Energy-momentum tensor in scalar QED. II

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We consider the renormalization of the energy-momentum tensor in scalar QED in which the $(\phi^*\phi)^2$ interaction is purely electromagnetically induced. This is a theory with scalar fields and a *single* coupling constant e and it may be expected in an analogy with $\lambda\phi^4$ theory that the finite improvement programs, of the kind considered in paper I, may work here. Indeed, we find that the finite improvement programs work to a high order, viz., up to $O(e^{10})$. But, we also find that both kinds of finite improvement programs fail at least in $O(e^{14})$ and most probably in $O(e^{12})$.

I. INTRODUCTION

In Ref. 1 (henceforth referred to as I) we considered the renormalization of the energy-momentum tensor in scalar QED. It was shown that an improvement term was needed for the energy-momentum tensor. The addition of such an improvement term in $\theta_{\mu\nu}$ is equivalent to the addition of a term in the action (in the presence of an external gravitational field) of the form $\frac{1}{2}K_0\int R\phi^*\phi\, d^nx$. As explained in detail in Ref. 2, the question we focused our attention on in I was whether the coefficient K_0 is renormalized independently or whether it was fixed by the flat-space theory. If K_0 is renormalized independently, this requires a new experimental input to fix the renormalized parameter K. A new experimental input is not needed in case the finite improvement program(s) goes through. Finite improvement programs are of two kinds. (i) The improvement coefficient K_0 is a finite function of bare quantities at $\epsilon\equiv 4-n=0$. In this case, no renormalization counterterms which would correspond to the renormalization of the coefficient of $\frac{1}{2}\int R\phi^*\phi\, d^nx$ are needed (apart from flat-space renormalizations). (ii) The improvement coefficient is a finite function of renormalized quantities at $\epsilon=0$. In this case the renormalization counterterms are uniquely fixed by the requirement (imposed *ad hoc*) that the improvement coefficient is a finite function of renormalized quantities.

In I it was established that neither kind of improvement program goes through to $O(e^2\lambda^n)$. A similar result was derived in the other renormalizable field theories with scalar fields and two coupling constants, in Refs. 2–4. The proofs depended crucially upon the fact that two independent coupling constants were present. In this paper we shall consider the other alternative: viz., the theory has scalar fields but only one independent coupling constant. This can be arranged if one assumes that, in scalar QED, the $(\phi^*\phi)^2$ coupling is induced only through electromagnetic interactions. In this paper we wish to address ourselves to the question of whether finite improvement programs go through in such a theory. In an analogy with the $\lambda\phi^4$ theory, a theory with a scalar field and one coupling constant, one may hope that they do go through.

We find that finite improvement programs of either kind do go through up to $O(e^{10})$. But both of them necessarily fail in $O(e^{14})$ and most probably in $O(e^{12})$ itself. [To ascertain whether finite improvement programs break down in $O(e^{12})$ itself requires a tedious calculation of a renormalization constant, which we have not done.] Thus, in either case, K_0 is independently renormalized at least from $O(e^{14})$; and flat-space parameters are insufficient to specify the theory completely in the presence of external gravity from this order.

As a technical point, it should be noted that the mathematical steps in the present case are not just a special case of those in I.

II. PRELIMINARIES

We shall work with a complex scalar field coupled to an Abelian gauge field A_μ described by the Lagrange density

$$\mathcal{L}'=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(D_\mu\phi)^*(D^\mu\phi)-\tfrac{1}{2}m_0^2\phi^*\phi-\tfrac{1}{2}\xi_0(\partial\cdot A)^2\ . \tag{2.1}$$

(The notation is as in I.)

There is no $(\phi^*\phi)^2$ interaction in the lowest order, and thus $\mathcal{L}'$ contains only one independent coupling constant e_0. However, in $O(e_0^4)$ and higher, a $(\phi^*\phi)^2$ coupling is induced and there are counterterms needed of the form

$$\mu^\epsilon\delta\lambda(e^2,\epsilon)\int\frac{(\phi^*\phi)^2}{4!}d^nx\ .$$

Thus the original Lagrange density $\mathcal{L}'$ must be modified to contain a counterterm

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(D_\mu\phi)^*(D^\mu\phi)-\tfrac{1}{2}m_0^2\phi^*\phi-\mu^\epsilon\delta\lambda(e^2,\epsilon)\frac{(\phi^*\phi)^2}{4!}-\tfrac{1}{2}\xi_0(\partial\cdot A)^2\ . \tag{2.2}$$

This Lagrange density is a special case of the Lagrange density

$$\mathcal{L}''=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(D_\mu\phi)^*(D^\mu\phi)-\tfrac{1}{2}m_0^2\phi^*\phi-\frac{\lambda_0}{4!}(\phi^*\phi)^2-\tfrac{1}{2}\xi_0(\partial\cdot A)^2 \tag{2.3}$$

used in I. $\int \mathcal{L}'' d^n x$ is an action containing two independent coupling constants λ_0 and e_0 and it is often convenient to look upon $\mathcal{L}$ of Eq. (2.2) as a limiting case of $\mathcal{L}''$ of Eq. (2.3), the limit being specified below. However, we should remark that in this work we are interested in the Lagrange density $\mathcal{L}$ of Eq. (2.2) which has only *one* independent coupling constant e_0 and that $\mathcal{L}''$ of Eq. (2.3) is being introduced only for certain technical reasons.

We shall use dimensional regularization together with the minimal subtraction scheme for regularizing proper vertices of the theory and of operators. The renormalization transformations are

$$\phi = Z^{1/2}\phi^R, \quad m_0^2 = m^2 Z_m ,$$
$$\lambda_0 = \mu^\epsilon[\lambda Z_\lambda(\lambda, e, \epsilon) + \delta\lambda(e^2, \epsilon)] , \tag{2.4}$$
$$e_0^2 = \mu^\epsilon e^2 Z_e^2, \quad A_\mu = Z_3^{1/2} A_\mu^R, \quad \xi_0 = Z_\xi \xi = Z_3^{-1}\xi .$$

In I we had introduced two independent mass parameters μ_1 and μ_2 in the relations for λ_0 and e_0. But for the purpose of this paper, this is unnecessary. So we set $\mu_1 = \mu_2 = \mu$. The renormalization-group quantities are also now similarly redefined. For example,

$$\beta^\lambda(\lambda, e, \epsilon) = \mu\frac{\partial}{\partial\mu}\lambda = \mu_1\frac{\partial}{\partial\mu_1}\lambda + \mu_2\frac{\partial}{\partial\mu_2}\lambda\bigg|_{\mu_1=\mu_2=\mu}$$
$$= (\beta_1^\lambda + \beta_2^\lambda)|_{\mu_1=\mu_2=\mu} , \tag{2.5}$$

etc.

The Lagrange density of Eq. (2.2) is obtained from that of Eq. (2.3) by putting $\lambda = 0$. So we shall deal with $\mathcal{L}''$ of Eq. (2.3) first, and obtain results for $\mathcal{L}$ of Eq. (2.2) by setting $\lambda = 0$.

III. CONDITIONS FOR A FINITE ENERGY-MOMENTUM TENSOR

The improved energy-momentum tensor for $\mathcal{L}''$ is given by Eq. (3.5) of I. Its trace is, as seen from Eq. (3.6) of I,

$$\langle \theta_\mu^{\text{imp}\ \mu} \rangle = \text{finite} + (n-4)\left\langle -\frac{\lambda_0}{4!}(\phi^*\phi)^2 + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\xi_0(\partial\cdot A)^2 \right\rangle + \tilde{g} Z_m^{-1}\langle \partial^2(\phi^*\phi)\rangle^R . \tag{3.1}$$

As shown in I, the right-hand side of Eq. (3.1) simplifies to

$$\langle \theta_\mu^{\text{imp}\ \mu} \rangle = \text{finite} - \epsilon X \langle \partial^2(\phi^*\phi)\rangle^R \tag{3.2}$$

with

$$X = Z_{17} - \frac{1}{\epsilon}\tilde{g} Z_m^{-1} \equiv Z_{17} + g Z_m^{-1} . \tag{3.3}$$

Here Z_{17} is the coefficient of mixing of operator O_1 (defined in I) that appears in Eq. (3.1) with $O_7 = \partial^2(\phi^*\phi)$.

As shown in I, $Z_{17}(\lambda, e^2, \epsilon)$ satisfies the renormalization-group (RG) equation

$$(-\lambda\epsilon + \beta^\lambda)\frac{\partial Z_{17}}{\partial\lambda} + \left[-\frac{e\epsilon}{2} + \beta^e\right]\frac{\partial Z_{17}}{\partial e} - 2\gamma_m Z_{17} = Z_{11}\gamma_{17} + Z_{15}\gamma_{57} . \tag{3.4}$$

We are interested in the Lagrange density $\mathcal{L}$ of Eq. (2.2). The results for this Lagrange density are obtained by setting $\lambda = 0$ in the above equation and noting that Z_{17}, being a power series in λ has a smooth limit as $\lambda \to 0$, we arrive at the RG equation satisfied by $Z_{17}(\lambda = 0, e^2, \epsilon) \equiv Z_{17}(e^2, \epsilon)$:

$$\left[-\frac{e\epsilon}{2} + \beta^e\right]\frac{\partial Z_{17}}{\partial e} - 2\gamma_m Z_{17} = Z_{11}\gamma_{17} + Z_{15}\gamma_{57} - \beta^\lambda Y , \tag{3.5}$$

where every quantity in Eq. (3.5) is evaluated at $\lambda = 0$ and

$$Y \equiv \frac{\partial Z_{17}}{\partial\lambda}\bigg|_{\lambda=0}$$

is a series in e^2. [It should be noted that Z_{11} and Z_{15} individually contain terms proportional to $1/\lambda$. However, the combination $Z_{11}\gamma_{17} + Z_{15}\gamma_{57}$ does not contain such terms. This is seen by substituting expressions for γ_{17} and γ_{57} given in Eq. (A3) of I (the γ_{17} here is $\gamma_{17} + \gamma'_{17}$ of I, etc.) and noting that $Z_{17} - Z_{57}$ is proportional to λ.] Hence X of Eq. (3.3) now satisfies an equation (assuming g to depend only on e and ϵ)

$$\left[-\frac{e\epsilon}{2} + \beta^e\right]\frac{\partial X}{\partial e} - 2\gamma_m X = Z_{11}\gamma_{17} + Z_{15}\gamma_{57} - \beta^\lambda Y + \left[\mu\frac{\partial}{\partial\mu}g\right] Z_m^{-1} \equiv Z_{11}\gamma_{17} + Z_{15}\gamma_{57} + T . \tag{3.6}$$

Equation (3.6) is valid for an arbitrary choice of $g(e^2, \epsilon)$. Now, the first two terms on the right-hand side of Eq. (3.6) contain only simple poles,[1] while T may contain double and higher-order poles in ϵ. We now formulate the necessary and sufficient conditions for the finiteness of $\theta_{\mu\nu}^{\text{imp}}$.

To this end we note that finiteness of $\theta_\mu^{\text{imp}\ \mu}$ is necessary and sufficient[5] for the finiteness of $\theta_{\mu\nu}^{\text{imp}}$. $\theta_\mu^{\text{imp}\ \mu}$ is finite, as seen from Eq. (3.2), iff X has no worse than simple poles in ϵ. We now prove the necessary and sufficient conditions for X to have no worse than simple poles.

Theorem. The necessary and sufficient conditions for X to have no worse than simple poles are (i) T has no worse than simple poles and (ii) $X^{(2)}$ is made zero, where X and T have been expanded as

$$X=\sum_{r=-\infty}^{\infty}\frac{X^{(r)}}{\epsilon^r},\quad T=\sum_{r=-\infty}^{\infty}\frac{T^{(r)}}{\epsilon^r}\ . \tag{3.7}$$

Proof. (i) First we prove the necessity of the two conditions. Let X have no worse than simple poles. Then $X^{(2)}=0$. Further, the left-hand side of Eq. (3.6) as well as the first two terms on its right-hand side have at worst simple poles and hence T cannot have worse than simple poles. (ii) Next we prove the sufficiency of these conditions. Suppose that $X^{(2)}$ is zero and that T has no worse than simple poles. We then compare the coefficient of ϵ^{-p} $(p\geq 2)$ in Eq. (3.6) and obtain

$$-\frac{e}{2}\frac{\partial}{\partial e}X^{(p+1)}+\beta_e\frac{\partial X^{(p)}}{\partial e}-2\gamma_m X^{(p)}=0,\quad p\geq 2\ . \tag{3.8}$$

These equations, starting from $p=2$, successively yield $X^{(3)}=X^{(4)}=\cdots=0$, noting that $X^{(p)}(e=0)=0$; $p\geq 2$. Hence X has no worse than simple poles in ϵ.

Thus, the question is whether g can be chosen so that the two conditions of the theorem above can be satisfied. In the subsequent sections we try to choose g of either forms of the finite improvement program to see whether these two conditions are satisfied.

IV. IMPROVEMENT COEFFICIENT OF THE FORM $\tilde{g}(e_0^2\mu^{-\epsilon},\epsilon)$

In this section we shall consider an improvement coefficient $\tilde{g}$ which is a finite function (at $\epsilon=0$) of the bare coupling e_0. Noting that $g=-\tilde{g}/\epsilon$, g has an expansion

$$\begin{aligned}g(e_0^2\mu^{-\epsilon},\epsilon)&=\sum_{n=0}^{\infty}g_n(\epsilon)(e_0^2\mu^{-\epsilon})^n\\&=\sum_{n=0}^{\infty}\sum_{k=-1}^{\infty}g_n^k(e_0^2\mu^{-\epsilon})^n\epsilon^k\end{aligned} \tag{4.1}$$

and consider X order by order.

In $O(e^0)$:

$$X=Z_{17}+\sum_{k=-1}^{\infty}g_0^k\epsilon^k=Z_{17}+g_0(\epsilon)\ .$$

Now $Z_{17}=0$ in $O(e^0)$. Hence X has no worse than simple poles.

In $O(e^2)$:

$$X=Z_{17}+g_0(\epsilon)Z_m^{-1}+g_1(\epsilon)e^2\ .$$

Z_{17}, to $O(e^2)$, vanishes as seen by an explicit calculation. Z_m^{-1} does have a simple pole in $O(e^2)$. The last term in the above equation has, at worst, a simple pole. Hence the double pole in X is proportional to g_0^{-1}. Thus, X has no worse than simple poles iff $g_0^{-1}=0$.

In $O(e^4)$:

$$X=Z_{17}+g_0(\epsilon)Z_m^{-1}+g_1(\epsilon)e^2Z_e^2Z_m^{-1}+g_2(\epsilon)e^4\ .$$

Z_{17} has only a simple pole to $O(e^4)$. We expand

$$Z_e^{2q}Z_m^{-1}=1+\sum_{l=1}^{\infty}\sum_{p=1}^{l}\frac{Y_p^{ql}e^{2l}}{\epsilon^p}\ . \tag{4.2}$$

Then, the double poles in X are (noting that $g_0^{-1}=0$)

$$\frac{e^4}{\epsilon^2}(g_0^0Y_2^{02}+g_1^{-1}Y_1^{11})\ .$$

As seen from Appendix A neither Y_2^{02} nor Y_1^{11} vanish. Thus, these double poles vanish if

$$g_0^0=-g_1^{-1}\frac{Y_1^{11}}{Y_2^{02}}\ . \tag{4.3}$$

In $O(e^6)$: From this order onward, it is convenient to deal with X indirectly, making use of the theorem in the preceding section. First we require that T has no worse than simple poles:

$$\begin{aligned}T&=\left[\mu\frac{\partial}{\partial\mu}\sum_{n=0}^{\infty}g_n(\epsilon)(e_0^2\mu^{-\epsilon})^n\right]Z_m^{-1}-\beta^\lambda Y\\&=-\epsilon\sum_{n=1}^{\infty}ng_n(\epsilon)(e_0^2\mu^{-\epsilon})^nZ_m^{-1}-\beta^\lambda Y\\&=-\epsilon\sum_{n=1}^{\infty}ng_n(\epsilon)e^{2n}Z_e^{2n}Z_m^{-1}-\beta^\lambda Y\ .\end{aligned} \tag{4.4}$$

Now β^λ (at $\lambda=0$) is $O(e^4)$. Y to $O(e^2)$ has only a simple pole.[1] Hence the second term on the right-hand side has no double poles to this order.

To this order the first term is

$$-\epsilon g_1(\epsilon)e^2Z_e^2Z_m^{-1}-2\epsilon g_2(\epsilon)e^4(Z_e^4Z_m^{-1})-3\epsilon g_3(\epsilon)e^6\ .$$

This again has a double pole coming from the first term. It is proportional to g_1^{-1} (with a nonzero proportionality constant Y_2^{12}; see Appendix A). Hence,

$$g_1^{-1}=0 \tag{4.5}$$

and Eq. (3.2) then implies that

$$g_0^0=0\ . \tag{4.6}$$

Now $X^{(2)}$ to this order is [here $Z_{17,q}^{(p)}$ is the coefficient of e^{2q}/ϵ^p in $Z_{17}(e^2,\epsilon)$]

$$X^{(2)}=Z_{17,3}^{(2)}e^6+g_0^1Y_3^{03}e^6+g_1^0e^6Y_2^{12}+g_2^{-1}e^6Y_1^{21}\ ;$$

$X^{(2)}=0$ implies (4.7)

$$g_0^1Y_3^{03}=-g_1^0Y_2^{12}-g_2^{-1}Y_1^{21}-Z_{17,3}^{(2)}\ .$$

In $O(e^8)$: In this order T is

$$-\epsilon g_1(\epsilon)e^2(Z_e^2Z_m^{-1})-2\epsilon g_2(\epsilon)e^4(Z_e^4Z_m^{-1})-3\epsilon g_3(\epsilon)e^6(Z_e^6Z_m^{-1})-4\epsilon g_4(\epsilon)e^8-\beta^\lambda Y\ .$$

As β^λ is $O(e^4)$, $\beta^\lambda Y$ does not have triple poles to this order, nor do the remaining terms [noting Eq. (4.5)]. Equating the double poles in T to zero one obtains

$$-2g_2^{-1}Y_2^{22}-g_1^0Y_3^{13}=a\tilde{Y}_2^2\ , \tag{4.8}$$

where

$$\beta^\lambda=ae^4+be^6+\cdots$$

and $[Z_{17}(\lambda,e^2,\epsilon)$ is zero in $O(\lambda)$; hence Y has no term of $O(e^0)]$

$$Y=\sum_{r=1}^{\infty}\sum_{p=1}^{r}\frac{\tilde{Y}_p^r e^{2r}}{\epsilon^p} . \tag{4.9}$$

Now, X to this order is

$$X=Z_{17}+g_0(\epsilon)Z_m^{-1}+g_1(\epsilon)e^2Z_e^2Z_m^{-1} +g_2(\epsilon)e^4Z_e^4Z_m^{-1}+g_3(\epsilon)e^6Z_e^6Z_m^{-1}+g_4(\epsilon)e^8 .$$

Double poles in X cancel if

$$0=Z_{17,4}^{(2)}+g_0^2Y_4^{04}+g_1^1Y_3^{13}+g_2^0Y_2^{22} ,$$

i.e., if

$$g_0^2Y_4^{04}=-Z_{17,4}^{(2)}-g_1^1Y_3^{13}-g_2^0Y_2^{22} . \tag{4.10}$$

In $O(e^{10})$: In this order T is

$$-\epsilon g_1(\epsilon)e^2(Z_e^2Z_m^{-1})-2\epsilon g_2(\epsilon)e^4(Z_e^4Z_m^{-1}) -3\epsilon g_3(\epsilon)e^6(Z_e^6Z_m^{-1})-4\epsilon g_4(\epsilon)e^8Z_e^8Z_m^{-1} -5\epsilon g_5(\epsilon)e^{10}-\beta^\lambda Y . \tag{4.11}$$

In view of Eqs. (4.5) and (4.6), no term has a quadruple pole. The triple poles in T cancel iff

$$-g_1^0Y_4^{14}-2g_2^{-1}Y_3^{23}-a\tilde{Y}_3^3=0 . \tag{4.12}$$

The determinant

$$\begin{vmatrix} Y_4^{14} & 2Y_3^{23} \\ Y_3^{13} & 2Y_2^{22} \end{vmatrix} \neq 0$$

(see Appendix A). Hence Eqs. (4.12) and (4.8) uniquely determine g_1^0 and g_2^{-1}. Then Eqs. (4.7) uniquely determines g_0^1.

Double poles in T cancel iff

$$g_1^0Y_3^{14}+g_1^1Y_4^{14}+2g_2^{-1}Y_2^{23}+2g_2^0Y_3^{23}+3g_3^{-1}Y_2^{32} +a\tilde{Y}_2^3+b\tilde{Y}_2^2=0 . \tag{4.13}$$

g_1^0,g_2^{-1} have been determined already and the equation contains three new free parameters g_1^1, g_2^0, g_3^{-1}. We rewrite the equation as

$$g_1^1Y_4^{14}+2g_2^0Y_3^{23}+3g_3^{-1}Y_2^{32}=\text{known} . \tag{4.14}$$

Double poles in X to this order cancel if

$$Z_{17,5}^{(2)}+g_0^2Y_4^{05}+g_0^1Y_3^{05}+g_0^3Y_5^{05}+g_1^0Y_2^{14}+g_1^1Y_3^{14}+g_1^2Y_4^{14} +g_2^{-1}Y_1^{23}+g_2^0Y_2^{23}+g_2^1Y_3^{23}+g_3^{-1}Y_1^{32}+g_3^0Y_2^{32} +g_4^{-1}Y_1^{41}=0 . \tag{4.15}$$

In Eq. (4.15), the coefficients g_0^1, g_1^0, g_2^{-1} are already determined. One could eliminate g_1^1 between Eqs. (4.14) and (4.15). One could substitute for g_0^2 in Eq. (4.15) from Eq. (4.10). One then obtains an equation containing the as yet arbitrary parameters

$$g_0^3,g_1^2,g_2^1,g_3^0,g_4^{-1},g_2^0,g_3^{-1} .$$

Therefore, this constraint can be satisfied in an infinite number of ways. Thus the energy-momentum tensor *can* be made finite by the improvement term of the form assumed in this section to $O(e^{10})$.

Next we shall show that the improvement term of the kind assumed in this section cannot definitely work beyond $O(e^{12})$. To this end we consider the constraints obtained on g_1^0 and g_2^{-1} by the requirement that the quartic poles in T in $O(e^{12})$ and quintic poles in T in $O(e^{14})$ must cancel, in particular, if the program is to work in $O(e^{14})$. This places, as we shall see, constraints on the already fixed parameters g_1^0 and g_2^{-1} which are inconsistent with those placed by Eqs. (4.8) and (4.12).

That in $O(e^{12})$ the quartic poles in T cancel requires

$$-g_1^0Y_5^{15}-2g_2^{-1}Y_4^{24}=a\tilde{Y}_4^4 . \tag{4.16}$$

That in $O(e^{14})$, the quintic poles in T cancel requires

$$-g_1^0Y_6^{16}-2g_2^{-1}Y_5^{25}=a\tilde{Y}_5^5 . \tag{4.17}$$

Consider now the four equations (4.8), (4.12), (4.16), and (4.17). The coefficients occurring in it, viz., Y_p^{1p} $(p\geq 4)$ are related to Y_3^{13} and Y_p^{2p} to Y_2^{22} via renormalization-group equations (see Appendix A). $\tilde{Y}_p^p$ are also related ultimately (see Appendix B) to the simple pole divergences in Z_{17} in orders λe^2 and e^4. We have calculated the simple pole terms in $O(\lambda e^2)$ in Z_{17} but not in $O(e^4)$. We shall treat the latter as an unknown. One thus has four equations in three unknowns: viz.,

$$(g_0^1Y_3^{13}),\ (g_2^{-1}Y_2^{22}),\ \text{and } e^4/\epsilon \text{ terms in } Z_{17} .$$

We have verified that these inhomogeneous equations are inconsistent.

This inconsistency, most probably, is in the set of the first three equations [Eqs. (4.8), (4.12), and (4.16)] itself. To show this requires a tedious calculation of simple pole divergences in $O(e^4)$ in Z_{17}, which we have not done.

V. IMPROVEMENT COEFFICIENT OF THE FORM $\tilde{g}(e^2,\epsilon)$

In this section we shall consider an improvement coefficient $\tilde{g}(e^2,\epsilon)$ which is a finite function of e^2 at $\epsilon=0$. Noting that $g=-\tilde{g}/\epsilon$, g has an expansion

$$g(e^2,\epsilon)\equiv\sum_{n=0}^{\infty}h_n(\epsilon)e^{2n}\equiv\sum_{n=0}^{\infty}\sum_{k=-1}^{\infty}h_n^ke^{2n}\epsilon^k . \tag{5.1}$$

In this case T of Eq. (3.6) is

$$T=-\beta^\lambda Y+\left[\beta^e(e)-\frac{e\epsilon}{2}\right]\frac{\partial}{\partial e}gZ_m^{-1} =-\beta^\lambda Y+\left[\frac{\beta^e(e)}{e}-\frac{\epsilon}{2}\right]\sum_{n=1}^{\infty}2nh_n(\epsilon)e^{2n}Z_m^{-1} . \tag{5.2}$$

We consider X and T, order by order, as in Sec. IV. We shall write, directly, the constraints placed on h_n^k's. They are

$O(e^0)$: no constraint ,

$$O(e^2):\ h_0^{-1}=0 , \tag{5.3}$$

$$O(e^4):\ h_0^0 Y_2^{02} + h_1^{-1} Y_1^{01} = 0\ , \tag{5.4}$$

$$O(e^6):\ h_1^{-1}[\beta_3^e Y_0^{01} - \tfrac{1}{2} Y_2^{02}] = 0\ . \tag{5.5}$$

Here β_3^e is the coefficient of the $O(e^3)$ term in β^e. As the quantity in the square brackets is not zero (see Appendix A) Eq. (5.5) and hence Eq. (5.4) imply that

$$h_1^{-1} = 0 = h_0^0\ , \tag{5.6}$$

$$O(e^8):\ h_1^0[2\beta_3^e Y_2^{02} - Y_3^{03}] + h_2^{-1}[4\beta_3^e Y_1^{01} - 2Y_2^{02}] - a\tilde{Y}_2^2 = 0\ , \tag{5.7}$$

$$h_0^2 Y_4^{04} = -Z_{17,4}^{(2)} - h_1^1 Y_3^{03} - h_2^0 Y_2^{22}\ , \tag{5.8}$$

$$O(e^{10}):\ h_1^0[2\beta_3^e Y_3^{03} - Y_4^{04}] + h_2^{-1}[4\beta_3^e Y_2^{02} - 2Y_3^{03}] - a\tilde{Y}_3^3 = 0\ . \tag{5.9}$$

Equations (4.7) and (4.9) have a unique solution for h_1^0 and h_2^{-1}:

$$h_1^1[2\beta_3^e Y_3^{03} - Y_4^{04}] + 2h_2^0[2\beta_3^e Y_2^{02} - Y_3^{03}] + 3h_3^{-1}[2\beta_3^e Y_1^{01} - Y_2^{02}] = \text{known}\ . \tag{5.10}$$

Equation (5.10) is analogous to Eq. (4.14) and has an infinite number of solutions for h_1^1, h_2^0, h_3^{-1}. In a similar manner, one obtains an equation analogous to Eq. (4.15), which can be satisfied by infinite choices for

$$h_0^3, h_1^2, h_2^1, h_3^0, h_4^{-1}, h_2^0, h_3^{-1}\ .$$

Thus the energy-momentum tensor can be made finite by this kind of improvement program also up to $O(e^{10})$. Next we shall show, in a manner analogous to the previous section, that this kind of an improvement program definitely fails beyond $O(e^{12})$.

Cancellation of quartic poles in T to $O(e^{12})$ requires

$$h_1^0[2\beta_3^e Y_4^{04} - Y_5^{05}] + 2h_2^{-1}[2\beta_3^e Y_3^{03} - Y_4^{04}] = a\tilde{Y}_4^4 \tag{5.11}$$

and cancellation of quintic poles in T to $O(e^{14})$ requires

$$h_1^0[2\beta_3^e Y_5^{05} - Y_6^{06}] + 2h_2^{-1}[2\beta_3^e Y_4^{04} - Y_5^{05}] = a\tilde{Y}_5^5\ . \tag{5.12}$$

As in the previous section it can be verified that Eqs. (5.7), (5.9), (5.11), and (5.12) are inconsistent equations, proving the failure of this kind of a finite improvement program in $O(e^{14})$ and beyond. As noted in the previous section the breakdown takes place most probably in $O(e^{12})$ itself.

APPENDIX A

We have defined [see Eq. (4.2)]

$$Z_e^{2q} Z_m^{-1} = 1 + \sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{Y_s^{qr} e^{2r}}{\epsilon^s}\ . \tag{A1}$$

In this appendix we shall derive the relations among Y_s^{qr} used in the text. β^e is defined by

$$[\beta^e(e) - \tfrac{1}{2}e\epsilon] = \mu\frac{\partial}{\partial\mu}e = \mu\frac{\partial}{\partial\mu}(e_0\mu^{-\epsilon/2}Z_e^{-1}) = -\frac{e\epsilon}{2} - e\mu\frac{\partial}{\partial\mu}\ln Z_e\ .$$

This implies

$$\mu\frac{\partial}{\partial\mu}\ln Z_e = -\frac{\beta^e(e)}{e}. \tag{A2}$$

Further Z_m^{-1} satisfies

$$\mu\frac{\partial}{\partial\mu}\ln Z_m^{-1} = 2\gamma_m(e)\ . \tag{A3}$$

Thus, we have

$$\mu\frac{\partial}{\partial\mu}\ln(Z_e^{2q}Z_m^{-1}) = \left[2\gamma_m(e) - 2q\frac{\beta^e(e)}{e}\right]$$

or equivalently

$$\mu\frac{\partial}{\partial\mu}(Z_e^{2q}Z_m^{-1}) = [\beta^e(e) - \tfrac{1}{2}e\epsilon]\frac{\partial}{\partial e}(Z_e^{2q}Z_m^{-1}) = \left[2\gamma_m(e) - 2q\frac{\beta^e(e)}{e}\right](Z_e^{2q}Z_m^{-1})\ . \tag{A4}$$

Now consider the coefficient of e^{2r+2}/ϵ^r on both sides of Eq. (A4). Defining

$$\beta^e(e) = \beta_3^e e^3 + O(e^5),\quad \gamma_m(e) = \gamma_{m2}e^2 + O(e^4)\ , \tag{A5}$$

we obtain

$$-\tfrac{1}{2}(2r+2)Y_{r+1}^{q,r+1} + \beta_3^e Y_r^{qr}(2r) = (2\gamma_{m2} - 2q\beta_3^e)Y_r^{qr}\ ,$$

i.e.,

$$Y_{r+1}^{q,r+1} = \frac{1}{r+1}[(2r+2q)\beta_3^e - 2\gamma_{m2}]Y_r^{qr}\ . \tag{A6}$$

To obtain Y_r^{qr} using Eq. (A6), one needs to only know Y_1^{q1}. Y_1^{q1} is easily related to β_3^e and γ_{m2}. The result is

$$Y_1^{q1} = (2q\beta_3^e - 2\gamma_{m2})\ . \tag{A7}$$

As seen from the values of β_3^e and γ_{m2} stated in Eq. (B9) Y_r^{qr} is always nonzero (here, $q \geq 0$).

We now deduce a number of related results used in the text:

$$\text{(i)}\quad \begin{vmatrix} Y_4^{14} & 2Y_3^{23} \\ Y_3^{13} & 2Y_2^{22} \end{vmatrix} = 2Y_3^{13}Y_2^{22} \begin{vmatrix} \tfrac{1}{4}(8\beta_3^e - 2\gamma_{m2}) & \tfrac{1}{3}(8\beta_3^e - 2\gamma_{m2}) \\ 1 & 1 \end{vmatrix} \neq 0\ .$$

This has been used below Eq. (4.12).

$$\text{(ii)}\quad \beta_3^e Y_1^{10} - \tfrac{1}{2}Y_2^{02} = Y_1^{01}[\beta_3^e - \tfrac{1}{2}\ \tfrac{1}{2}(2\beta_3^e - 2\gamma_{m2})] = Y_1^{01}[\tfrac{1}{2}\beta_3^e + \tfrac{1}{2}\gamma_{m2}] \neq 0\ .$$

This has been used below Eq. (5.5).

APPENDIX B

We have defined

$$Y=\frac{\partial}{\partial\lambda}Z_{17}\bigg|_{\lambda=0}=\sum_{r=1}^{\infty}\sum_{p=1}^{\infty}\frac{\tilde{Y}_p^r(e^2)^r}{\epsilon^p}\ .$$

We wish to relate the quantities $\tilde{Y}_2^2$, $\tilde{Y}_3^3$, $\tilde{Y}_4^4$, $\tilde{Y}_5^5$ that appear in Eqs. (4.8), (4.12), (4.16), and (4.17) which we wish to show are inconsistent. These relations are obtained via the renormalization-group equation satisfied by Z_{17}, viz.,

$$[-\lambda\epsilon+\beta^{\lambda}(\lambda,e)]\frac{\partial}{\partial\lambda}Z_{17}+\left[\frac{-e\epsilon}{2}+\beta^{e}(\lambda,e)\right]\frac{\partial Z_{17}}{\partial e}-2\gamma_m Z_{17}=Z_{11}\gamma_{17}+Z_{15}\gamma_{57}\ . \tag{B1}$$

We expand X_{17} (showing only the necessary terms)

$$Z_{17}=a'\frac{\lambda e^2}{\epsilon}+b'\frac{\lambda e^4}{\epsilon^2}+c\frac{\lambda e^6}{\epsilon^3}+d\frac{\lambda e^8}{\epsilon^4}+f\frac{\lambda e^{10}}{\epsilon^5}+\frac{ke^4}{\epsilon}+h\frac{e^6}{\epsilon^2}+g\frac{e^8}{\epsilon^3}+m'\frac{e^{10}}{\epsilon^4}+d'\frac{\lambda^2e^6}{\epsilon^4}+f'\frac{\lambda^2e^4}{\epsilon^3}$$
$$+c'\frac{\lambda^2e^2}{\epsilon^2}+j\frac{\lambda^3e^2}{\epsilon^3}+\cdots\ . \tag{B2}$$

We also expand $\beta^{\lambda}(\lambda,e)$, $\beta^{e}(\lambda,e)$, $\gamma_m(\lambda,e)$ as

$$\beta^{\lambda}(\lambda,e)=\lambda^2\frac{\partial}{\partial\lambda}Z_{\lambda}^{(1)}+\frac{e\lambda}{2}\frac{\partial}{\partial e}Z_{\lambda}^{(1)}+\frac{e}{2}\frac{\partial}{\partial e}\delta\lambda^{(1)}-\delta\lambda^{(1)}\equiv\beta_1^{\lambda}e^4+\beta_2^{\lambda}\lambda e^2+\tilde{\beta}_1^{\lambda}\lambda^2+\cdots\ ,$$
$$\beta^{e}(\lambda,e)=\frac{e^2}{2}\frac{\partial}{\partial e}Z_e^{(1)}+e\lambda\frac{\partial Z_e^{(1)}}{\partial\lambda}\equiv\beta_3^{e}e^3+\cdots\ ,\quad \gamma_m(\lambda,e)=\gamma_{m1}\lambda+\gamma_{m2}e^2+\cdots\ . \tag{B3}$$

Next we compare the coefficients of $\lambda e^{10}/\epsilon^4$, e^{10}/ϵ^3, $\lambda e^8/\epsilon^3$, $\lambda e^6/\epsilon^2$, e^8/ϵ^2, e^6/ϵ, $\lambda e^4/\epsilon$, λ^2e^6/ϵ^3, λ^2e^4/ϵ^2, and λ^3e^2/ϵ^2 on both sides of Eq. (B1) and obtain successively

$$-6f+(8\beta_3^e+\beta_2^{\lambda}-2\gamma_{m2})d-2\gamma_{m1}m'+2\beta_1^{\lambda}d'=0\ , \tag{B4}$$

$$\beta_1^{\lambda}c-5m'+(8\beta_3^e-2\gamma_{m2})g=0\ , \tag{B5}$$

$$-5d+(6\beta_3^e+\beta_2^{\lambda}-2\gamma_{m2})c-2\gamma_{m1}g+2\beta_1^{\lambda}f'=0\ , \tag{B6}$$

$$-4c+(4\beta_3^e+\beta_2^{\lambda}-2\gamma_{m2})b'-2\gamma_{m1}h+2\beta_1^{\lambda}c'=0\ , \tag{B7}$$

$$\beta_1^{\lambda}b'-4g+(6\beta_3^e-2\gamma_{m2})h=0\ , \tag{B8}$$

$$\beta_1^{\lambda}a'-3h-2\gamma_{m2}k=0\ , \tag{B9}$$

$$-3b'+(\beta_2^{\lambda}-2\beta_3^e-2\gamma_{m2})a'-2\gamma_{m1}k=0\ , \tag{B10}$$

$$-5d'+3\beta_1^{\lambda}j+(\tilde{\beta}_1^{\lambda}-2\gamma_{m1})c+(2\beta_2^{\lambda}+4\beta_3^e-2\gamma_{m2})f'=0\ , \tag{B11}$$

$$-4j+(2\beta_1^{\lambda}-2\gamma_{m1})c'=0\ , \tag{B12}$$

$$-4f'+(2\beta_2^{\lambda}+2\beta_3^e-2\gamma_{m2})c'+(\tilde{\beta}_1^{\lambda}-2\gamma_{m1})b'=0\ . \tag{B13}$$

This, together with

$$-3c'=(2\gamma_{m1}-\tilde{\beta}_1^{\lambda})a' \tag{B14}$$

obtained in I and the following values calculated explicitly,

$$\beta_1^{\lambda}=\frac{9}{8\pi^2},\quad \tilde{\beta}_1^{\lambda}=\frac{1}{8\pi^2},\quad \beta_2^{\lambda}=-\frac{1}{\pi^2},\quad \gamma_{m1}=\frac{1}{48\pi^2},$$
$$\gamma_{m2}=-\frac{3}{16\pi^2},\quad \beta_3^e=\frac{1}{48\pi^2},\quad a'=\frac{1}{3(16\pi^2)^2}\ , \tag{B15}$$

enables one to express $\tilde{Y}_2^2$, $\tilde{Y}_3^3$, $\tilde{Y}_4^4$, and $\tilde{Y}_5^5$ (denoted by b', c, d, and f, respectively, in the above equations) in terms of "k," the simple pole terms in Z_{17} of $O(e^4)$, a quantity which we have not calculated. This enables one to look upon Eqs. (4.8), (4.12), (4.16), and (4.17) as four equations in three unknowns $(g_0^1Y_3^{13})$, $(g_2^{-1}Y_2^{22})$, and k and verify their inconsistency.

[1]S. D. Joglekar and Anuradha Misra, Phys. Rev. D **38**, 2546 (1988); **39**, 3181(E) (1980). See also references in this paper.

[2]S. D. Joglekar and Anuradha Misra, Phys. Rev. D **39**, 1716 (1989).

[3]Anuradha Misra and S. D. Joglekar, Phys. Rev. D **39**, 1724 (1989).

[4]Anuradha Misra and S. D. Joglekar, Phys. Rev. D **39**, 1730 (1989).

[5]D. Z. Freedman, I. Muzinich, and E. J. Weinberg, Ann. Phys. (N.Y.) **87**, 95 (1974).