C^a}{\sqrt{\mathrm{Tr}C^2}} , \tag{23a}$$

and we note that, in the gauge of Eqs. (13) and (15), $\hat{C}^a \to -\delta^{a3}$ as $\rho\to\infty$. The expression in Eq. (22) is therefore equivalent to the functional

$$\Omega[\hat{q}] \equiv \frac{1}{2\pi}\oint_{\rho\to\infty} d\phi\, \rho\, \mathrm{Tr}(C\hat{C}) , \tag{23b}$$

which, as we have shown, gives the winding number of $\hat{q}$:

$$\Omega[\hat{\mathbf{q}}_n] = n . \tag{23c}$$

Finally, we observe that since $\hat{q}$ and $\mathcal{D}_\mu\hat{q}$ both transform covariantly, so does their commutator C. The functional $\Omega[\hat{q}]$ is therefore manifestly gauge invariant.

We have therefore established the fact that the winding number n is a gauge-invariant property of our solutions. We conclude this section by noting that the gauge transformation that connects the gauge of Eqs. (13) and (15), to the gauge in which $\hat{q}^a=\delta^{a3}$ [cf. Eqs. (8) and (9)] is singular, as it must be,[3] since the winding number of Eq. (14) vanishes identically in the latter gauge, $\omega[\delta^{a3}]=0$. However, the singular term acquired by $\mathbf{A}^a$ under this gauge transformation results in $\Omega[\delta^{a3}]=n$ (Ref. 13).

V. APPLICATION TO A SPHERICAL SHELL OF CHARGE

To illustrate the properties of our set of topologically nontrivial axisymmetric solutions, we now solve in detail the equations of motion for a system containing a spherical shell of charge

$$Q(\rho,z) = Q(r) = \frac{Q_S}{4\pi a^2}\delta(r-a) , \tag{24}$$

where Q_S is the total, gauge-invariant charge of the shell [cf. Eq. (5)].

We note that our axially symmetric solutions obviously break the spherical symmetry of this source.[5] Of course, the (nonspherical) energy density of this source is independent of the direction along which the spherical symmetry is broken. This implies that the system can have rotations about an axis perpendicular to the symmetry axis, just as in deformed nuclei.

Using the perturbative methods described in Sec. III, we can obtain the solution for this system in closed form, for a weak source. Since the charge density enters into the equations of motion through the product $g^2Q(r)$ [cf. Eq. (16c)], the actual expansion parameter for the perturbative solution is[14]

$$\bar{\alpha} \equiv Q_S \frac{g^2}{4\pi} . \tag{25}$$

Following JJR, we first note that we can actually obtain a useful bound on the total energy $\mathcal{E}_n$ of the system for small $\bar{\alpha}$, without knowing the actual perturbative solution. To leading order in $\bar{\alpha}$, we have

$$\mathcal{E}_n = \mathcal{E}_n^{[2]} + O(\bar{\alpha}^4/g^2) , \tag{26a}$$

where $\mathcal{E}_n^{[2]}$ is the energy to $O(\bar{\alpha}^2)$ (Ref. 15):

$$\mathcal{E}_n^{[2]} = \frac{g^2}{8\pi}\int d^3r\, d^3r'\, \hat{q}_n^a(\phi)\hat{q}_n^a(\phi')\frac{Q(r)Q(r')}{|\mathbf{r}-\mathbf{r}'|} . \tag{26b}$$

We note that $\mathbf{B}^a = O(\bar{\alpha}^2)$ only contributes to the next-to-leading term in $\mathcal{E}_n$. Equation (26b) follows by expressing the solution to Eq. (10b) (which is just Poisson's equation) for A^{a0} to $O(\bar{\alpha})$ in integral form, and then using Eq. (10a) for $\mathbf{E}^a$ in Eq. (18) for the energy. Comparing Eq. (26b) with the energy $\mathcal{E}_{\mathrm{coul}}$ of the Abelian Coulomb solution

$$\mathcal{E}_{\mathrm{Coul}} = \frac{g^2}{8\pi}\int d^3r\, d^3r'\, \frac{Q(r)Q(r')}{|\mathbf{r}-\mathbf{r}'|} , \tag{27}$$

shows that $\mathcal{E}_n^{[2]} < \mathcal{E}_{\mathrm{coul}}$ for $n\neq 0$ [since $\hat{q}_n^a(\phi)\hat{q}_n^a(\phi') \le 1$]. This also shows that $\mathcal{E}_{(n=0)}^{[2]} = \mathcal{E}_{\mathrm{Coul}}$. We note that this analysis applies to *any* positive-definite, axisymmetric charge density $Q(\rho,z)$.

We obtain considerably more information about the n dependence of the energy by explicitly solving Eq. (10b) for Φ_n to leading order in $\bar{\alpha}$. Equation (10b) is just Poisson's equation, and is easily solved for our problem in integral form. The result is most conveniently expressed in spherical coordinates. We find

$$\Phi_n(r,\theta) = \bar{\alpha}\sum_{l=n}^{\infty} \frac{r_<^l}{r_>^{l+1}} C_l^n P_l^n(\theta) + O(\bar{\alpha}^3) , \tag{28a}$$

where $r_< \equiv \min(r,a)$, $r_> \equiv \max(r,a)$, $P_l^n(\theta)$ is the associated Legendre polynomial, and where

$$C_l^n \equiv \frac{(l-n)!}{2(l+n)!}\int_0^\pi d\theta\, \sin\theta P_l^n(\theta) . \tag{28b}$$

We note that $C_l^n=0$ if $l+n=$odd. We have also obtained a closed expression for A_n to leading order in $\bar{\alpha}$, but it is not particularly illuminating, and we do not need it to get the $O(\bar{\alpha}^2)$ energy.

We now make use of a very convenient expression for the energy, due to JJR, which is valid for any static solution to the Yang-Mills equations, provided that $\mathbf{E}^a$ and $\mathbf{B}^a$ fall off faster than $r^{-3/2}$ at infinity[3]:

$$\mathcal{E}=\int d^3r\, Q^a(\mathbf{r})\mathbf{r}\cdot\mathbf{E}^a \ . \tag{29}$$

We then obtain the following (exact) expression for the energy of our solutions for the source of Eq. (24):

$$\mathcal{E}_n=-\frac{aQ_S}{2}\int_0^\pi d\theta\sin\theta\frac{\partial\Phi_n}{\partial r}(r=a,\theta)\ . \tag{30a}$$

Since the source is a delta function at $r=a$, the radial derivative of Φ has a discontinuity there [as follows by integrating both sides of Eq. (16c) from $r=a-\epsilon$ to $r=a+\epsilon$, for $\epsilon\to 0^+$]. Thus, the energy in this case becomes

$$\mathcal{E}_n=-\frac{aQ_S}{4}\int_0^\pi d\theta\sin\theta\left[\frac{\partial\Phi_n}{\partial r}(a^+,\theta)+\frac{\partial\Phi_n}{\partial r}(a^-,\theta)\right]\ . \tag{30b}$$

We now do the integration in Eq. (30b) using Eq. (28) for Φ_n to $O(\bar{\alpha})$, to obtain an analytical expression for the $O(\bar{\alpha}^2)$ energy $\mathcal{E}_n^{[2]}$:

$$\mathcal{E}_n^{[2]}=\frac{\bar{\alpha}Q_S}{4a}\sum_{l=n}^{\infty}C_l^n\int_0^\pi d\theta\sin\theta P_l^n(\theta)$$
$$=\mathcal{E}_{\rm coul}\times\sum_{l=n}^{\infty}\frac{(l+n)!}{(l-n)!}(C_l^n)^2\ , \tag{31}$$

where we used Eq. (28b) to rewrite the integral in the first line above in terms of C_l^n, and where for convenience we have expressed the energy in terms of the Coulomb energy $\mathcal{E}_{\rm Coul}=\bar{\alpha}Q_S/(2a)$.

We have explicitly summed the series in Eq. (31) to compute the numerical value of $\mathcal{E}_n^{[2]}$. However, we first note that we can turn Eq. (31) into a simple and very useful upper bound for $\mathcal{E}_n^{[2]}$. We do this by using the fact that the Legendre polynomials P_l^n form a complete set with respect to l, for any n. We can therefore write

$$1=\sum_{l=n}^{\infty}O_l^nP_l^n(\theta)\ , \tag{32a}$$

where we find that the coefficients O_l^n in this Legendre series are related to C_l^n by [cf. Eq. (28b)]

$$O_l^n=(2l+1)C_l^n\ . \tag{32b}$$

We then square Eq. (32a) and integrate with respect to θ to find

$$1=\sum_{l=n}^{\infty}(2l+1)\frac{(l+n)!}{(l-n)!}(C_l^n)^2\ , \tag{32c}$$

using Eq. (32b) to express O_l^n in terms of C_l^n.

Then, inserting $1=(2l+1)/(2l+1)$ into the summand in Eq. (31), using the fact that $1/(2l+1)\le 1/(2n+1)$, and using Eq. (32c), we arrive at the following upper bound for $\mathcal{E}_n^{[2]}$:

$$\mathcal{E}_n^{[2]}\le\mathcal{E}_{\rm Coul}\frac{1}{2n+1} \tag{33}$$

[the equality in Eq. (33) only applies to the $n=0$ solution; recall that we have already shown that $\mathcal{E}_{(n=0)}^{[2]}=\mathcal{E}_{\rm Coul}$]. We have thus proven, at least in the perturbative regime $\bar{\alpha}\ll 1$, that the energy of a shell of charge in our axisymmetric solutions tends to zero as the topological quantum number n of the solution tends to infinity.

We have explicitly summed the series in the exact expression for $\mathcal{E}_n^{[2]}$, Eq. (31), to obtain the actual value of the $O(\bar{\alpha}^2)$ energy, for a few values of n:

$$\mathcal{E}_n^{[2]}=\mathcal{E}_{\rm coul}\times\begin{cases}0.317, & n=1,\\ 0.126, & n=3,\\ 0.039, & n=10.\end{cases} \tag{34}$$

It is interesting to compare these results with the energy of a spherical shell of charge in the spherically symmetric solution due to JJR. As described in Sec. III, their ansatz takes the form

$$Q^a(\mathbf{r})=\hat{\mathbf{r}}^aQ(r),\quad A^{a0}(\mathbf{r})\equiv\hat{\mathbf{r}}^a\Phi_{\rm JJR}(r)\ . \tag{35a}$$

Their solution is therefore characterized by the unit winding number of the vector $\hat{\mathbf{r}}^a$ over the sphere in gauge space. We have solved Eq. (10b) analytically for the perturbative limit of $\Phi_{\rm JJR}$ for a shell of charge [Eq. (24)]. We find

$$\Phi_{\rm JJR}(r)=\frac{\bar{\alpha}}{3}\frac{r_<}{r_>^2}+O(\bar{\alpha}^3)\ . \tag{35b}$$

Using Eq. (10a) for $\mathbf{E}^a$ to $O(\bar{\alpha})$, we then derive the $O(\bar{\alpha}^2)$ energy $\mathcal{E}_{\rm JJR}^{[2]}$ of their solution for the charge shell:

$$\mathcal{E}_{\rm JJR}^{[2]}=\tfrac{1}{3}\mathcal{E}_{\rm Coul}\ . \tag{35c}$$

Comparison with Eq. (34) shows, in particular, that our solution with unit winding number has lower energy than the spherically symmetric solution of JJR, at least in the perturbative regime.

To extend the results of our perturbative analysis, we have used relaxation methods to solve the coupled, nonlinear, partial-differential equations of motion [Eqs. (16c) and (16d)] numerically, for arbitrary $\bar{\alpha}$. The use of relaxation methods to solve similar problems in classical and semiclassical gauge field theories has been thoroughly described in an excellent review paper by Adler and Piran.[16] We give here a brief summary of these methods, as we have applied them to our problem.

Relaxation methods proceed by discretizing the differential equations [Eqs. (16c) and (16d)] on a finite mesh, and solving the resulting algebraic equations iteratively. The most convenient discretization links each point on the mesh only to its nearest neighbors. An initial guess for the potentials at each lattice point is made (in our case, the iterative method converges even with a relatively crude initial guess), and a sequence of estimates of the potentials is then generated by repeatedly sweeping through the lattice. On each sweep, the updated potentials are obtained by solving the discretized equation for the potential at a given node, in terms of the current values of the potentials at the neighboring nodes. The nonlinear coupling between Eqs. (16c) and (16d) is handled by sweeping through the entire lattice for one potential while the other potential is held fixed. Convergence of the iterative method can be accelerated by systematically overestimating the change in the values of the potential from one iteration to the next ("successive overrelaxation"). Once convergence has been obtained on a lattice with a given cell size, the cell size is halved, and the iterative procedure is repeated on the new lattice, using the values of the potentials on the cruder mesh as initial estimates. The mesh size is repeatedly halved until the quantities of interest (in our case, the energy), converge to the desired accuracy.

In our case, we found that it was most convenient to solve Eqs. (16c) and (16d) in spherical coordinates, as it turned out that the energy could be computed to reasonable accuracy ($\approx 1\%$ or better) using only a small, *fixed* number of points along the θ axis. In this way, our problem is effectively reduced to a one-dimensional system, since the computing time only doubles with each halving of the $[r]$ lattice cell size (as compared to the quadrupling of the computing time for a two-dimensional lattice that must be halved along both dimensions in order to converge to the true solution to the desired accuracy). Furthermore, the source term in Eq. (16c) is easily handled in spherical coordinates, by simply imposing a discontinuity condition on the radial derivative at $r=a$:

$$\frac{\partial \Phi_n}{\partial r}(r=a^+,\theta)-\frac{\partial \Phi_n}{\partial r}(r=a^-,\theta)=-\frac{\bar{\alpha}}{a^2} \tag{36}$$

for all θ. Our problem then reduces to the source-free equations for all $r\neq a$.

We typically solved these equations by starting on a mesh with 20×20 points in $r\times\theta$, and iterating up to a mesh with 320×20 points, obtaining the energy in most cases to better than 1% [we note in this connection that we found that the accuracy of the numerical estimate of the energy on a lattice of a given size could be considerably improved by taking the average value of Eqs. (18) and (29)]. Only 8 min of CPU time on a VAX 8820 computer was required.

We plot our results for the energy as a function[14] of $\bar{\alpha}$ in Fig. 1, for our solutions with topological quantum numbers $n=0,1,3,10$ [we recall that $\mathcal{E}_{-n}=\mathcal{E}_n$; see Eq. (17)]. We have also included the Coulomb parabola in Fig. 1, for comparison.

The curves in Fig. 1 clearly show that the decrease in energy with increasing n, which we found analytically in the perturbative regime, holds for all values of $\bar{\alpha}$. In fact, our detailed results for large $\bar{\alpha}$ show that the bound of Eq. (33) on the $O(\bar{\alpha}^2)$ energy is actually satisfied for all values of the coupling

$$\mathcal{E}_n \leq \frac{g^2 Q_S^2}{8\pi a}\frac{1}{2n+1} . \tag{37}$$

It is also interesting to note that our results for $n=0$ (corresponding to the explicitly axisymmetric solution of Sikivie and Weiss[2]), exhibit a critical coupling at $\bar{\alpha}=\frac{3}{2}$. Below this coupling, $\mathcal{E}_{(n=0)}=\mathcal{E}_{\text{Coul}}$ [compare to the perturbative result given below Eq. (33)]. We have actually been able to derive the value of this critical coupling in the $n=0$ ansatz by analytical methods.[17] We also note that our numerical results for $\mathcal{E}_{(n=0)}$ agree with the charge-shell calculation in Ref. 6.

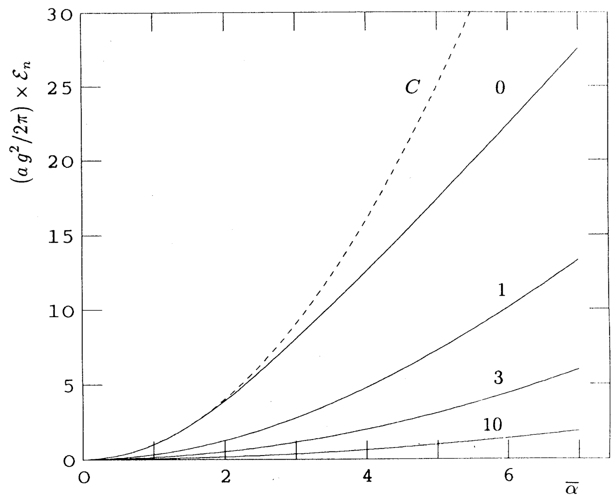

FIG. 1. Energy $\mathcal{E}_n$ in units of $2\pi/(ag^2)$ of a spherical shell of charge of radius a and total charge Q_S vs $\bar{\alpha}\equiv Q_S g^2/(4\pi)$. g is the gauge coupling (Ref. 14). The curves show the energy of the shell in our axisymmetric solutions with topological numbers $n=0,1,3,10$. The curve labeled by C is the Coulomb energy of the shell, which in these units is given by $\bar{\alpha}^2$.

We have also repeated the numerical calculations of JJR for the charge shell in their spherically symmetric solution, using the methods they described in Ref. 3. We find that our analytical result, that $\mathcal{E}_{\text{JJR}}>\mathcal{E}_{(n=1)}$ in the perturbative regime, also holds for arbitrary $\bar{\alpha}$.

VI. CONCLUSIONS AND OUTLOOK

We have discovered that our topologically nontrivial axisymmetric solutions to the Yang-Mills equations possess two remarkable properties. First, we showed that the total charge of the system (source plus gauge fields) vanishes *identically* for our solutions with topological number $n\neq 0$, even if the source carries net charge. The charge carried by the gauge fields thus completely screens the charge carried by the source. Second, we showed that the energy of a spherical shell of charge in our solutions tends to zero as n tends to infinity.

We note that similar properties have been observed in other solutions to the Yang-Mills equations.[4] In particular, there exists a family of (nontopological) spherically symmetric solutions (originally discovered by Sikivie and Weiss), which are labeled by a continuous parameter, whose energy can be made arbitrarily small by taking an appropriate limit of the parameter labeling the solutions.[18]

However, while the energy of this family of spherically symmetric solutions and the energy of our axisymmetric solutions can both be made arbitrarily small, we expect there to be an important difference between the two sets of solutions having to do with their stability. Since the spherically symmetric family depends on a continuous parameter, these solutions are clearly unstable with respect to perturbations to the gauge fields that change the parameter. On the other, we expect that our axisymmetric solutions are stable, due to their topologically nontrivial properties.

Jackiw and Rossi performed a thorough analysis of the stability of the topologically charged solution due to JJR, and found it to be gyroscopically stable, at least in the perturbative regime.[19] We are currently investigating the possibility of generalizing their methods to study the stability of our axisymmetric solutions.

In addition to the stability problem, other avenues of investigation which we are currently undertaking include a search for possible bifurcating axisymmetric solutions (JJR found bifurcating solutions in their spherically symmetric ansatz in Ref. 3), and for possible nontrivial embeddings of our SU(2) solutions in SU(3).

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[1]As is well known, solutions for a theory with a given symmetry group can be trivially embedded in a theory with a "higher" symmetry, having the given group as a subgroup; see, for example, J. E. Mandula, Phys. Rev. D **14**, 3497 (1976); A. Actor, Rev. Mod. Phys. **51**, 461 (1979).

[2]P. Sikivie and N. Weiss, Phys. Rev. Lett. **40**, 1411 (1978); Phys. Rev. D **18**, 3809 (1978).

[3]R. Jackiw, L. Jacobs, and C. Rebbi, Phys. Rev. D **20**, 474 (1979).

[4]Complete screening solutions were first discovered by Sikivie and Weiss (Ref. 2), and by Jackiw, Jacobs, and Rebbi (Ref. 3). However, we note that the explicitly axisymmetric solution of Sikivie and Weiss (corresponding to our solution with $n=0$), only partially screens a charged source (see Ref. 6).

[5]Several authors have considered solutions that break the symmetry of a spherically symmetric external source. For example, the fact that Coulomb solution for a point charge becomes unstable (above a critical coupling) to axisymmetric perturbations in the gauge fields was discovered by J. E. Mandula, Phys. Lett. **67B**, 175 (1977) [see also Mandula (Ref. 1)]. Furthermore, the explicitly axisymmetric solution due to Sikivie and Weiss (Ref. 2), which corresponds to our solution with $n=0$, was solved numerically for spherically symmetric charges in Ref. 6.

[6]The explicitly axisymmetric solution of Sikivie and Weiss (Ref. 2) was solved numerically for a point charge by J. E. Mandula, D. I. Meiron, and S. A. Orszag, Phys. Lett. **124B**, 365 (1983), and for a spherical shell of charge by L. J. Carson, R. Goldflam, and L. Wilets, Phys. Rev. D **28**, 385 (1983). See also L. J. Carson, *ibid.* **29**, 2355 (1984).

[7]See, e.g., Mandula (Ref. 1).

[8]C. H. Lai and C. H. Oh, Phys. Rev. D **29**, 1805 (1984); **33**, 1825 (1986), and references therein.

[9]Moreover, in the Abelian approximation for the gauge fields in the Yang-Mills equations for a general time-dependent external current density $j^{a\mu}(\mathbf{r},t)$, J_3^μ reduces to the Abelian conserved current of the source.

[10]Strictly speaking, Eq. (2b) only implies that $A^{a0}\propto\hat{q}^a$ in the region where the charge density $Q(\mathbf{r})$ of the external source is nonvanishing. Therefore, Eq. (10d) does not lead to the most general solution. However, it *does* lead to a large class of valid solutions. In this connection, we note that it might be more accurate to interpret $\hat{q}^a(\mathbf{r})$ as the direction in gauge space of the scalar potential A^{a0}, rather than of the external charge density.

[11]J. Hadamard, note in J. Tannery, *Introduction à la Théorie des Fonctions* (Herman, Paris, 1910), pp. 437–477; C. Allendoerfer, Am. J. Math. **62**, 243 (1940). See also J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. **16**, 433 (1975).

[12]Symmetry transformations of this kind are discussed by R. Jackiw, in *Current Algebra and Anomalies,* edited by S. B. Treiman, R. Jackiw, Bruno Zumino, and E. Witten (World Scientific, Singapore, 1985), and in references therein.

[13]For the solution with winding number n, the gauge transformation that connects the gauge of Eqs. (13) and (15) to the gauge in which $\hat{q}^a=\delta^{a3}$ [cf. Eqs. (8) and (9)] is $U=e^{-i(\pi/4)\sigma^a\hat{\phi}_n^a}$, where $\hat{\phi}_n^a\equiv-\sin(n\phi)\delta^{a1}+\cos(n\phi)\delta^{a2}$. Under this transformation, $\mathbf{A}^a$ acquires the singular term $n\rho^{-1}(\hat{\rho}_n^a+\delta^{a3})\hat{\boldsymbol{\phi}}$, where $\hat{\rho}_n^a=\cos(n\phi)\delta^{a1}+\sin(n\phi)\delta^{a2}$.

[14]We note that, in units of $1/g^2$, the energy of the charge shell depends on its total charge Q_S and the gauge coupling g only through the combination $g^2Q_S\propto\bar{\alpha}$.

[15]Equation (26b) is the $O(\bar{\alpha}^2)$ term in the following exact expression for the energy $\mathscr{E}_n$ for arbitrary $\bar{\alpha}$:

$$\mathscr{E}_n=\frac{g^2}{8\pi}\sum_{\lambda=0,\pm}\int d^3r\,d^3r'\hat{q}_n^\lambda(\phi)Q(r)\times\left\langle\mathbf{r}\left|\frac{1}{-\nabla^2+A_n^2-i\lambda(A_n\nabla_\varphi+\nabla_\varphi A_n)}\right|\mathbf{r}'\right\rangle\times\hat{q}_n^\lambda(\phi')Q(r')+\frac{1}{2g^2}\int d^3r[\nabla\times(\hat{\boldsymbol{\phi}}A_n)]^2\,,$$

where $\nabla_\varphi\equiv\hat{\boldsymbol{\phi}}\cdot\nabla$, and $\hat{q}_n^\lambda(\phi)\equiv u_a^\lambda\hat{q}_n^a(\phi)$, where u_a^λ are the eigenfunctions of the matrix $i\epsilon^{abc}\delta^{c3}$ (coming from the $\epsilon^{abc}\mathbf{A}^c$ term in the covariant derivative), and $\hat{q}_n^a(\phi)$ is the direction of the charge density in the internal gauge space, in the gauge of Eq. (13a). We note that the functions $\hat{q}_n^\lambda(\phi)$ are gauge invariant. For analogous expressions involving Green's functions in a variational approximation to the quantized SU(2) gauge theory, see A. K. Kerman and D. Vautherin, Ann. Phys. (N.Y.) **192**, 408 (1989).

[16]S. L. Adler and T. Piran, Rev. Mod. Phys. **56**, 1 (1984).

[17]Mandula found an instability in the Coulomb solution above the critical coupling $\bar{\alpha}=\frac{3}{2}$ (see Ref. 5). The same critical coupling was found analytically in an explicitly axisymmetric

solution to the Yang-Mills equations by W. B. Campbell, Phys. Lett. **122B**, 293 (1983), and was obtained independently for our $n=0$ ansatz by the present authors (unpublished). This critical coupling in the Sikivie and Weiss ansatz was also obtained in the numerical calculations of Ref. 6.

[18]This family of solutions is a generalization of the "total screening" solutions discovered by Sikivie and Weiss (Ref. 2). While their original set of solutions was labeled by an integer parameter (not of topological origin), it was subsequently noted by several authors that the parameter labeling this family of solutions is, in fact, a continuous variable. These generalized solutions are thoroughly analyzed in Ref. 19.

[19]R. Jackiw and P. Rossi, Phys. Rev. D **21**, 426 (1980).