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Global chiral-invariant formulation of lattice gauge theory

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A global chiral-invariant lattice fermionic action which solves the species-doubling problem of free Dirac fermions on a lattice for (even) dimensions of space-time less than 8 is systematically studied in perturbation theory in two space-time dimensions when a gauge interaction is present. The decoupling of the replica fermions requires one to give up gauge symmetry for a finite lattice spacing but the quantum continuum limit is gauge invariant and reproduces the results of the continuum Schwinger model with one flavor.

I. INTRODUCTION

The most important shortcoming of lattice gauge theories at present comes from the difficulties to put a Weyl fermion on the lattice. This is a key ingredient in many theories such as the standard model of weak interactions, grand unified theories, many composite models, and supersymmetric theories. None of the standard methods to put fermions on the lattice[1,2] has been considered so far, in the case of chiral gauge theories, with enough details to know if one recovers in the quantum continuum limit all the features of these theories.[3]

For a chiral gauge theory in the continuum the fermionic measure cannot be regularized in a gauge-invariant way. The only way to mimic this breaking on the lattice (where the measure is invariant) is to work with a non-gauge-invariant action. We escape this way the no-go theorem[4] and it becomes possible to build an action with a global chiral symmetry. This possibility, could also be used to construct a global chiral-invariant action for vector gauge theories. Then an alternative to working in the manner of Wilson is to maintain the continuous global chiral symmetry if one is forced to break gauge invariance in the regularization procedure. Of course, we have to be sure we shall recover the usual Ward identities in the quantum continuum limit.

Looking for this alternative, we have proposed[5] a free fermion lattice action having the minimum number (i.e., only one) of replica fermion consistent with a continuous global chiral invariance, short-range couplings, and reflection positivity. A gauge field is then introduced keeping the replica fermion uncoupled. We need to give up gauge invariance on the lattice but the symmetry is recovered in the continuum limit.

This model will be here systematically studied in perturbation theory for the two-dimensional vector case. The result is a formulation with the following properties.

(i) The continuum limit is gauge invariant despite a breaking of the gauge symmetry in the regularization procedure.

(ii) Two fermions are present, but only one is coupled to the gauge field and all the results of the continuum Schwinger model with one fermion are recovered.

(iii) There is a manifest invariance under the point-independent chiral transformations acting on the lattice fermionic field. (The corresponding Noether current has, of course, no anomaly: the contributions coming from the two fermions cancel each other.) Nevertheless, we can identify a lattice axial-vector current which, in the continuum limit, picks up contributions from only one fermion. As it must be[6] this current has an anomaly, which is the correct gauge-invariant one.

So the continuum limit has all the desired properties including an anomalous axial-vector current, while a global chiral symmetry of the lattice action is preserved at every step.

This paper is devoted only to the vector gauge theories case and is organized as follows. In Sec. II we present the formulation on the lattice of a free fermion theory with global chiral invariance. Section III is devoted to the discussion of the gauge coupling and the derivation of the lattice Schwinger model based on the Ward identities. In Sec. IV the continuum limit of the model, when studied in perturbation theory, is shown to reproduce the results of the continuum theory for the mass gap, the U(1) anomaly, and the chiral-symmetry-breaking order parameter $\langle \bar{\psi}\psi \rangle$. We conclude with a summary and possible extension of the present work.

II. A FREE FERMION LATTICE ACTION

The naive way to put a fermion field on the lattice is[1]

$$I_-(\psi)=\frac{a^{D-1}}{2}\sum_{n,\mu}[\bar{\psi}_n\gamma_\mu(\psi_{n+\mu}-\psi_n)-(\bar{\psi}_{n+\mu}-\bar{\psi}_n)\gamma_\mu\psi_n] , \quad (2.1)$$

which is the only quadratic, Hermitian, lattice-rotational-invariant action involving couplings only between nearest neighbors. The corresponding fermion propagator in momentum space is

$$D_F^-(\theta)=a\left[i\sum_\mu\gamma_\mu P_\mu^-(\theta)\right]^{-1} , \quad (2.2)$$

where a is the lattice spacing; the angular variables θ_μ, with $-\pi\leq\theta_\mu\leq\pi$, are the momentum components in units of a^{-1} and

$$P_\mu^-(\theta)=\sin\theta_\mu . \quad (2.3)$$

Then the free fermion propagator vanishes in the continuum ($a\to 0$) limit except in the regions of size of order a centered around the 2^D poles of (2.2) located at the points with $\theta_\mu=0$ or π, where it reproduces the standard free fermion propagator up to similarity transformation of the gamma matrices.[7]

The action (2.1) is invariant under the U(1)×U(1) global transformations

$$\psi_n\to e^{i\alpha}\psi_n,\quad \psi_n\to e^{i\beta\gamma_5}\psi_n ,$$
$$\bar{\psi}_n\to\bar{\psi}_n e^{-i\alpha},\quad \bar{\psi}_n\to\bar{\psi}_n e^{i\beta\gamma_5} , \quad (2.4)$$

and the associated currents

$$J_{\mu(5)}(n)=\frac{i}{2}[\bar{\psi}_{n+\mu}\gamma_\mu(\gamma_5)\psi_n+\bar{\psi}_n\gamma_\mu(\gamma_5)\psi_{n+\mu}] \quad (2.5)$$

are conserved as a consequence of the U(1)×U(1) symmetry of the quantum theory. The coupling of these currents to the fermions is given by the vertex (in momentum space)

$$V_{\mu(5)}(\theta,\theta')=-i\gamma_\mu(\gamma_5)\cos\left[\frac{\theta_\mu+\theta'_\mu}{2}\right] , \quad (2.6)$$

which reduces, for the 2^D regions selected by the propagator, to the vertex in the continuum theory, up to signs which are in fact responsible for the cancellation of the axial anomaly.[7]

In order to have a lattice formulation of a theory with one fermion one has to decouple the replica fermions, i.e., one should find a set of lattice operators which are coupled to just one fermion $[\theta_\mu=O(a)]$ in the $a\to 0$ limit and which reproduce the correlation functions of the corresponding continuum theory operators. If one starts, for instance, with the currents (2.5) one sees that the problem with the naive action (2.1) is that all (2^D) fermions are coupled to them. One way to try to avoid this problem, which is the one explored here, is to add a new term to the action in (2.1) in order to change the location of the poles of the fermion propagator in such a way that only one of them $[\theta_\mu=O(a)]$ is coupled to the current in the continuum limit. The additional poles (there is at least one in order to cancel the axial anomaly for the Noether axial-vector current) should have all components $\theta_\mu=\pm\pi/2$.

We consider the simplest case with just one replica fermion located at $\theta_\mu=\pi/2$. If we work only with couplings between nearest neighbors and add the condition that the additional terms in the action do not disturb the fermion propagator for $\theta_\mu=O(a)$ then one has

$$P_\mu(\theta)=\sin\theta_\mu+\sum_\nu C_{\mu\nu}(1-\cos\theta_\nu) \quad (2.7)$$

the corresponding action being

$$I=I_-(\bar{\psi},\psi)+I_+(\bar{\psi},\psi) , \quad (2.8)$$

where

$$I_+(\bar{\psi},\psi)=-i\frac{a^{D-1}}{2}\sum_{n,\mu,\nu}C_{\mu\nu}[\bar{\psi}_n\gamma_\mu(\psi_{n+\nu}-\psi_n)+(\bar{\psi}_{n+\nu}-\bar{\psi}_n)\gamma_\mu\psi_n] \quad (2.9)$$

and the $D\times D$ matrix of coefficients should satisfy the condition

$$\sum_\nu C_{\mu\nu}=-1 \quad (2.10)$$

in order for the fermion propagator to have a pole at $\theta_\mu=\pi/2$.

Note that the new action does not respect the lattice rotational symmetry for any choice of $C_{\mu\nu}$. The reason is that the only quadratic, Hermitian, lattice rotational-invariant and chiral-invariant action involving neighboring couplings exclusively is I_-, for which the propagator does not have any pole at $\pi/2$. As now we are placing a pole at $\pi/2$, maintaining that the action will be quadratic, Hermitian, and chiral invariant, by force the lattice rotational symmetry must be broken. The only transformation which one can ask to be symmetries of the new action are those that leave invariant the selected location of the poles of the fermion propagator. Among these transformations, those leaving invariant each pole separately are any permutation of axis which requires[5] that

$$C_{ii}=C_{jj},\quad C_{ij}=C_{ji},\quad C_{ki}=C_{kj}\quad k\neq i,j , \quad (2.11)$$

which together with the condition (2.10) determine the matrix C of elements $C_{\mu\nu}$ up to an arbitrary constant λ:

$$C_{\mu\nu}=\lambda\delta_{\mu\nu}-\frac{1}{D}(1+\lambda) . \quad (2.12)$$

If together with this symmetry one requires the action I to be invariant under the transformation which interchanges the two poles of the fermion propagator one obtains $|\lambda|=1$. Now, the requirement of having only one replica fermion at $\theta_\mu=\pi/2$ is satisfied (in $D=2,4,6$) by $\lambda=1$. So we have

$$P_\mu(\theta)=\sin\theta_\mu+(1-\cos\theta_\mu)-\frac{2}{D}\sum_\nu(1-\cos\theta_\nu) , \quad (2.13)$$

the correction to the naive action being

$$I_+(\bar\psi,\psi)=i\frac{a^{D-1}}{2}\sum_{n,\mu}[\bar\psi_n\gamma_\mu(\psi_{n+\mu}-\psi_n)+(\bar\psi_{n+\mu}-\bar\psi_n)\gamma_\mu\psi_n]-i\frac{a^{D-1}}{D}\sum_{n,\mu,\nu}[\bar\psi_n\gamma_\mu(\psi_{n+\nu}-\psi_n)+(\bar\psi_{n+\nu}-\bar\psi_n)\gamma_\mu\psi_n]\ . \tag{2.14}$$

The difference between this formulation and the naive way to put a fermion field on the lattice is that now there is a natural way to decouple the replica fermion. In fact when one associates to any local operator $\bar\psi(X)\Gamma\psi(X)$ the lattice operator

$$O_-(n)=\tfrac{1}{2}(\bar\psi_n\Gamma\psi_{n+\mu}+\bar\psi_{n+\mu}\Gamma\psi_n]\ , \tag{2.15}$$

then one finds that only the pole at $p_\mu=0$ contributes to the correlation functions of these lattice operators in the continuum limit because, due to the location of the additional zero mode in momentum space, the coupling of these operators to the additional fermion is of the order of a. There is nothing similar to this simple mechanism of decoupling of the replica fermion in the case of the naive action.

The final result is that the action (2.8), with I_+ given by (2.14), solves the fermion doubling problem via decoupling of the replica fermion. For the moment, of course, this method works only for free fermions. Usually we are not free to choose the propagators and the vertices independently of one another without troubles with the gauge invariance. The coupling with a gauge field will be studied in the next section.

The continuum limit of this model can be studied explicitly in perturbation theory. By standard power-counting arguments[8,9] one finds that after subtraction of the first terms in the Taylor expansion of a Green's function, the ultraviolet finite part can be calculated as a sum of contributions each one calculated by replacing the vertices and propagators by their expansion around each pole. There is an additional contribution where the limit $a\to 0$ cannot be taken until the end. This additional contribution is present only when the superficial degree of divergence is not negative.

These general arguments have been discussed in detail in the context of two-dimensional models in Ref. 9 and a detailed calculation using this method will be presented in the next section when the gauge interaction is taken into account. In the present context of two-dimensional models, the only nontrivial case, where the continuum limit can differ from the previous naive classical arguments, is the one-loop contribution to the current-current correlation function which is the same calculation involved in the current Ward identities.

III. COUPLING TO THE ABELIAN GAUGE FIELD

The standard method to introduce a gauge interaction in the lattice formulation of a field theory is[1] through a new dynamical variable $U_{n,\mu}$, associated to the links of the lattice, which is inserted in the action in order to have invariance under local phase transformations. In particular the gauge-invariant version of the naive action is

$$I_-(\psi,U)=\frac{a^{D-1}}{2}\sum_{n,\mu}[\bar\psi_n\gamma_\mu(U_{n,\mu}\psi_{n+\mu}-\psi_n)-(\bar\psi_{n+\mu}U^\dagger_{n,\mu}-\bar\psi_n)\gamma_\mu\psi_n]\ .$$

When this general procedure is applied to the action (2.8) the result is

$$I(\psi,U)=I_-(\psi,U)+I_+(\psi,U)+I_W(U)\ , \tag{3.1}$$

where

$$I_W(U)=\frac{1}{2g^2}\sum_{n,\mu,\nu}(1-U_{n,\mu}U_{n+\mu,\nu}U^\dagger_{n+\nu,\mu}U^\dagger_{n,\nu}) \tag{3.2}$$

is the standard Abelian gauge lattice action.

The vertex which describes the interaction of the fermion with the gauge field is a sum of two contributions coming from to the two terms $I_-(\psi,U)$ (responsible for the coupling to the fermion at $\theta_\mu=0$) and $I_+(\psi,U)$ (which gives the coupling to the replica fermion). The final result is that the action (3.1) describes a gauge field coupled simultaneously to both fermions and the gauge interaction ruins the solution of the doubling problem we had before.

The obvious thing to try to avoid this problem is to start with a different action:

$$I'(\psi,U)=I_-(\psi,U)+I_+(\psi)+I_W(U)\ . \tag{3.3}$$

Now the gauge field, which is coupled to the fermion only through the first term, will respect the decoupling of the replica fermion at least at the classical level. The action (3.3) differs from the naive action one would have written to describe the Abelian gauge theory with one fermion field by the presence of the additional term $I_+(\psi)$ irrelevant in the naive continuum limit which breaks explicitly the lattice rotational symmetry and the gauge invariance. The remaining symmetries are vector and chiral global transformations of the fermionic field.

So the most difficult question, but necessary if we want to solve the fermion-doubling problem via decoupling, is to see how gauge invariance can be reobtained when the action (3.3) is taken as a starting point. This question can be studied explicitly in perturbation theory looking for a lattice realization in the continuum limit of the relations among Green's functions (Ward identities) which reflect the gauge invariance of the theory in the continuum. The Ward identities corresponding to Green's functions with a negative superficial degree of divergence are automatically satisfied because one can calculate them directly by replacing the vertex and propagators by their approximation around $P_\mu=0$, i.e., by their continuum expressions. There is no contribution from the pole around $\pm\pi/2$ provided we use a lattice current operator of the type (2.15) (naive current). In other words, the term $I_+(\psi)$ in the action (3.3) is completely negligible in this calculation and the remaining terms are gauge invariant. An alternative way to understand the result is to realize that once the I_+ part of the action has been neglected the naive current is the conserved Noether current associated to

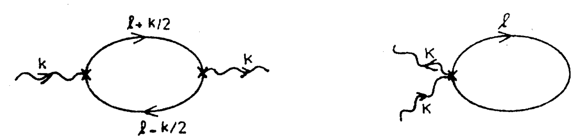

FIG. 1. The Feynman graphs that give the vacuum-polarization contribution.

the global symmetry of the naive action.

Then the only quantity one has to consider in detail is, in two dimensions, $\Pi_{\mu\nu}$, the vacuum polarization. In the continuum limit $\Pi_{\mu\nu}$ receives contributions only from the two diagrams depicted in Fig. 1 and the result of applying the Feynman rules corresponding to the action (3.3) is

$$\Pi_{\mu\nu}(k)=-\int_{-\pi/a}^{\pi/a}\frac{d^2l}{(2\pi)^2}[I_{\mu\nu}^{(2)}(l,k)+I_{\mu\nu}^{(1)}(l)]\ , \quad (3.4)$$

with

$$I_{\mu\nu}^{(2)}(l,k)=\mathrm{Tr}\left[V_{\mu}^{(1)}(l,l+k)\frac{1}{i\boldsymbol{P}(l+k)}\times V_{\nu}^{(1)}(l,l+k)\frac{1}{i\boldsymbol{P}(l)}\right]\ , \quad (3.5a)$$

$$I_{\mu\nu}^{(1)}(l)=\mathrm{Tr}\left[V_{\mu\nu}^{(2)}(l,l+k)\frac{1}{i\boldsymbol{P}(l)}\right]\ , \quad (3.5b)$$

the vertices being

$$V_{\mu}^{(1)}(p,q)=-i\gamma_{\mu}\cos\left[\frac{a}{2}(p+q)_{\mu}\right]\ , \quad (3.6a)$$

$$V_{\mu\nu}^{(2)}(p,q)=ia\delta_{\mu\nu}\gamma_{\mu}\sin\left[\frac{a}{2}(p+q)_{\mu}\right]\ , \quad (3.6b)$$

and, using (2.13) for $D=2$,

$$aP_1(l)=\sin(al_1)-[1-\cos(al_2)]\ ,$$

$$aP_2(l)=\sin(al_2)-[1-\cos(al_1)]\ .$$

The next step in a perturbative calculation of a diagram with a superficial degree of divergence D is to separate the contribution of the first $D+1$ terms in the Taylor expansion around $k=0$ from the rest. In the present case $D=0$ but due to infrared divergences around the pole at $l_{\mu}=0$ an intermediate IR regularization has to be introduced in order to give some meaning to the terms evaluated at zero external momentum. It is convenient, for example, to introduce a mass term m in the fermion propagators. Then one can write

$$\Pi_{\mu\nu}(k)=\lim_{m\to 0}[\overline{\Pi}_{\mu\nu}^{(m)}(k)+\Pi_{\mu\nu}^{(m)}(0)]\ , \quad (3.7)$$

where

$$\overline{\Pi}_{\mu\nu}^{(m)}(k)=-\int_{\pi/a}^{-\pi/a}\frac{d^2l}{(2\pi)^2}[I_{\mu\nu}^{(2)}(l,k,m)-I_{\mu\nu}^{(2)}(l,k=0,m)]\ , \quad (3.8)$$

$$\Pi_{\mu\nu}^{(m)}(0)=-\int_{-\pi/a}^{\pi/a}\frac{d^2l}{(2\pi)^2}[I_{\mu\nu}^{(2)}(l,k=0,m)+I_{\mu\nu}^{(1)}(l,m)]\ . \quad (3.9)$$

In $\overline{\Pi}_{\mu\nu}^{(m)}(k)$ we can replace the vertices and the propagators by their expansions around the poles, and the calculation reduces to the standard calculation of the vacuum polarization in the continuum theory for one fermion. There is no contribution from the replica fermion because the vertex vanishes when $l_{\mu}=\pi/2a$. The result is

$$\lim_{m\to 0}\lim_{a\to 0}\overline{\Pi}_{\mu\nu}^{(m)}(k)=-\frac{i}{\pi}\left[\delta_{\mu\nu}-\frac{k_{\mu}k_{\nu}}{k^2}\right]\ . \quad (3.10)$$

One still has to consider the contribution (3.9). After the IR regularization (3.9) reads

$$\Pi_{\mu\nu}^{(m)}(0)=-\int_{-\pi/a}^{\pi/a}\frac{d^2l}{(2\pi)^2}\mathrm{Tr}[V_{\mu}^{(1)}(l,l)S_m(l)\times V_{\nu}^{(1)}(l,l)S_m(l)+V_{\mu\nu}^{(2)}(l,l)S_m(l)]\ ,$$

where

$$S_m(l)=\frac{1}{i\not{p}(l)+m}\ .$$

This part of the calculation can be simplified if one integrates by parts using the relations

$$V_{\mu\nu}^{(2)}(l,l)=\frac{\partial V_{\mu}^{(1)}(l)}{\partial l_{\nu}}\ ,$$

$$\frac{\partial S_m(l)}{\partial l_{\nu}}=-S_m(l)\left[i\sum_{\mu}\gamma_{\mu}\frac{\partial P_{\mu}(l)}{\partial l_{\nu}}\right]S_m(l)\ .$$

After this has been done one has

$$\Pi_{\mu\nu}^{(m)}(0)=-\int_{-\pi/a}^{\pi/a}\frac{d^2l}{(2\pi)^2}\mathrm{Tr}\left[V_{\mu}^{(1)}S_m(l)\left[V_{\nu}^{(1)}+i\sum_{\mu}\gamma_{\mu}\frac{\partial P_{\mu}(l)}{\partial l_{\nu}}\right]S_m(l)\right]\ .$$

If the lattice action were gauge invariant, then the factor in brackets would vanish identically due to the relation between the vertex and the propagator of a gauge-invariant theory and the result of the continuum theory (3.10) would be recovered in the lattice formulation. In the present case the term $I_{+}(\psi)$ breaks explicitly gauge invariance and one gets

$$\Pi_{\mu\nu}^{(m)}(0)=-\int_{-\pi/a}^{\pi/a}\frac{d^2l}{(2\pi)^2}(-i\cos al_{\mu})(-i\sin al_{\nu})\times\mathrm{Tr}[\gamma_{\mu}S_m(l)\gamma_{\bar{\nu}}S_m(l)]\ ,$$

where $\bar{\nu}=2$ (1) when $\nu=1$ (2). Because of the fact that the momentum-dependent factor coming from the vertices vanishes on all the poles of the fermion propagator the computation of

$$\lim_{m\to 0}\Pi^{(m)}_{\mu\nu}(0)$$

can be done directly with $m=0$. The final answer is

$$\lim_{a\to 0}\Pi^{(0)}_{\mu\nu}(0)=c_0+c_1\delta_{\mu\nu}\ , \tag{3.11}$$

where

$$c_0=\frac{1}{\pi}-\tfrac{3}{8}\ , \tag{3.12a}$$

$$c_1=\tfrac{1}{16}\ . \tag{3.12b}$$

As the vertex vanishes for the replica pole, the replica fermion is not coupled and contributes only with contact terms. One can deal with such terms and eliminate the replica fermion by using the renormalization procedure. So we are led to introduce a counterterm in the action (3.3):

$$\tilde{I}(\psi,U)=I'(\psi,U)-\frac{c_0}{8}\sum_{n\mu\nu}(U_{n,\mu}-U^{\dagger}_{n,\mu})(U_{n,\nu}-U^{\dagger}_{n,\nu})$$
$$-\frac{c_1}{8}\sum_{n,\mu}(U_{n,\mu}-U^{\dagger}_{n,\mu})^2\ , \tag{3.13}$$

which is the action corresponding to our lattice global chiral-invariant Schwinger model. Note that the last modification introduced in the action does not affect the coupling of the fermion to the gauge field and all the previous discussion based on it remains valid for the new action. With the action (3.13) the vacuum-polarization tensor, given in terms of Feynman diagrams in Fig. 2, is then

$$\Pi_{\mu\nu}(k)=-\frac{1}{\pi}\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right] \tag{3.14}$$

which is the result for the gauge-invariant continuum theory with only one fermion.

As expected we have to introduce a counterterm to restore the gauge invariance broken by the action in Eq. (3.3); an additional counterterm is also needed to restore the rotational symmetry. (See the works in Ref. 10 where similar problems have been considered.)

The last step is the identification of the lattice operator corresponding to the vector current in the continuum theory. If one considers, for example, the diagrammatic expansion of $\langle J_\mu J_\nu\rangle$ it is easy to see the operator we look for is

$$\tilde{J}_\mu(n)=J_\mu(n)-\frac{i}{2a}c_0\sum_{\nu}(U_{n\nu}-U^{\dagger}_{n\nu})$$
$$-\frac{i}{2a}c_1(U_{n\mu}-U^{\dagger}_{n\mu})\ , \tag{3.15}$$

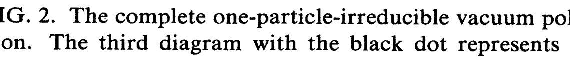

FIG. 2. The complete one-particle-irreducible vacuum polarization. The third diagram with the black dot represents the counterterm defined in Eq. (3.13).

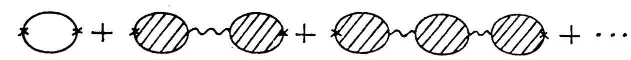

FIG. 3. The expansion of the current-current correlation function $\langle \tilde{J}_\mu \tilde{J}_\nu\rangle$ in terms of Feynman diagrams.

when $J_\mu(n)$ is the naive current

$$J_\mu(n)=+\frac{i}{2}(\bar{\psi}_n\gamma_\mu U_{n,\mu}\psi_{n+\mu}+\bar{\psi}_{n+\mu}\gamma_\mu U^{\dagger}_{n,\mu}\psi_n)\ . \tag{3.16}$$

In the expansion of $\langle \tilde{J}_\mu \tilde{J}_\nu\rangle$ in terms of Feynman diagrams given in Fig. 3 one recognizes the expression for the continuum correlation function except for the first diagram which differs from the continuum one only by unimportant contact terms which can be omitted. In the definition (3.15) a "normal-ordered" prescription is understood, i.e., one must subtract the expectation value of $\tilde{J}_\mu$. With the action (3.13) and the definition (3.15) for the vector current all the Green's functions involving vector currents are those of a gauge- and Lorentz-invariant continuum theory with only one fermion.

IV. COMPARISON OF THE GLOBAL CHIRAL-INVARIANT LATTICE SCHWINGER MODEL AND THE CONTINUUM SCHWINGER MODEL

Because of the controversy among different studies of lattice fermion formulations, especially when applied to the nonlocal formulations, one strategy which has proved to be useful[9] is to study these formulations in the context of the exactly solvable Schwinger model. The result of these studies is that no nonlocal formulation can reproduce the results of the continuum theory, the Kogut-Susskind[2] action corresponds to a theory with two flavors, and only the Wilson[1] action is consistent with the continuum theory.

Motivated by these results and once the global chiral-invariant Schwinger model with lattice action (3.13) has been obtained it seems interesting to see whether or not the continuum limit of the perturbation expansion based on this model is consistent. The main properties of the Schwinger model which are going to be studied are the mass of the lowest-lying particle in the spectrum, which can be obtained from the vacuum-polarization calculation, the U(1) anomaly related to the construction of the lattice axial-vector-current operator, and the chiral order parameter $\langle\bar{\psi}\psi\rangle$.

In the preceding section it has been shown that the vacuum-polarization tensor $\Pi_{\mu\nu}$ is the same as in the continuum theory with one fermion, then the mass gap for the Schwinger model is automatically reproduced with the lattice model.

The second property of the lattice Schwinger model we are going to study is to see whether one can understand in this formulation the U(1) anomaly of the continuum theory. The starting point in this case is $\Pi_{\mu 5\nu}$ the analogue of the vacuum-polarization tensor but with an axial vertex which can be calculated following exactly the same steps we used in Sec. III for the Ward identity associated to the vector current. The result for $\overline{\Pi}_{\mu 5\nu}$ can be obtained directly from the result for the vector current in (3.10) using the relation $\gamma_\mu\gamma_5=\epsilon_{\mu\nu}\gamma_\nu$ typical to the two-dimensional case. In any case it is clear the evaluation of this part reduces once more to the standard perturbation calculation in the continuum theory. The result is

$$\lim_{a\to 0}\lim_{m\to 0}\overline{\Pi}^{(m)}_{\mu 5\nu}(k)=-\frac{1}{\pi}\epsilon_{\mu\nu}\left[\delta_{\alpha\nu}-\frac{k_\alpha k_\nu}{k^2}\right] . \tag{4.1}$$

If this were the total answer then one would have (in a gauge where $D^{-1}_{\alpha\beta}$ exists)

$$ik_\mu\Pi_{\mu 5\nu}(k)=-\frac{1}{\pi}ik_\mu\epsilon_{\mu\nu}=-\frac{1}{\pi}ik_\mu\epsilon_{\mu\alpha}D_{\alpha\beta}D^{-1}_{\beta\nu} , \tag{4.2}$$

and using the definition of the A_μ propagator,

$$D_{\alpha\beta}(k)=a^2\sum_n e^{iank}\langle A_\alpha(n)A_\beta(0)\rangle ,$$

we would have the axial Ward identity:

$$ik_\mu\Pi_{\mu 5\nu}(k)=a^2\sum_n e^{iank}\left\langle\left[-\frac{1}{\pi}\epsilon_{\mu\alpha}\frac{A_\alpha(n+\mu)-A_\alpha(\mu)}{a}\right]A_\beta(0)\right\rangle D^{-1}_{\beta\nu}(k) , \tag{4.3}$$

where one recognizes in the term in large parentheses the two-dimensional chiral anomaly with the derivative replaced by its finite difference representation. The Noether axial-vector current we can derive from the action (3.3) receives contributions from the two fermions and has no anomaly. Note that the nonvanishing result in Eq. (4.3) is due to the fact that the current involved in $\Pi_{\mu 5\nu}$ is not the conserved Noether current derived from the action (3.3), but the axial-vector current associated to the I_- part (i.e., the naive one) of this action (3.3).

Then in order to show that the correct U(1) anomaly is reproduced in the lattice all one needs is to show how to get rid of the remaining contributions to $\Pi_{\mu 5\nu}$ which come from the expression similar to (3.9) for the axial case:

$$\Pi^{(m)}_{\mu 5\nu}(0)=-i\int_{-\pi/a}^{+\pi/a}\frac{d^2l}{(2\pi)^2}\cos(al_\mu)\sin(al_\nu)\times\mathrm{Tr}\left[\gamma_\mu\gamma_5\frac{1}{\not{P}(l)}\gamma_{\bar\nu}\frac{1}{\not{P}(l)}\right] . \tag{4.4}$$

This integration, once more, can be done directly with $m=0$ and the result is

$$\lim_{a\to 0}\Pi^{(0)}_{\mu 5\nu}(0)=\frac{1}{2\pi}\epsilon_{\mu\nu} . \tag{4.5}$$

This term spoils both the gauge invariance and the chiral properties one would like for $\langle\tilde J_\mu J^5_\nu\rangle$. It reflects nothing but the naive lattice axial-vector current; J^5_ν does not converge to the continuum gauge-invariant axial-vector current when the lattice spacing goes to zero. We can follow the same procedure we used for the vector current in Sec. III to construct the correct lattice axial-vector current:

$$\tilde J^5_\mu(n)=J^5_\mu(n)-\frac{i}{2a}\sum_\nu\frac{\epsilon_{\mu\nu}}{2\pi}(U_{n,\nu}-U^\dagger_{n,\nu}) , \tag{4.6}$$

where $J^5_\mu(n)$ is the naive axial-vector current,

$$J^5_\mu(n)=+\frac{i}{2}(\bar\psi_n\gamma_\mu\gamma_5U_{n,\mu}\psi_{n+\mu}+\bar\psi_{n+\mu}\gamma_\mu\gamma_5U^\dagger_{n,\mu}\psi_n) \tag{4.7}$$

and again the normal-ordered prescription is understood in the definition (4.6).

The expansion of $\langle\tilde J_\mu\tilde J^5_\nu\rangle$ in Feynman diagrams is given in Figs. 4 and 5. Again we recognize the usual continuum expression, up to unimportant contact terms in the first diagram. So one recovers the gauge invariance and simultaneously gets the correct anomaly

$$\partial^\mu\tilde J^5_\mu=-\frac{g}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} . \tag{4.8}$$

The last property of the Schwinger model we shall investigate with the lattice formulation is the chiral order parameter $\langle\bar\psi\psi\rangle$. It is known that the one-flavor massless Schwinger model undergoes a breaking of the chiral symmetry. This can be seen by noticing that[9,11]

$$G(x,0)=\langle\bar\psi_{(x)}\psi_{(x)}\bar\psi_{(0)}\psi_{(0)}\rangle\underset{x\to\infty}{\longrightarrow}\tfrac{1}{2}|\langle\bar\psi\psi\rangle|^2=\frac{g^2}{8\pi^3}e^{2\gamma} , \tag{4.9}$$

where γ is the Euler constant. For more than one flavor $\langle\bar\psi\psi\rangle$ vanishes.

We shall follow closely the method described in Refs. 9 and 11 to compute $G_L(n,m)$ defined by

$$G_L(n,m)=\left\langle\tfrac{1}{4}\sum_\mu(\bar\psi_n\psi_{n+\mu}+\bar\psi_{n+\mu}\psi_n)\times\tfrac{1}{4}\sum_\nu(\bar\psi_m\psi_{m+\nu}+\bar\psi_{m+\nu}\psi_m)\right\rangle , \tag{4.10}$$

where according to our previous discussion we have introduced for $\bar\psi\psi$ an operator which cannot be a source for the replica fermion. The diagrams which contribute to $G_L(n,m)$ when a goes to zero are shown in Fig. 6.

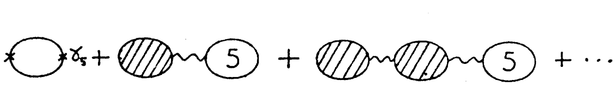

FIG. 4. The expansion of the axial-current–vector-current correlation function $\langle\tilde J_\mu\tilde J^5_\nu\rangle$. The axial bubble is defined in Fig. 5.

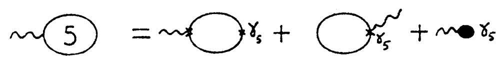

FIG. 5. This axial bubble is made with the two graphs in Fig. 1 when one vertex picks up γ_5 matrix. The third graph represents the counterterm introduced in the definition of the axial-vector current $\tilde{J}^5_\nu$ (4.6).

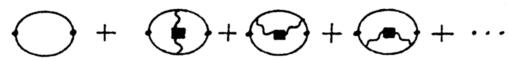

FIG. 6. The diagrams giving a contribution to $G_L(x,0)$ when the lattice spacing goes to zero. The wavy line with a black square is the complete photon propagator.

The power-counting arguments tell us every diagram of Fig. 6 except the first one can be obtained in the limit a goes to zero by replacing the propagators and the vertices by their expansion around each pole. As in the previous cases the contributions of the replica fermion to these diagrams are destroyed by the vertices and we can write

$$\lim_{a\to 0} G_L(x,0) = G(x,0) - I(x) + \lim_{a\to 0} I_L(x) , \qquad (4.11)$$

where $I(x)$ and $I_L(x)$ are the values of the first diagram for the continuum theory and for the lattice model, respectively. With the technique used in Secs. III and IV one can show that these two terms cancel each other up to an irrelevant contact term. The result for the order parameter follows:

$$\lim_{a\to 0} G_L(x,0) \underset{x\to\infty}{\longrightarrow} \frac{g^2}{8\pi^3} e^{2\gamma} . \qquad (4.12)$$

V. SUMMARY AND DISCUSSION

We have shown that it is possible to have a global chiral-invariant lattice formulation of a vector gauge theory with one fermion by adding a term to the naive action which decouples the replica fermion. This mechanism of decoupling requires us to break explicitly the rotational and gauge invariance at the level of the action but when applied to the Schwinger model is has been shown, by using weak-coupling perturbative expansion, that the gauge and rotational symmetries are recovered in the quantum continuum limit and a lattice Schwinger model with all the desired properties is obtained by adding appropriate counterterms. In particular, the relevant axial-vector current, including counterterms, reproduces the correct anomaly of the effective one fermion theory, in contrast with the conserved axial-vector current associated with the global chiral invariance of the full action (3.3)

The extension of the present analysis to the case of a non-Abelian gauge symmetry as well as to four-dimensional theories and the determination of the structure of counterterms in these cases is presently under investigation. One should take into account that the possibility to determine the counterterms perturbatively is specific to the two-dimensional case. In four dimensions there would be contributions from arbitrary high orders in the coupling constant. In principle one should fix the counterterms with a nonperturbative calculation of some appropriate physical quantities and/or a nonperturbative check of Ward identities. In a first step, the Schwinger model could also be used to test if the present formulation works with the standard nonperturbative method.

The use of a formulation which respects a global chiral symmetry on the lattice at every step should be very appropriate to understand those aspects of the standard model of strong interactions related with this symmetry.

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