Solvable two-dimensional models and the Virasoro algebra

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The Poisson brackets of various solvable two-dimensional models are specified by the Virasoro algebra. As a result, their equations of motion result from appropriate evolution equations. These models share an infinite number of conserved quantities and the same central charge and are related by suitable changes of dynamical variables. In the quantum version, the conserved quantities are modified differently but the central charge is common.

I. INTRODUCTION

Conformal field theory in two dimensions (2D) plays an important role in string theory and solvable 2D models.[1] We have noted in the past that the solution of various classical nonlinear 2D models can be obtained from the condition that the appropriate curvature two-form $\Omega=0$ (Ref. 2). This suggests that these models are closely related among themselves; here we further explore the close relations among the models discussed in Ref. 2. The condition for solvability is the existence of an infinite number of conserved quantities.[3,4] We focus our attention on these conserved quantities and the central charge c in the classical version of the Virasoro algebra which specifies the Poisson brackets.

Here is an outline of our paper. Following Gervais,[3] we begin with the Poisson brackets of the Virasoro generators L_n and obtain the Poisson brackets of their Fourier transform $u(x)$. The Korteweg–de Vries (KdV) equation for the amplitude $u(x)$ then follows from the evolution equation given a suitable Hamiltonian. The infinite set of commuting conserved quantities involved (which include the Hamiltonian) and the central charge are obtained. We proceed to transform amplitudes from $u(x)$ to $p(x)$ amplitudes via $u(x)=p^2(x)+p_x(x)$, where the subscript denotes a derivative with respect to the x variable, and obtain the Poisson brackets of $p(x)$. As a consequence, the modified Korteweg–de Vries (MKdV) equation, sine-Gordon (SG) equation,[5] and Liouville (L) equation are obtained from the evolution equation for a suitable Hamiltonian. When $v(x)=p^2+ip_x$ is formed, another form of the KdV equation follows. The conserved quantities and c are shared by the KdV, MKdV, SG, and L models. Finally, the additional transformation $p(x)=\psi_x(x)+\psi^\dagger(x)$ is made, and the nonlinear Schrödinger (NLS) equation is obtained from the evolution equation. Comments are made on the ϕ^4 model and the quantum version of the aforementioned models.

We wish to emphasize that the Virasoro algebra employed below is not a degeneracy symmetry; i.e., its generators do not represent charges that commute with the Hamiltonian. Instead, they constitute the dynamical variable (amplitude) of the system, and the algebra serves to specify its dynamics through their nontrivial commutators with the Hamiltonian.

II. CLASSICAL THEORY

The Virasoro algebra for classical systems is expressed in terms of the Poisson brackets

$$i\{L_n,L_m\}=(n-m)L_{n+m}+\tfrac{1}{12}c(n^3-n)\delta_{n+m,0}\ , \tag{1}$$

where L_n are the Virasoro generators and c is the central charge.

We introduce the field $u(x)$, when the time dependence is suppressed:

$$u(x)=2\hbar\sum_{n=-\infty}^{\infty}L_n e^{-inx}-\tfrac{1}{4}\ , \tag{2}$$

where the scale factor $2\hbar$ is a parameter chosen to yield the appropriate commutation relation in the quantized field theory, and the constant $\frac{1}{4}$ is included to remove the δ' term from the Poisson brackets of $u(x)$ that results from (1). The periodic boundary condition $u(t,x+2\pi)=u(t,x)$ is imposed throughout, and it is assumed that $u(x)$ has continuous x derivatives of any order. From (1) and (2), we obtain,[3] with the aid of $\delta'(x)=-\delta(x)/x$,

$$\{u(x),u(y)\}=2\pi\hbar[-\delta'''(x-y)+2u(x)\delta'(x-y)+2u(y)\delta'(x-y)]\ , \tag{3}$$

where the primes on the δ function are derivatives with respect to x, provided the central charge is chosen to be

$$c=3/\hbar\ . \tag{4}$$

Observe the formal analogy of Eq. (3) with the quantum commutator of stress-energy tensors, which also represent the Virasoro algebra in the context of conformal field theory.

The infinite set of conserved quantities H_n of the KdV model are given by[3,6,7]

$$H_n=\frac{1}{4\pi\hbar}\int_0^{2\pi}dx\ w_{2n+1}(u),\quad n\ge 0\ , \tag{5}$$

where

$$w_1=u,\quad w_{n+1}=-\sum_{r=1}^{n-1}w_r w_{n-r}-w_n'\ . \tag{6}$$

The w_2 to w_5 that result from (6) are

$$w_2=-u',\quad w_3=u''-u^2\,,$$
$$w_4=-u'''+4uu',\quad w_5=u''''-4uu''-3u'^2+2u^3\,. \tag{7}$$

It is shown that[3,6,7]

$$\{H_n,H_m\}=0\,. \tag{8}$$

Indeed, the KdV equation follows from the evolution equation

$$u_t=\{u,H\}\,, \tag{9}$$

provided H_1 is picked as the Hamiltonian so that

$$u_t=\left\{u,\frac{1}{4\pi\hbar}\int dx\,-u^2\right\}=u_{xxx}-6u(x)u_x(x)\,. \tag{10}$$

We pause here and remark on the physical significance of the conserved properties of the KdV equation. The equation of motion of the KdV Lagrangian[8]

$$\mathcal{L}=\tfrac{1}{2}\phi_x\phi_t-\phi_x^3-\tfrac{1}{2}\phi_{xx}^2$$

yields

$$\phi_{xt}-6\phi_x\phi_{xx}-\phi_{xxxx}=0$$

or, for $u=\phi_x$,

$$u_t-6uu_x-u_{xxx}=0\,.$$

This is another form of the KdV equation that follows from (10) upon $u\to -u$. The Hamiltonian density for $\mathcal{L}$ is

$$H=\phi_x^3+\tfrac{1}{2}\phi_{xx}^2=u^3+\tfrac{1}{2}u_x^2\,.$$

Therefore the integral corresponds to H_2 of (5):

$$H_2=\frac{1}{4\pi\hbar}\int dx\,w_5=\frac{1}{2\pi\hbar}\int dx(u^3+\tfrac{1}{2}u_x^2)\,.$$

The KdV equation describes shallow water waves, where the amplitude u is proportional to the depth. The H_2 expresses the conservation of energy of shallow water waves. On the other hand, H_0 and H_1 express conservation of mass and horizontal momentum, respectively.[8] The KdV equation is invariant under the scale transformation $u\to\lambda^2U$, $t\to T/\lambda^3$, $x\to X/\lambda$, $\varphi\to\lambda\phi$, as this symmetry holds for the evolution equation (9) and the Poisson brackets (3).

In order to derive the MKdV equation and the SG equation from the evolution equation (9), introduce the variable[3] $u(x)=p^2+p_x$ and obtain the following Poisson brackets of $p(x)$ from (3):

$$\{p(x),p(y)\}=2\pi\hbar\delta'(x-y),\quad p(x)=p(x+2\pi)\,. \tag{11}$$

Consequently the Poisson brackets (11) is also equivalent to the Virasoro algebra.

We substitute $u(x)=p^2+p_x$ in the H_1 of (5) and to obtain

$$H_1=-\frac{1}{4\pi\hbar}\int_0^{2\pi}dx(p^4+p_x^2)\,. \tag{12}$$

Then, from (9) with u supplanted by p, and (11), the MKdV equation follows:

$$p_t=\{p,H_1\}=p_{xxx}-6p^2p_x\,. \tag{13}$$

The conserved quantities of the MKdV model $H_n(p)$ can be obtained from (5) and (9) or from

$$Y_{n+1}=-\partial_x Y_n-p\sum_{k=1}^{n-1}Y_kY_{n-k},\quad Y_1=p\,, \tag{14}$$

$$H_n=\frac{1}{4\pi\hbar}\int_0^{2\pi}dx\,pY_{2n-1}(p)\,. \tag{15}$$

In order to obtain the SG equation in the light-cone frame, we identify the field $\phi(x)$ with

$$p(x)=\sqrt{\hbar}\partial_x\phi\,. \tag{16}$$

The Poisson brackets of $p(x)$ and $\phi(y)$ follows from (11) and (16):

$$\{p(x),\phi(y)\}=-\sqrt{\hbar}2\pi\delta(x-y)\,,$$

or, generally,

$$\{p(x),\phi^n(y)\}=-2\pi\sqrt{\hbar}n\phi^{n-1}\delta(x-y)\,. \tag{17}$$

The Hamiltonian

$$H_{\rm SG}=\frac{1}{8\pi\hbar}\int dy(1-\cos\beta\phi),\quad \beta=2\sqrt{\hbar} \tag{18}$$

and $p_t=\{p,H_{\rm SG}\}$ with (17) lead to the SG equation

$$\partial_{tx}\phi=-\frac{1}{\beta}\sin(\beta\phi)\,. \tag{19}$$

The conserved quantities of the SG and MKdV equation are obtained from the conserved quantities of the KdV given by (5) in the following way.

We substitute $u\to -v$ in (7) and define

$$H_n(-v)\equiv K_{n-1}[v(x)=p^2+ip_x]\,. \tag{20}$$

It is straightforward to show

$$\left\{p^2+ip_x,\int dy\,e^{i\beta\phi(y)}\right\}=0\,. \tag{21}$$

If F is an integral of a polynomial of $v,v_y,v_{yy},\dots$ and real, i.e., $F(v)=F^\dagger(v)$, (21) implies[5]

$$\left\{F(v),\int dy\,e^{i\beta\phi(y)}\right\}=\left\{F(v),\int dy\,e^{-i\beta\phi(y)}\right\}$$
$$=0\,. \tag{22}$$

Then $\{F(v),H_{\rm SG}\}=0$, and

$$(K_n)_t=\{K_n(v),H_{\rm SG}\}=0\,. \tag{23}$$

$K_n(v)$ are the conserved quantities and one obtains the commutation relation of v by putting $u\to -v$ in (3):

$$\{v(x),v(y)\}=-2\pi\hbar[\delta'''(x-y)+2v(x)\delta'+2v(y)\delta']\,. \tag{24}$$

Another form of the KdV equation is obtained from the evolution equations (9) and (24) for the Hamiltonian K_1:

$$v_t=\{v,K_1\}=v_{xxx}+6vv_x \ . \quad (25)$$

In a similar way, we obtain the Liouville equation in the light-cone frame with the Hamiltonian

$$H_{\rm L}=-\frac{1}{8\pi\hbar}\int e^{\beta\phi}dy \ , \quad (26)$$

which satisfies

$$(K_n)_t=\{K_n(v),H_{\rm L}\}=0 \ . \quad (27)$$

From the evolution equation, one obtains

$$p_t=\sqrt{\hbar}\partial_{xt}\phi=\{p,H_{\rm L}\}=\tfrac{1}{2}e^{\beta\phi} \ ,$$

or

$$\partial_{xt}\phi=\frac{1}{\beta}e^{\beta\phi} \ . \quad (28)$$

In the previous classical models, the central charge has been chosen to be $c=3/\hbar$.

In order to introduce the classical nonlinear Schrödinger (NLS) equation we normalize (11) with a factor i on the right-hand side as we anticipate an extension to commutators,

$$\{p(x),p(y)\}=i2\pi\hbar\delta'(x-y) \ , \quad (29)$$

and further substitute $p(x)=\psi_x(x)+\psi^\dagger(x)$, where $p(x)$ is not Hermitian, in (29) and find the following Poisson brackets[9] consistent with (29):

$$\begin{aligned}&\{\psi(x),\psi^\dagger(y)\}=i\pi\hbar\delta(x-y) \ ,\\ &\{\psi(x),\psi(y)\}=\{\psi^\dagger(x),\psi^\dagger(y)\}=0 \ , \quad 0\le x\le 2\pi \ .\end{aligned} \quad (30)$$

The conserved quantities H_n are obtained similarly from (14) and (15): i.e.,

$$Y_{n+1}=-\partial_x Y_n-K\sum_{k=1}^{n-1}Y_k Y_{n-k}\psi, \quad Y_1=\psi^\dagger \ , \qquad n=1,2,3,\ldots, \quad (31)$$

$$H_n=\frac{1}{\pi\hbar}\int_0^{2\pi}dx\ Y_n\psi \ , \quad (32)$$

where K is a real positive parameter, and Y_n is a polynomial in $\psi,\psi^\dagger$ and their derivatives.

The H_n for $n=1,2,3,4$ that result from (31) and (32) are[10]

$$\begin{aligned}&H_1=\int dx\ \psi^\dagger\psi/\pi\hbar \ ,\\ &H_2=-\int dx\ \psi_x^\dagger\psi/\pi\hbar \ ,\\ &H_3=-\int dx(\psi_x^\dagger\psi_x+K\psi^{\dagger 2}\psi^2)/\pi\hbar \ ,\\ &H_4=\int dx(\psi_{xx}^\dagger\psi_x+3K\psi^\dagger\psi_x\psi^2)/\pi\hbar \ .\end{aligned} \quad (33)$$

If we choose H_3 as the Hamiltonian $H_{\rm NL}$, the evolution equation yields the NLS equation

$$\dot{\psi}=\{\psi,H_3\}=-i(-\psi_{xx}+2K\psi^\dagger\psi^2) \quad (34)$$

or

$$i\dot{\psi}+\psi_{xx}-2K_\psi^\dagger\psi^\dagger=0 \ . \quad (35)$$

To the extent that the Poisson brackets (30) is obtained from a suitable change of variables of the Virasoro algebra (3), the corresponding central charge is identical with that of the aforementioned models, i.e., $c=3/\hbar$. However, the conserved quantities (33) are different.

We comment on the ϕ^4 model that is also solvable.[2] If we choose as the Hamiltonian

$$H=\frac{1}{32\pi}\int_0^{2\pi}(\phi^4-2\phi^2)dy$$

then

$$p_t=\{p,H\}=\sqrt{\hbar}\phi_{xt}$$

yields, with the aid of (17),

$$4\phi_{xt}+\phi^3-\phi=0 \ .$$

In this case, however, it is not possible to obtain an infinite number of conserved quantities. This may be traceable to the fact that in two dimensions the ϕ^4 theory is not conformally invariant.

III. QUANTUM THEORY

The canonical quantization of the $p(x)$ field is obtained by replacing the Poisson brackets (11) by the commutation relation

$$[p(x),p(y)]=i2\pi\hbar\delta'(x-y) \ . \quad (36)$$

If we expand $p(x)$, which we take to be Hermitian, in a Fourier series[11]

$$p(x)=\sum_{n=-\infty}^{\infty}a_n e^{inx}, \quad a_n^\dagger=a_{-n} \ , \quad (37)$$

we obtain

$$[a_n,a_m]=n\hbar\delta_{n+m,0} \ . \quad (38)$$

The a_n is a creation operator for $n<0$ and an annihilation operator for $n>0$. The vacuum $|0\rangle$ is defined as $a_n|0\rangle=0$, $n>0$. The normal ordering designated by :: is specified by putting all annihilation operators to the right of creation operators.

The quantum SG equation, for example, is formulated by introducing the ϕ field via (16) and expanding it in the complex z plane:

$$\phi(z)=q-ia_0\ln z+i\sum_{n\neq 0}\frac{a_n}{n}z^{-n} \ , \quad (39)$$

where

$$[q,a_n]=i\hbar\delta_{n,0} \ . \quad (40)$$

In the classical case, each Poisson-brackets operation of a polynomial of fields and their derivatives acts once, leading to $\{P,Q\}=\hbar R$. In the quantum case, because of the multiple application of commutation relations, one obtains the more general form $[:P:,:Q:]=\hbar^n:R_n:$. That is to say, terms with higher powers of $\hbar$ and singularities appear. It is shown in Ref. 9 for the NLS that an infinite set of quantum polynomial conserved quantities $\overline{H}_n$ corresponding to the classical case H_n can be constructed, of the form[11]

$$\overline{H}_n = :H_n: + \sum_{k=1} \hbar^k :H_n^k: , \tag{41}$$

such that

$$[\overline{H}_n(x), \overline{H}(y)] = 0 . \tag{42}$$

The added terms are quantum corrections.

We return to the SG model where the Hamiltonian $H_{\rm SG}$ of (18) is defined in the quantum version as

$$H = \frac{1}{\beta^2}\frac{1}{4\pi i}\int \frac{dz}{z}(2 - e^{i\beta\phi(z)} - c^{-i\beta\phi(z)}) \tag{43}$$

with $\beta = 2\sqrt{\hbar}$. It is shown in Ref. 5 that there is an infinite number of corresponding conserved quantities $\overline{H}_n$, and that the central charge is

$$c = 1 - \frac{12}{\hbar}\lambda^2, \quad \lambda = \frac{1}{\beta}(1 - \tfrac{1}{2}\beta^2\hbar) \tag{44}$$

or

$$c = 13 - \frac{12}{\hbar\beta^2} - 3\hbar\beta^2 . \tag{45}$$

The Liouville equation can be incorporated by the quantum version of $H_{\rm L}$ given in (26).

The central charge for the Liouville model was obtained in Ref. 12. We make the adjustment of their parameter $\beta_c \to i/\beta\sqrt{\hbar}$, and obtain

$$\begin{aligned} c &= 1 + 12\left[\beta_c + \frac{1}{2\beta_c}\right]^2 \\ &= 1 - 12\left[\frac{1}{\beta\sqrt{\hbar}} - \frac{\beta\hbar}{2}\right]^2 \\ &= 13 - \frac{12}{\hbar\beta^2} - 3\hbar\beta^2 \end{aligned} \tag{46}$$

in agreement with (49).

IV. REMARKS

The Poisson brackets for the amplitudes $u(x)$ for KdV and $p(x)$ for MKdV, SG, and L models may all be obtained from the Virasoro algebra. In the classical version they share the infinite number of conserved quantities and the central charge. In the quantum version, however, the conserved quantities are modified, but the central charge of the SG and L models is still common. For the NLS model, the Poisson brackets of the amplitude $\psi(x)$ is consistent with the Virasoro algebra but a comparison of (7) and (33) indicates that the conserved quantities of the NLS are different from the other models.

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