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On the finite-temperature quantum electrodynamics of gravitational acceleration

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The temperature-dependent quantum-electrodynamic corrections to the Helmholtz free energy F of a particle at rest, and to its inertial mass m_{inert}, are the same: $\Delta F = \Delta m_{\text{inert}} = \pi e^2 (kT)^2/3m$. By contrast, the correction to the total energy $U = F + TS$ is $\Delta U = -\Delta F$. Donoghue, Holstein, and Robinett have pointed out that if (as the equivalence principle appears to imply) weight is proportional to total energy, then the gravitational acceleration of a particle inside a blackbody cavity becomes $g(m+\Delta U)/(m+\Delta F) \approx g(1-2\Delta F/m) < g$. However, while F represents the random kinetic energy of (and is thereby localized on) the particle, further analysis now suggests that the entropic energy difference $T\Delta S = \Delta U - \Delta F$ is distributed over the cavity uniformly and independently of the particle position. If so, then the gravitational pull on $T\Delta S$ cannot affect the motion of the particle well within the cavity, so that it will, after all, experience the universal Galilean acceleration g.

Consider the quantum-electrodynamic corrections of order e^2 to the motion of a charged particle which is at finite temperature, in the limited sense that it is exposed to blackbody radiation described by Planck's formula.[1] In other words, the average values of the photon occupation numbers are maintained at $\langle n_\lambda \rangle = 1/[\exp(\beta\omega_\lambda) - 1]$ by an external reservoir of heat (i.e., of photons). By contrast, the particle is not in thermal equilibrium. On the contrary, to begin with we take it to be in a state specified by its momentum $\mathbf{p}$. [Later we shall, without further comment, apply the results to particles described by quasiclassical wave packets having a reasonably well-defined (mean) position $\mathbf{r}$ as well as a reasonably well-defined (mean) momentum.[2]]

For the equation of motion one requires the radiative correction ΔF to the Helmholtz free energy of the particle, since at finite temperature work done on the particle by external forces (e.g., electric or gravitational) is registered as a change in F rather than in the total energy U. Conformably with our scenario, F is in fact the free energy of a state of *constrained* equilibrium (the particle is formally constrained to momentum $\mathbf{p}$); all thermodynamic quantities in the paper refer to such constrained equilibrium states, whose thermodynamic status has been discussed in detail elsewhere.[3,4] Further, our radiative shifts $\Delta F, \Delta U$ are defined to vanish at $T=0$; they are unplagued by divergences since we suppose the usual (zero-temperature) renormalizations to have been performed already (so that m, for instance, is the physical mass observed at $T=0$).

For a nonrelativistic particle under such conditions one has

$$\Delta F = \Delta_0(T) - \frac{\mathbf{p}^2}{2m}\frac{\Delta_0(T)}{m} + \cdots , \tag{1}$$

where

$$\Delta_0(T) = \pi e^2 (kT)^2/3m . \tag{2}$$

In fact, provided $kT \ll m$, the total free energy and the momentum are related in a pseudo-Lorentz-invariant way best displayed as

$$[m + \Delta_0(T)]^2 = (F + \Delta F)^2 - \mathbf{p}^2 , \tag{3}$$

even though (3) is warranted by perturbation theory only to leading order in Δ_0 and ΔF.

Equation (3) and the consequent equation (1) exhibit the fact that the shifts in the free energy at rest and in the inertial mass are the same, both being given by Δ_0. Thus, the equation of motion under an applied force $\mathbf{f}$ is

$$(m + \Delta_0) d^2\mathbf{r}/dt^2 = \mathbf{f} .$$

In other words, Hamilton's equations on a classical level and Heisenberg's equations on a quantum level yield $\mathbf{p} = (m+\Delta_0)\dot{\mathbf{r}}$, $\dot{\mathbf{p}} = \mathbf{f}$. All these results have long been established, though recent discussions have tended to put more explicit stress on the status of Δ_0 as a correction specifically to F. Early papers can be traced from Refs. 3 and 4, which also discuss the thermodynamic aspects. Equations (3) and (1) (which break down when $kT \gtrsim m$) will be taken for granted in the present paper. One proof (along with a systematic approach through Feynman diagrams) has been evolved by Donoghue, Holstein, and Robinett[5] (DHR); a more pedestrian one based on the optical theorem, along the lines of Ref. 3, will be given elsewhere. Δ_0 itself will be calculated below.

By virtue of the standard thermodynamic relation

$$U = (1 - T\partial/\partial T)F , \tag{5}$$

Eqs. (1) and (2) entail for the total energy at rest the shift

$$\Delta U = (1 - T\partial/\partial T)\Delta F = -\Delta F = -\Delta_0 . \tag{6}$$

[The momentum- or velocity-dependent contributions to ΔU are too small (i.e., of too high an order in $1/m$) to affect the subsequent argument.]

In light of these results, DHR (Refs. 6 and 7) have made the following most remarkable and stimulating observation about the way such a particle would fall under

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gravity (e.g., about the fall of an electron inside a blackbody cavity on the surface of the Earth). If one assumes (as at first sight seems almost self-evident) that the gravitational pull acts on the total energy U (rather than on F), then the weight of the particle is $(m+\Delta U)g=(m-\Delta_0)g$, and the equation of motion (with z measured vertically upward) is

$$(m+\Delta_0)d^2z/dt^2=-(m-\Delta_0)g \ ,$$
$$\frac{d^2z}{dt^2}=-\frac{m-\Delta_0}{m+\Delta_0}g\approx -g(1-2\Delta_0/m) \ . \tag{7}$$

As DHR put it,[7] this suggests that "Aristotle was right: heavier objects fall faster." [This at least is what (7) entails for charged particles in principle: in practice, the effect is far too small to be measured.]

DHR's view of the problem appears natural from an approach via Feynman diagrams, which tend to direct attention to plane-wave states, i.e., to totally delocalized particles, and which do not, therefore, automatically challenge one to investigate the distribution in space of the various energies involved. In fact ΔF is accounted for by the extra kinetic energy of the particle due to rapid vibrations driven by the random electric field of blackbody radiation. These vibrations are centered on the ordinary (formally, zero-temperature) trajectory of the particle, and are established practically at once when the particle enters the cavity [in sharp contrast to the enormous time scale needed to establish the true (unconstrained) equilibrium[2]]. The relevant point is that all the energy represented by ΔF is localized at the particle: the two necessarily move together. That $\Delta F=\Delta_0$ indeed consists of kinetic energy is shown in the Appendix.

By contrast, as regards the difference

$$\Delta U-\Delta F=T\Delta S \ , \tag{8}$$

a more pedestrian alternative[3,8] to the diagrammatic approach suggests that $T\Delta S$, the entropic part of ΔU, is totally delocalized, in the sense that it is spread out essentially uniformly through the volume V of the cavity or quantization volume. The argument will be given presently. If the conclusion is correct, then the spatial distribution of $T\Delta S$ remains unchanged as the particle falls. Hence, the gravitational pull undoubtedly exerted on the energy $T\Delta S$ would perhaps be felt by the walls of the cavity or by the heat reservoir responsible for maintaining the Planck distribution; but the pull would not be transmitted to the particle itself while the particle remains well within the cavity. Accordingly, its fall would be governed not by (7), but by the familiar equation

$$(m+\Delta_0)d^2z/dt^2=-(m+\Delta_0)g \ ,$$
$$d^2z/dt^2=-g \ . \tag{9}$$

Thus the particle would fall normally, i.e., according to Galileo rather than Aristotle: in the language of DHR (Refs. 6 and 7), one would not after all be faced with a violation (whether real or merely apparent) of the equivalence principle.

It remains to verify the uniform spatial distribution of the energy $T\Delta S$ (i.e., of the entropy). The argument will automatically supply a convenient form of Δ_0. Start from the radiative shift $\delta E(i,\{n_\lambda\})$ of the pure state which in zero order (in the absence of any radiative coupling) is specified as having the particle in an energy eigenstate $|i\rangle$, and the Maxwell field likewise in an eigenstate $|\{n_\lambda\}\rangle$ of the set $\{n_\lambda\}$ of all photon occupation numbers. The shift δE can be calculated either by conventional perturbation theory[9] (e.g., nonrelativistically from the coupling $-e\,\mathbf{A}\cdot\mathbf{p}/m+e^2\mathbf{A}^2/2m$), or via a shortcut through the optical theorem.[3,8] The results are the same:

$$\delta E(i,\{n_\lambda\})=\sum_\lambda n_\lambda \delta\omega_\lambda^{(i)} \ , \tag{10}$$

with

$$\delta\omega_\lambda^{(i)}=-2\pi f^{(i)}(\omega_\lambda)/\omega_\lambda V \ . \tag{11}$$

Here, $f^{(i)}$ is the polarization-averaged forward-scattering amplitude of light from the particle in state $|i\rangle$. Evidently $\delta\omega_\lambda^{(i)}$ could be envisaged as the frequency shift of the field oscillator λ due to the particle. (We consider only particles far enough from the cavity walls for the usual position-dependent image forces to be negligible.)

In our case the index i specifies the momentum of the particle. To apply Eqs. (1)–(3), one need evaluate only Δ_0 (i.e., ΔF for a particle at rest), so that $f^{(i)}=-e^2/m$ is just the forward Thomson amplitude:

$$\delta\omega_\lambda=2\pi e^2/m\omega_\lambda V \ . \tag{12}$$

Standard thermodynamic perturbation theory[10] now yields ΔF at once, as the average of the level shifts over the *unperturbed* canonical ensemble (constrained to given $\mathbf{p}$). Thus ΔF is found simply by replacing the n_λ in (10) by their thermal averages $\langle n(\omega_\lambda)\rangle$:

$$\Delta F=\sum_\lambda \langle n(\omega_\lambda)\rangle\delta\omega_\lambda=\sum_\lambda \frac{\delta\omega_\lambda}{e^{\beta\omega_\lambda}-1} \ . \tag{13}$$

From this one obtains $\Delta F=\Delta_0$ as in (2) by substituting for $\delta\omega_\lambda$ from (12) and performing $\sum_\lambda$ (Ref. 1).

The relation between ΔU and ΔF now becomes obvious and explicit. Because the total energy U is the expectation value of the perturbed energies over the *perturbed* canonical ensemble, one has

$$(U+\Delta U)=\sum_\lambda \langle n(\omega_\lambda+\delta\omega_\lambda)\rangle(\omega_\lambda+\delta\omega_\lambda) \ ,$$

whence

$$\Delta U=\sum_\lambda [\langle n(\omega_\lambda)\rangle\delta\omega_\lambda+\omega_\lambda\delta\langle n(\omega_\lambda)\rangle] \tag{14a}$$
$$=\Delta F+T\Delta S \ . \tag{14b}$$

Here $\Delta F=\sum_\lambda\langle n(\omega_\lambda)\rangle\delta\omega_\lambda$ as in (13), while

$$T\Delta S=\sum_\lambda \omega_\lambda\delta\langle n(\omega_\lambda)\rangle=\sum_\lambda \omega_\lambda\frac{\partial\langle n(\omega_\lambda)\rangle}{\partial\omega_\lambda}\delta\omega_\lambda \ . \tag{15}$$

Since $\omega\partial\langle n\rangle/\partial\omega=\beta\partial\langle n\rangle/\partial\beta=-T\partial\langle n\rangle/\partial T$, the relations (5) and (6) are, of course, satisfied identically.

Next, we observe that the energy $T\Delta S$ stems wholly from a change of photon numbers: the reservoir responds to the introduction of the particle by reabsorbing a number of photons, counted by the $|\delta\langle n\rangle|$ [note that $\partial\langle n\rangle/\partial\omega$ is negative].

The argument concludes by recalling that photons are just excitations of the normal modes (of the coupled system cavity plus particle); and that their energy densities, governed by the normal-mode amplitudes, are essentially uniform throughout the cavity (provided only that the temperature is not too low, i.e., that the dominant wavelengths $2\pi/K_{\max}\sim 2\pi/kT$ are much shorter than the linear dimensions of the cavity). We say "essentially uniform" because, in principle, though in fact only marginally, the amplitudes are, of course, affected by the location of the particle, just as they would be by small changes in the cavity shape. The point is that, the energy $T\Delta S$ being wholly due to the presence of the particle, small nonuniformities in its distribution that are themselves due to the particle are only a negligible secondary effect. Such effects are of the same relative order as are any other deviations from the asymptotics dictated by Weyl's theorem (number of normal modes proportional to V). It may help to forestall confusion to contrast $T\Delta S$ in this respect with the shift ΔF: Eq. (13) shows that ΔF stems from a small relative shift in the energy of the preexisting photons (present even when the particle is not); and we know *a posteriori* that *these* small relative corrections, far from being uniformly distributed, actually do reside on the particle itself, constituting as they do a shift in its kinetic energy.

To sum up, we have outlined a plausibility argument suggesting that the distribution in space of the entropic difference $T\Delta S$ between the energies ΔU and ΔF is independent of the position of the particle within the cavity. If so, then the gravitational force on $T\Delta S$ cannot contribute to the downward pull on the particle, which would, therefore, fall according to (9), i.e., with the ordinary, universal, Galilean accelaration g (Ref. 11).

Of course, a wholly convincing demonstration would need to introduce the local stress tensor, and then to analyze explicitly the forces acting on an arbitrarily small region that contains the particle. Neither our argument nor the earlier contrary suggestion by DHR (Refs. 6 and 7) can confidently preempt such an analysis. As compared with DHR, we would claim only that our formalism does at least invite the question of the possibly different localizations of ΔF and of $T\Delta S$, while in their diagrammatic method, though in many respects it is more flexible than ours, this question has not, so far, been raised at all. Since the question, though irrelevant to feasible experiments, is challenging in principle, it is to be hoped that quantum field theorists better equipped than the present writer to tackle the full calculation will soon resolve it conclusively.[12]

APPENDIX: IDENTIFICATION OF ΔF AS KINETIC ENERGY

Since ΔF is a (canonical) average of the level shifts δE defined just above (10), we need merely identify δE itself as an expectation value (in a pure state) of the kinetic energy operator. Recall that $\Delta F=\Delta_0$ relates to a particle at rest (in zero order, i.e., until driven to vibrate by the blackbody electric field). The zero-order Hamiltonian H_0 and the coupled Hamiltonian H are (in the Coulomb gauge)

$$H_0=\mathbf{p}^2/2m+H_{\rm rad}\ , \tag{A1}$$

$$H=\pi^2/2m+H_{\rm rad}\ , \tag{A2a}$$

$$\pi\equiv(\mathbf{p}-e\,\mathbf{A})\ , \tag{A2b}$$

$$H=H_0+H_{\rm int}^{(1)}+H_{\rm int}^{(2)}\ , \tag{A3a}$$

$$H_{\rm int}^{(1)}=-e\,\mathbf{A}\cdot\mathbf{p}/m,\quad H_{\rm int}^{(2)}=e^2\mathbf{A}^2/2m\ . \tag{A3b}$$

Here $H_{\rm rad}=\int dV(\mathbf{E}_{\rm rad}^2+\mathbf{B}_{\rm rad}^2)/8\pi=\sum_\lambda\omega_\lambda n_\lambda$; the zero-point energies have been dropped because they are irrelevant to the argument. With $[a_\lambda,a_{\lambda'}^\dagger]=\delta_{\lambda\lambda'}$, $n_\lambda=a_\lambda^\dagger a_\lambda$, $\lambda\equiv(\mathbf{k},s)$, and $\mathbf{K}\cdot\epsilon_s(\mathbf{K})=0$, $\epsilon_s^2=1$, we have[1]

$$\mathbf{A}=\sum_\lambda(a_\lambda\mathbf{A}_\lambda+a_\lambda^\dagger\mathbf{A}_\lambda^*)\ , \tag{A4a}$$

$$\mathbf{A}_\lambda=\left[\frac{2\pi}{VK}\right]^{1/2}\epsilon_s(\mathbf{K})\exp(i\mathbf{K}\cdot\mathbf{r})\ . \tag{A4b}$$

Since our particle is at rest ($\mathbf{p}=0$), $H_{\rm int}^{(1)}$ annihilates the zero-order state vector $|\mathbf{p}=0\rangle|\{n_\lambda\}\rangle$, and drops out of the proceedings altogether; by the same token, there are no single-photon admixtures. But then, to order e^2, our result follows trivially. On one hand, the energy shift is just the zero-order expectation value of $H_{\rm int}$, which now reduces to $\langle H_{\rm int}^{(2)}\rangle=\langle e^2\mathbf{A}^2/2m\rangle$. On the other hand, the kinetic-energy operator is $\pi^2/2m$, and (given $\mathbf{p}=0$) its expectation likewise reduces to $\langle e^2\mathbf{A}^2/2m\rangle$. Thus the entire perturbative shift is indeed identically the same as the expectation value of the kinetic energy. Evaluating it one finds

$$\langle\pi^2/2m\rangle=\frac{e^2}{2m}\langle\{n_\lambda\}|\mathbf{A}^2|\{n_\lambda\}\rangle$$
$$=\frac{e^2}{2m}\sum_\lambda 2n_\lambda|\mathbf{A}_\lambda|^2=\frac{e^2}{m}\sum_\lambda n_\lambda|\mathbf{A}_\lambda|^2\ . \tag{A5}$$

The thermal average $[n_\lambda\to(e^{\beta\omega_\lambda}-1)^{-1}]$ then becomes[1]

$$\frac{2e^2}{m\pi}\int_0^\infty\frac{dK\,K}{e^{\beta K}-1}=\pi e^2/3m\beta^2\ , \tag{A6}$$

reproducing Δ_0 as given by (2).

The same expression emerges from a classical model featuring the normal-mode electric field amplitude $\mathbf{E}_\lambda$. In response to $\mathbf{E}_\lambda$ a classical particle vibrates at frequency $\omega_\lambda=K_\lambda$ with an amplitude $\mathbf{x}_\lambda=-e\mathbf{E}_\lambda/m\omega_\lambda^2$. The resulting time-averaged kinetic energy T_λ is

$$T_\lambda=\tfrac{1}{2}m|\dot{\mathbf{x}}_\lambda|^2=\frac{e^2}{2m}\frac{|\mathbf{E}_\lambda|^2}{\omega_\lambda^2}\ . \tag{A7}$$

By considering the energy density of the normal mode one can see that $\omega_\lambda n_\lambda/V=2|\mathbf{E}_\lambda|^2/8\pi$, whence

$$|E_\lambda|^2/\omega_\lambda^2 = 4\pi n_\lambda/\omega_\lambda V\ . \tag{A8}$$

It is worth noting that both sides are Lorentz scalars.

Substitution into (A7) and summation over normal modes yields

$$\sum_\lambda T_\lambda = \frac{2\pi e^2}{m}\frac{1}{V}\sum_\lambda \frac{n_\lambda}{\omega_\lambda}\ , \tag{A9}$$

which is precisely the level shift given by (10) and (12). A direct connection with (A5) can be made via $\mathbf{E}_\lambda = -\dot{\mathbf{A}}_\lambda = i\omega_\lambda \mathbf{A}_\lambda$.

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[1]The particle has charge e and mass m. We use natural and unrationalized Gaussian units: $\hbar = 1 = c, e^2 \approx \frac{1}{137}$. Only nonrelativistic particles are considered, and only temperatures such that $kT \ll m$. Photon normal modes are defined as usual in a (real or fictitious) quantization volume V. They are labeled by λ, which embraces both the polarization index s and the wave vector $\mathbf{K}$, with $K_\lambda \equiv \omega_\lambda$. Sums over normal modes are written $\sum_\lambda$; since blackbody radiation is isotropic and unpolarized, we have $V^{-1}\sum_\lambda = 2(2\pi)^{-3}\int d^3K = \pi^{-2}\int_0^\infty dK\, K^2$.

[2]J. Wainwright (Sussex M. Phil. thesis, 1988) has described, through a Langevin-type equation, how such particles do eventually reach equilibrium, with $\langle \mathbf{p}\rangle = 0$ and $\langle \mathbf{p}^2\rangle = 3mkT$. However, by then one's interest in their dynamics is much diminished [and not only because the relaxation time is $\tau = 135m^3/32\pi^3 e^4 (kT)^4 \approx 5\times 10^{11}(300/T)^4$ sec for electrons, and the original observer long since dead].

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[11]In their 1986 paper (Ref. 6) DHR suggest and evaluate an additional effect, ascribed to a supposed vertical variation of the equilibrium blackbody temperature T [and thereby of $\Delta_0(T)$], due to the variation of the gravitational potential. If operative, this effect would upset the Galilean result (9) once again. However, to the present writer the argument seems to depend, essentially, on the maintenance of strict *local* equilibrium within the cavity. If the cavity is empty of matter (except for the particle under study), the radiation behaves like a dilute gas with mean free path much longer than the cavity dimensions; consequently there is no local equilibrium, and DHR's argument does not apply. We shall ignore the effect elsewhere in the present paper; likewise we ignore any possible consequences of the variation of the equilibrium temperature up the walls of the cavity. Indeed DHR themselves ignore all this in their 1987 (i.e., later) paper (Ref. 7).

[12]Though our reasoning suggests conclusions different from DHR's, it should be stressed that we have not managed to identify just where theirs fails (if indeed it does). However, by hindsight and very tentatively, the physics of our approach might direct suspicion to two specific points, both concerned with subtleties apparently inoffensive in other problems. (i) Since the spatial distribution of the energy is of the essence, it is perhaps conceivable that diagrammatic expressions routinely evaluated for plane waves require some unconventional adaptation to particles realistically described by wave packets. (If so, it would simply be by luck that our more elementary approach can dispense with such elaboration.) Put more technically, DHR's $q = 0$ (zero-momentum graviton) limit might be unexpectedly accident prone (nonuniform?) in the presence of the heat bath. (ii) The physically crucial diagram is evidently the vertex coupling the graviton to the photon. But in the real-time formalism natural to the problem, precisely this diagram suffers, at $q = 0$, from a mathematically undefined coincidence of singularities (Ref. 6). DHR define their answer by an analytic continuation from the imaginary-time formalism. Their procedure, though popular, entails an element of choice, which may need to be exercised unconventionally in order to fit the physics of this particular problem.